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Fundamental solutions of the generalized axially symmetric Helmholtz equation

Maged G. Bin-Saad[1], **Anvar Hasanov**[2] **and Jihad A. Younis**[34]

Abstract. This paper deals with the fundamental solutions of two-dimensional elliptic equations with two singular coefficients. We construct the fundamental solutions of the generalized axially symmetric Helmholtz equation in terms of a confluent hypergeometric function in two variables.

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1. Introduction

Fundamental solutions play an important role in solving many problems in theory of elliptic equations. Dirichlet and Neumann problems for elliptic equation in some part of a ball were solved [21, 22]. Hasanov [10] constructed fundamental solutions of the generalized Helmholtz equation. In the paper [11], fundamental solutions were constructed for the generalized bi-axially symmetric Helmholz equation expressed by confluent hypergeometric functions of Kummer of three variables. Fundamental solutions for various modified Helmholtz equation were investigated by several authors (see, e.g.[1, 6, 7, 12, 14, 15, 18, 19, 20]).

In the domain $\Omega = \{(x,y) : -\infty < x < +\infty, y > 0\}$ consider the generalized axially symmetric Helmholtz equation

$$u_{xx} + u_{yy} + \frac{2\nu}{y} u_y + k^2 u = 0, \nu > 0. \tag{1.1}$$

The equation (1.1) was considered by Gilbert and Howard [9] and Kumar and Singh [17]. In particular the case $k = 0$ was studied by Erdelyi [4], Henrici [13], Kumar and Arora [16] and Srivastava [23].

In [8, p. 214], the solution for axially symmetric Helmholtaz equation (1.1) was constructed

$$u(x,y) = \Gamma(2\nu)(kr)^{-\nu} \sum_{n=0}^{\infty} \frac{a_n n!}{\Gamma(2\nu+n)} J_{\nu+n}(kr) C_n^{\nu}(cos\theta), \tag{1.2}$$

[1]Department of Mathematics, Aden University, e-mail: majed.math.edu@aden-unv.net
[2]Institute of Mathematics, Uzbek Academy of Sciences; Department of Mathematics, Analysis, Logic and Discrete Mathematics Ghent University,
e-mail: anvarhasanov@yahoo.com
[3]Department of Mathematics, Aden University, e-mail: jihadalsaqqaf@gmail.com
[4]Corresponding author

where $x = rcos\theta, y = rsin\theta$ and $J_{\nu+n}(kr)$ are Bessel functions of first kind and $C_n^{\nu}(cos\theta)$ are Gegenbauer polynomials.

In this paper, we aim to construct fundamental solutions of the equation (1.1) which have logarithmic singularities by means of the confluent hypergeometric function of two variables H_3.

2. The confluent hypergeometric function H_3

For our purpose, we begin by recalling the confluent hypergeometric Horn function H_3 defined by (see [4, p.226, (31)])

$$\mathsf{H}_3(a,b;c;x,y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m}{(c)_m} \frac{x^m}{m!} \frac{y^n}{n!}, (|x| < 1), \tag{2.1}$$

where $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$ denotes the Pochhammer symbol. Integral representation of the Euler type for the function H_3 is given by (see [20, p.91, (3.19)],

$$\begin{aligned} &\mathsf{H}_3(a,b;c;x,y) \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \alpha^{b-1} (1-\alpha)^{c-b-1} (1-x\alpha)^{-a} \\ &\quad \times {}_0F_1\left(1-a; -(1-x\alpha)\, y\right) d\alpha, \left(Re(c) > 0, Re(c-b) > 0\right), \end{aligned} \tag{2.2}$$

where ${}_0F_1$ is the confluent hypergeometric function of Kummer defined by (see [24])

$${}_0F_1\left(-;c;x\right) = \sum_{m=0}^{\infty} \frac{1}{(c)_m} \frac{x^m}{m!}. \tag{2.3}$$

Using the formula of derivation

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} \mathsf{H}_3(a,b;c;x,y) = \frac{(a)_{i-j}(b)_i}{(c)_i} \mathsf{H}_3(a+i-j, b+i; c+i; x, y), \tag{2.4}$$

it is easy to show that the confluent hypergeometric function in (2.1) satisfies the system of hypergeometric equations

$$\left\{ \begin{aligned} x(1-x)\omega_{xx} + xy\omega_{xy} + [c-(a+b+1)x]\omega_x + by\omega_y - ab\omega = 0, \\ y\omega_{yy} - x\omega_{xy} + (1-a)\omega_y + \omega = 0, \end{aligned} \right. \tag{2.5}$$

where

$$\omega(x,y) = \mathsf{H}_3(a,b;c;x,y).$$

Now, by virtue of the derivation formula (2.4), we have the following expressions:

$$\omega_x = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m}{(c)_m} \frac{(a+m-n)(b+m)}{(c+m)} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{2.6}$$

$$xw_x = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m}{(c)_m} \frac{m}{1} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{2.7}$$

$$y\omega_y = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m}{(c)_m} \frac{n}{1} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{2.8}$$

$$xy\omega_{xy} = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m}{(c)_m} \frac{mn}{1} \frac{x^m}{m!} \frac{y^n}{n!}, \tag{2.9}$$

$$x^2\omega_{xx} = \sum_{m,n=0}^{\infty} \frac{(a)_{m-n}(b)_m}{(c)_m} \frac{m(m-1)}{1} \frac{x^m}{m!} \frac{y^n}{n!}. \tag{2.10}$$

Substituting equalities (2.6)-(2.10) into the first equation of the system (2.5), we are convinced that the function $\omega(x, y)$ satisfies this equation. Similarly, it is easy to convince oneself that $\omega(x, y)$ satisfies the second equation of the same system.

Having substituted $\omega(x, y) = x^\mu y^\tau \varphi(x, y)$ in the system of hypergeometric equations (2.5), it is possible to be convinced that for the values
$\mu : 0, \quad 1 - c,$
$\tau : 0, \quad\quad 0,$
the system has two linearly independent solutions

$$\omega_1(x, y) = \mathsf{H}_3(a, b; c; x, y),$$

$$\omega_2(x, y) = x^{1-c}\mathsf{H}_3(a + 1 - c, b + 1 - c; 2 - c; x, y).$$

On the other hand, we consider certain expansions for the confluent hypergeometric function H_3 as follows:

$$\mathsf{H}_3(a, b; c; x, y) = (1 - x)^{-a} \sum_{i=0}^{\infty} \frac{(-1)^i(a)_i(c - b)_i}{(c)_i i!} \left(\frac{x}{1 - x}\right)^i {}_0F_1(-; 1 - a - i; -(1 - x)y), \tag{2.11}$$

$$\mathsf{H}_3(a, b; c; x, y) = \sum_{i=0}^{\infty} \frac{(-1)^i y^i}{(1 - a)_i i!} {}_2F_1(a - i, b; c; x), \tag{2.12}$$

where ${}_2F_1$ is the Gauss hypergeometric function defined by (see [24])

$${}_2F_1(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m}{(c)_m} \frac{x^m}{m!}. \tag{2.13}$$

In addition, the following transformation formula for ${}_2F_1$ has been introduced in [4, p.64, (22)]:

$$ {}_2F_1(a,b;c;x) = (1-x)^{-b}{}_2F_1\left(c-a,b;c;\frac{x}{x-1}\right). \tag{2.14} $$

Also, the Gauss hypergeometric function has the analytic continuation formula [5, p.110, (12)],

$$ \begin{aligned} &\frac{1}{\Gamma(a+b+i)}{}_2F_1(a,b;a+b+i;x) \\ &\quad= \frac{\Gamma(i)}{\Gamma(a+i)\Gamma(b+i)}\sum_{j=0}^{i-1}\frac{(a)_j(b)_j}{(1-i)_j j!}(1-x)^j \\ &\qquad+\frac{(-1)^i(1-x)^i}{\Gamma(a)\Gamma(b)}\sum_{j=0}^{\infty}\frac{(a+i)_j(b+i)_j}{(i+j)!j!}[\Lambda_j-\log(1-x)](1-x)^j, \\ &\qquad -\pi<\arg(1-x)<\pi,\ a,b\neq 0,-1,-2,\ldots, \end{aligned} \tag{2.15} $$

where $\Lambda_j=\psi(j+1)+\psi(i+j+1)-\psi(a+i+j)-\psi(b+i+j)$, the function $\psi(x)$ has the form

$$ \psi(x)=\ln x-\sum_{n=0}^{\infty}\left[\frac{1}{n+x}-\ln\left(1+\frac{1}{n+x}\right)\right],\ x>0, \tag{2.16} $$

$$ \psi(x)=\int_0^{\infty}e^{-\alpha}\ln\alpha d\alpha+\int_0^1\frac{1-\alpha^{x-1}}{1-\alpha}d\alpha,\ Re(x)>0. \tag{2.17} $$

The proofs of formulas (2.11) and (2.12) are based on symbolical method of Burchnall-Chaundy [2, 3]. We need the expansion (2.12) to study certain properties of the fundamental solutions. Further, Hasanov [11] obtained some expansions for the confluent hypergeometric function of three variables A_2.

3. Fundamental solutions

Let us consider equation (1.1) in the domain R_+^2. We seek a solution of the equation (1.1) in the form

$$ u=P\omega(\sigma_1,\sigma_2), \tag{3.1} $$

where

$$ P=\left(r^2\right)^{-\nu},\sigma_1=1-\frac{r_1^2}{r^2},\sigma_2=\frac{k^2}{4}r^2, $$

$$ r^2=(x-x_0)^2+(y-y_0)^2, r_1^2=(x-x_0)^2+(y+y_0)^2. $$

Substituting (3.1) into equation (1.1), we get

$$ A_1\omega_{\sigma_1\sigma_1}+A_2\omega_{\sigma_1\sigma_2}+A_3\omega_{\sigma_2\sigma_2}+B_1\omega_{\sigma_1}+B_2\omega_{\sigma_2}+C\omega=0, \tag{3.2} $$

where

$$A_1 = P\left[(\sigma_1)_x^2 + (\sigma_1)_y^2\right],\ A_2 = 2P\left[(\sigma_1)_x(\sigma_2)_x + (\sigma_1)_y(\sigma_2)_y\right],\ A_3 = P\left[(\sigma_2)_x^2 + (\sigma_2)_y^2\right],$$

$$B_1 = P(\sigma_1)_{xx} + P(\sigma_1)_{yy} + 2P_x(\sigma_1)_x + 2P_y(\sigma_1)_y + \frac{2\nu}{y}P(\sigma_1)_y,$$

$$B_2 = P(\sigma_2)_{xx} + P(\sigma_2)_{yy} + 2P_x(\sigma_2)_x + 2P_y(\sigma_2)_y + \frac{2\nu}{y}P(\sigma_2)_y,$$

$$C = P_{xx} + P_{yy} + \frac{2\nu}{y}P_y + k^2 P.$$

After elementary evaluations, we have

(3.3)
$$\begin{aligned} A_1 &= -\frac{4Py^{-1}y_0}{r^2}\sigma_1(1-\sigma_1), \\ A_2 &= -Pk^2\sigma_1 - \frac{4Py^{-1}y_0}{r^2}\sigma_1\sigma_2, \\ A_3 &= Pk^2\sigma_2, \end{aligned}$$

(3.4)
$$\begin{aligned} B_1 &= -\tfrac{4Py^{-1}y_0}{r^2}\left[2\nu - (1+2\nu)\sigma_1\right], \\ B_2 &= Pk^2(1-\nu) - \tfrac{4Py^{-1}y_0}{r^2}\nu\sigma_2, \end{aligned}$$

(3.5)
$$C = \frac{4Py^{-1}y_0}{r^2}\nu^2 + Pk^2.$$

Substituting (3.3)-(3.5) into equation (3.2) we have the following system of hypergeometric equations

(3.6)
$$\begin{cases} \sigma_1(1-\sigma_1)\omega_{\sigma_1\sigma_1} + \sigma_1\sigma_2\omega_{\sigma_1\sigma_2} + [2\nu - (2\nu+1)\sigma_1]\omega_{\sigma_1} + \nu\sigma_2\omega_{\sigma_2} - \nu^2\omega = 0, \\ \sigma_2\omega_{\sigma_2\sigma_2} - \sigma_1\omega_{\sigma_1\sigma_2} + (1-\nu)\omega_{\sigma_2} + \omega = 0. \end{cases}$$

The system of hypergeometric equations (3.6) has the following solutions

(3.7)
$$\omega_1(\sigma_1, \sigma_2) = \mathsf{H}_3(\nu, \nu; 2\nu; \sigma_1, \sigma_2),$$

(3.8)
$$\omega_2(\sigma_1, \sigma_2) = \sigma_1^{1-2\nu}\mathsf{H}_3(1-\nu, 1-\nu; 2-2\nu; \sigma_1, \sigma_2).$$

Substituting the solutions (3.7) and (3.8) into relation (3.1), we find two solutions for the generalized axially symmetric Helmholtz equation (1.1) in the forms

(3.9)
$$u_1(x, y; x_0, y_0) = k_1\left(r^2\right)^{-\nu}\mathsf{H}_3\left(\nu, \nu; 2\nu; 1 - \frac{r_1^2}{r^2}, \frac{k^2}{4}r^2\right),$$

(3.10)
$$u_2(x, y; x_0, y_0) = k_2\left(r^2\right)^{-\nu}\sigma_1^{1-2\nu}\mathsf{H}_3\left(1-\nu, 1-\nu; 2-2\nu; 1 - \frac{r_1^2}{r^2}, \frac{k^2}{4}r^2\right),$$

where k_1, k_2 are constants which are defined by solving the boundary value problems for equation (1.1).

Now, we show that the fundamental solution (3.9) possess a logarithmic singularity at $r \to 0$. For our purpose, we use the expansion (2.12) for the confluent hypergeometric function (2.1) and the formula (2.14). As a result, solution (3.9) can be written as follows:

$$\begin{aligned} & u_1(x,y;x_0,y_0) \\ (3.11) \quad = \; & k_1 \left(r_1^2\right)^{-\nu} {}_2F_1\left(\nu,\nu;2\nu;1-\frac{r^2}{r_1^2}\right) \\ & +k_1 \left(r_1^2\right)^{-\nu} \sum_{i=1}^{\infty} \frac{(-1)^i}{(1-\nu)i!} \left(\frac{k^2}{4}r^2\right)^i {}_2F_1\left(\nu+i,\nu;2\nu;1-\frac{r^2}{r_1^2}\right). \end{aligned}$$

Based on the formula (2.15), it follows from the expansion (3.11) that the constructed fundamental solution $u_1(x,y;x_0,y_0)$ has a logarithmic singularity at $r \to 0$. Similarly one can prove the fundamental solution $u_2(x,y;x_0,y_0)$ also has a logarithmic singularity at $r \to 0$.

It can be directly checked that constructed functions (3.9) and (3.10) possess the following properties:

$$(3.12) \qquad \begin{aligned} & \frac{\partial}{\partial x} u_1(x,y;x_0,y_0)|_{x=0} = 0, \quad && \frac{\partial}{\partial y} u_1(x,y;x_0,y_0)|_{y=0} = 0, \\ & u_2(x,y;x_0,y_0)|_{x=0} = 0, && \frac{\partial}{\partial y} u_2(x,y;x_0,y_0)|_{y=0} = 0. \end{aligned}$$

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