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Clausius theorem for hyperbolic scalar conservation laws

Sylvain Dotti[1]

Abstract. I give for hyperbolic scalar conservation laws with non-linear flux, the ideas and the computations behind the definitions of entropic solution and kinetic solution in different cases (homogeneous, deterministic source terms, stochastic source terms): they derive from the Clausius theorem. In conclusion, I give the definitions for a stochastic source term which is a general Lévy noise.

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1. Introduction

The partial differential equations called scalar conservation laws, and especially hyperbolic scalar conservation laws had been deeply studied since the nineteenth century. For a very complete inventory of the studies, see the introduction of the famous book of Constantine Dafermos [6].
The nineteenth century definition of solution required a function whose partial derivative exists in the classical sense with respect to the time variable and the space variable. In 1933, in order to allow discontinuous functions to be solutions of conservation laws, Jean Leray in [17], revolutionized the notion of solution by introducing test functions, that is smooth functions compactly supported, used with integration by parts. The name of the new solutions are 'weak solutions'.
Roughly speaking, Cauchy problems corresponding to hyperbolic scalar conservation laws had many solutions, without uniqueness, up to the famous article of Kruzkov [15]. His definition of solution is very important to solve the Cauchy problem. He solved the problem of uniqueness of weak solutions by adding a physical relevant condition called entropy condition with the so called vanishing viscosity method.
From that historical point, four directions are very interesting:

1. The improvement of the assumptions which give existence and uniqueness of the solution of a Cauchy problem by giving a definition of a generalized solution.

[1]CEMOI, Université de la Réunion, 15 Avenue René Cassin, CS 92003, 97744 Saint-Denis Cedex 9, La Réunion, France. e-mail: sylvain.dotti@univ-reunion.fr

2. The addition of a new variable modeling randomness or uncertainty and the resolution of the corresponding Cauchy problems.

3. The physical explanations, the links with physics, the particular cases from physics of the mathematical solutions of the Cauchy problems.

4. The methods which give the definition of a solution, or a generalized solution

For the item 1, I can quote the measure valued solutions of Di Perna in [8], the entropy process solutions of Eymard, Gallouët and Herbin in [13], the kinetic solutions of Lions, Perthame and Tadmor in [18].
For the item 2, I can quote the papers of E Mazel Khanin and Sinai [11], Kim [14], Debussche and Vovelle [7], Bauzet, Vallet and Wittbold [3], Biswas Karlsen and Majee [4], Dotti and Vovelle [9].
For the item 3, I can quote Dafermos with [5], Smoller with [21], Dubois with [10], Dafermos with [6], Evans with [12], and for a general review of the links between entropy and (ir)reversibility Villani with [22].
For the item 4, I can quote for the vanishing viscosity method: Kruzkov with [15], Lax with [16], Evans with [12], and for the method which can be called **Clausius theorem method**, I can quote Serre with [20], and Perthame with [19] for its mathematical part.
Note that the name 'Clausius theorem method' is mine, the method of Serre or Perthame is only mathematical. If Serre, in page 33 of his book, says that the solution of hyperbolic conservation laws has 'physical origin', Perthame in pages 3 or 61 uses this mathematical method without any mention of physical origin. Adding the link with the Clausius theorem, for a balance law of the type

$$\partial_t \left(u\left(x,t,\omega\right)\right) + \mathrm{div}_x \left(A\left(u\left(x,t,\omega\right)\right)\right) = G\left(u\left(x,t,\omega\right)\right), \tag{1.1}$$

with $A : \mathbb{R} \to \mathbb{R}^d, G : \mathbb{R} \to \mathbb{R}, u : \mathbb{R}^d \times [0;+\infty) \times \Omega \to \mathbb{R}$ where u is the unknown, $t \in [0;+\infty)$ is the time variable, $x \in \mathbb{R}^d$ is the space variable, ω is an elementary event of the probability space Ω, this method has two steps:

- If the quantity u, solution of a hyperbolic scalar conservation law of the type (1.1), models a reversible physical phenomenon (which can be only theoretical), then the function u is regular and the quantity of entropy $\eta(u)$ is conserved over time. Thus $\eta(u)$ is a solution of a hyperbolic scalar conservation law.

- If the quantity u, solution of a hyperbolic scalar conservation law of the type (1.1), models an irreversible physical phenomenon (which can be only theoretical), then the function u is not regular and the quantity of entropy $\eta(u)$ is decreasing over time. Thus $\eta(u)$ is a solution of an entropy inequality.

By applying this 'Clausius theorem method' in many cases of hyperbolic scalar conservation laws, I will try to convince the reader that this method is between

a physical principle and a mathematical postulate.
Of course, once the mathematical definition of a solution is given, then the mathematical proof of existence and uniqueness of the solution of a Cauchy problem validates the definition. But, with the complexification of the proofs, for example with the addition of a variable representing randomness, it is almost compulsory, in my opinion, to give the computations which leads to the definition of a solution. With those computations, I will explain why in [9] we changed the definition of [7] for the same equation and the same initial data. I will also explain why, with Kenneth Karlsen, we will slightly change (in an upcoming article) the definition of [4] for the same equation and the same initial data. More generally, the Clausius theorem method explains why it is more natural to work with a stochastic solution which is not weakly differentiated with the time variable t, but which is defined for all $t \in [0; +\infty)$. It is the reason why, in this article, I give for the first time the computations which leads to the definitions of the entropic and kinetic solutions of hyperbolic scalar conservation laws with different stochastic source terms with the **Clausius theorem method**. In conclusion of this article, I give for the first time the definition of an entropy solution and its kinetic formulation for hyperbolic scalar conservation laws driven by a general Lévy noise.

To end this introduction, once again I want to quote Evans with [12] who gives the most recent, complete study on the subject of this article. The ideas developed in my article differ from his ideas in this way: he uses the vanishing viscosity method to obtain the definitions of hyperbolic scalar conservation laws, and then, he notices (see remarks page 121) that 'We can regard the Lax entropy inequality as a form of Clausius-Duhem inequality, except that the sign is reversed'. Here, I say: there exists a law which is between the physical principle and the mathematical postulate, this law is formulated as follows: 'the mathematical entropies of hyperbolic scalar conservation laws follow the Clausius theorem'. This law gives the definitions of solutions to Cauchy problems. Furthermore, it is compulsory to begin by the definition of the deterministic homogeneous case (see the next Section 2) to have the definitions with a source term (deterministic or stochastic). The definition with a deterministic homogeneous source of Section 3 is a particular case of the definition with a deterministic source depending on time and space variables of Section 4. I give the computations of this particular case in detail because, they give the idea to replace the chain rule by the Itô Lemma to obtain the definition with a stochastic source term which can be a multiplicative brownian noise (see Section 5), which can be a pure jump Lévy noise (see Section 6), or which can be a general Lévy noise (see Section 7).

2. Case of the homogeneous hyperbolic scalar conservation law

2.1. Entropy solution

Let me explain the consequences of the Clausius theorem for the hyperbolic scalar conservation law

$$\partial_t\left(u\left(x,t\right)\right)+\mathrm{div}_x\left(A\left(u\left(x,t\right)\right)\right)=0,\quad t\in\mathbb{R}^+,x\in\mathbb{R}^d,d\in\mathbb{N}^*, \tag{2.1}$$

with $u:\mathbb{R}^d\times\mathbb{R}^+\mapsto\mathbb{R}$ called the conserved quantity and $A:\mathbb{R}\mapsto\mathbb{R}^d$ the flux. Without loss of generality, I suppose that $A(0)=0$.
When the physical phenomenon corresponding to (2.1) is reversible, the mathematical function u is regular. In this case, the quantity of entropy $\eta\left(u\left(x,t\right)\right)$ is conserved over time, and thus is a solution of the conservation law

$$\partial_t\left(\eta\left(u\left(x,t\right)\right)\right)+\mathrm{div}_x\left(\phi\left(u\left(x,t\right)\right)\right)=0.$$

$\eta:\mathbb{R}\mapsto\mathbb{R}$ is called the mathematical entropy, it is a convex function (see [15], [16]), $\phi:\mathbb{R}\mapsto\mathbb{R}^d$ is called the η entropy flux. Thermodynamic entropy is concave. The link between the two entropies is well described in the case of gas dynamics by Dubois [10].
If A, η and ϕ are regular functions, those two conservation laws can be written

$$\partial_t\left(u\left(x,t\right)\right)+A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)=0$$

and

$$\eta^{'}\left(u\left(x,t\right)\right)\partial_t\left(u\left(x,t\right)\right)+\phi^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)=0.$$

By identification of those two conservation laws, I get the definition (up to an additive constant) of the entropy flux

$$\phi^{'}\left(\xi\right)=A'\left(\xi\right)\eta'\left(\xi\right).$$

But when the physical phenomenon corresponding to (2.1) is irreversible, that is when u is not regular, a defect of mathematical entropy is created over time, that is written

$$\partial_t\left(\eta\left(u\left(x,t\right)\right)\right)+\mathrm{div}_x\left(\phi\left(u\left(x,t\right)\right)\right)\leq 0. \tag{2.2}$$

The inequality (2.2) is to be taken in the weak sense. It means to multiply by a test function $\varphi\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^+;\mathbb{R}^+)$, to integrate against the time variable $t\in\mathbb{R}^d$, against the space variable $x\in\mathbb{R}^d$, to integrate by parts $d+1$ times (against each real variable t and $x_i, i\in\{1,...,d\}$) to obtain under the initial data $u(x,0)=u_0(x)\in L^\infty(\mathbb{R}^d)$ the entropy inequality:

$$-\int_{\mathbb{R}^d\times\mathbb{R}^+}\left(\eta\left(u(x,t)\right)\partial_t\varphi\left(x,t\right)+\phi\left(u\left(x,t\right)\right).\nabla\varphi\left(x,t\right)\right)dxdt$$
$$-\int_{\mathbb{R}^d}\eta\left(u_0\left(x\right)\right)\varphi\left(x,0\right)dx\leq 0.$$

Remark 2.1. Kruzkov decided to use the mathematical entropies

$$\{u \in \mathbb{R} \mapsto |u-\xi| \text{ such that } \xi \in \mathbb{R}\}$$

to solve the problem of uniqueness of weak solutions of J. Leray [17]. They are convex functions. If he took the opposite of those functions, which are concave, the inequality (2.2) would be in the opposite sense, and we could say that mathematical entropy increase over time, just like physical entropy. But it is not the case !

Remark 2.2. The proof of the equivalence between Kruzkov entropies and convex functions (called mathematical entropies or Lax's mathematical entropies) in the entropy inequality (2.2) can be found for instance in [23].

2.2. Kinetic solution

2.2.1. A first method

Lions, Perthame and Tadmor replaced the entropy inequality by an equality called 'kinetic equality', introducing a supplementary variable $\xi \in \mathbb{R}$ thanks to the Kruzkov entropies and the corresponding entropy fluxes well chosen. More precisely, let us apply the inequality (2.2) to the entropies $\eta_\xi(u) = |u-\xi| - |\xi|$ and to the entropy fluxes $\phi_\xi(u) = \text{sgn}(u-\xi)\,(A(u) - A(\xi)) - \text{sgn}(\xi)A(\xi)$:

$$\begin{aligned} &\partial_t\big(|u\,(x,t)-\xi| - |\xi|\big) \\ &\quad + \text{div}_x\big(\text{sgn}(u(x,t)-\xi)\,(A(u(x,t)) - A(\xi)) - \text{sgn}(\xi)A(\xi)\big) \leq 0. \end{aligned} \tag{2.3}$$

The ξ was a constant for Kruzkov, it becomes a variable for Lions Perthame and Tadmor. They denote the left-hand side of (2.3) $-2m(x,t,\xi)$ and can write the entropy inequality

$$m(x,t,\xi) \geq 0.$$

The kinetic formulation of the entropy inequality (2.2) or simply of the hyperbolic scalar conservation law (2.1) is the equation giving the relation between $\partial_\xi m(x,t,\xi)$ and the partial derivatives (in the weak sense) of

$$\chi(\xi, u(x,t)) := \mathbf{1}_{u(x,t)>\xi} - \mathbf{1}_{0>\xi}.$$

From the equality

$$\begin{aligned} -2m(x,t,\xi) = {}& \partial_t\big(|u\,(x,t)-\xi| - |\xi|\big) \\ & + \text{div}_x\big(\text{sgn}(u(x,t)-\xi)\,(A(u(x,t)) - A(\xi)) - \text{sgn}(\xi)A(\xi)\big) \end{aligned}$$

and the partial derivatives

- $\partial_\xi\big(|u\,(x,t)-\xi| - |\xi|\big) = -2\chi(\xi, u(x,t))$
- $\partial_\xi\big(\text{sgn}(u(x,t)-\xi)\,(A_i(u(x,t)) - A_i(\xi)) - \text{sgn}(\xi)A_i(\xi)\big)$
  $= -2A_i^{'}(\xi)\chi(\xi, u(x,t))$

they obtain the kinetic formulation:

$$\partial_t \left(\chi\left(\xi, u\left(x,t\right)\right)\right) + A^{'}\left(\xi\right).\nabla_x \left(\chi\left(\xi, u\left(x,t\right)\right)\right) = \partial_\xi m(x,t,\xi)$$

where m is a bounded non-negative measure.

Remark 2.3. When the physical process is reversible, that is when u is regular, no entropy is absorbed over time, the kinetic entropy defect measure m is the null measure.
When the physical process is irreversible, that is when u is not regular, m (or rather $2m$) measures the entropy defect created over time, in other words the entropy absorption over time.

Remark 2.4. The choice of Lions, Perthame and Tadmor in [18] of couples entropy/entropy flux

$$|u\left(x,t\right) - \xi| - |\xi|, \quad \mathrm{sgn}(u(x,t) - \xi)\left(A(u(x,t)) - A(\xi)\right) - \mathrm{sgn}(\xi)A(\xi)$$

is done to find, differentiating with respect to ξ, the function $\chi(\xi, u(x,t))$ which belongs to $L^1\left(\mathbb{R}_\xi\right)$. The choice of Debussche and Vovelle in [7] of couples entropy/entropy flux

$$|u\left(x,t\right) - \xi| - \xi, \quad \mathrm{sgn}(u(x,t) - \xi)\left(A(u(x,t)) - A(\xi)\right) - A(\xi),$$

gives a kinetic formulation with the locally integrable function $\mathbf{1}_{u(x,t)>\xi}$ in place of $\chi(\xi, u(x,t))$. Other choices are possible.

2.2.2. A second method

For the conservation law

$$\partial_t \left(u\left(x,t\right)\right) + \mathrm{div}_x \left(A\left(u\left(x,t\right)\right)\right) = 0,$$

the second method to obtain the kinetic formulation consists in considering the corresponding equality of measures

$$\partial_t \left(u\left(x,t\right)\right) dtdx + \mathrm{div}_x \left(A\left(u\left(x,t\right)\right)\right) dtdx = 0$$

when u and A are regular, then in doing the operation $\otimes\delta_{u(x,t)}\left(d\xi\right)$ both sides to obtain the equality (weak in ξ)

$$\partial_t \left(\chi\left(\xi, u\left(x,t\right)\right)\right) + A^{'}\left(\xi\right).\nabla_x \left(\chi\left(\xi, u\left(x,t\right)\right)\right) = 0$$

by the use of the formulas

- $\partial_t \left(\chi\left(\xi, u\left(x,t\right)\right)\right) = \partial_t u\left(x,t\right) dtdx \otimes \delta_{u(x,t)}\left(d\xi\right)$
- $\partial_{x_i} \left(\chi\left(\xi, u\left(x,t\right)\right)\right) = \partial_{x_i} u\left(x,t\right) dtdx \otimes \delta_{u(x,t)}\left(d\xi\right)$
- $A^{'}\left(\xi\right).\nabla_x \left(\chi\left(\xi, u\left(x,t\right)\right)\right) = A^{'}\left(u\left(x,t\right)\right).\nabla_x \left(u\left(x,t\right)\right) dtdx \otimes \delta_{u(x,t)}\left(d\xi\right).$

When the quantity of entropy is conserved, that is when

$$\partial_t \left(\eta \left(u \left(x,t\right)\right)\right) + \mathrm{div}_x \left(\phi \left(u \left(x,t\right)\right)\right) = 0,$$

by doing the operation $dtdx \otimes \delta_{u(x,t)} \left(d\xi\right)$ both sides of the equality, I obtain the equality (weak in ξ)

$$\eta^{'} \left(\xi\right) \partial_t \left(\chi \left(\xi, u \left(x,t\right)\right)\right) + \phi^{'} \left(\xi\right) . \nabla_x \left(\chi \left(\xi, u \left(x,t\right)\right)\right) = 0. \tag{2.4}$$

When the physical process is irreversible, that is when u is not regular, an entropy defect is created which is written

$$\partial_t \left(\eta \left(u \left(x,t\right)\right)\right) + \mathrm{div}_x \left(\phi \left(u \left(x,t\right)\right)\right) \leq 0$$

or which is also written with the kinetic variable ξ

$$\int_{\mathbb{R}_\xi} \left(\eta^{'} \left(\xi\right) \partial_t \left(\chi \left(\xi, u \left(x,t\right)\right)\right) + \eta^{'} \left(\xi\right) A^{'}(\xi) . \nabla_x \left(\chi \left(\xi, u \left(x,t\right)\right)\right) \right) d\xi \leq 0,$$

or which is also written with an equality, defining the kinetic entropy defect measure m up to an additive constant by

$$\partial_\xi m(dx, dt, d\xi) = \partial_t \left(\chi \left(\xi, u \left(x,t\right)\right)\right) + A^{'} \left(\xi\right) . \nabla_x \left(\chi \left(\xi, u \left(x,t\right)\right)\right), \tag{2.5}$$

with

$$\int_{\mathbb{R}_\xi} \eta^{''}(\xi) m(dx, dt, d\xi) \geq 0, \quad \forall \eta : \mathbb{R} \to \mathbb{R} \text{ which is convex.}$$

Thus, it is the measure $\eta^{''}(\xi)m(dx, dt, d\xi)$ which measures the entropy of the quantity $u(x,t)$ over time. We can also say that measuring the entropy $\eta(u(x,t))$ over time is the same as measuring $\eta^{''}$ with m.
The equation (2.5) is called the kinetic formulation of the hyperbolic scalar conservation law (2.1), where the non-negativity of the measure m is equivalent to the entropy inequality (2.2).

3. Case of the hyperbolic scalar conservation law with a deterministic homogeneous source term

3.1. Entropy solution

For the conservation law with source term

$$\partial_t \left(u \left(x,t\right)\right) + \mathrm{div}_x \left(A \left(u \left(x,t\right)\right)\right) = G \left(u \left(x,t\right)\right), \tag{3.1}$$

here are the consequences of the Clausius theorem.
When the physical phenomenon corresponding to (3.1) is reversible, that is when u is regular, the quantity of entropy $\eta \left(u \left(x,t\right)\right)$ is conserved thus

$$\partial_t \left(\eta \left(u \left(x,t\right)\right)\right) + \mathrm{div}_x \left(\phi \left(u \left(x,t\right)\right)\right) = H \left(u \left(x,t\right)\right)$$

where $H\left(u\left(x,t\right)\right)$ is the source of entropy. If A, η and ϕ are regular functions, these two conservation laws can be written

$$\partial_t\left(u\left(x,t\right)\right)+A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)=G\left(u\left(x,t\right)\right)$$

and

$$\eta^{'}\left(u\left(x,t\right)\right)\partial_t\left(u\left(x,t\right)\right)+\phi^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)=H\left(u\left(x,t\right)\right).$$

By identification of these two conservation laws, I get the definition of the entropy source

$$H\left(\xi\right)=G\left(\xi\right)\eta'\left(\xi\right)$$

But when the physical phenomenon corresponding to (3.1) is irreversible, that is when u is not regular, a defect of entropy is created over time which is written

$$\partial_t\left(\eta\left(u\left(x,t\right)\right)\right)+\operatorname{div}_x\left(\phi\left(u\left(x,t\right)\right)\right)-\eta^{'}\left(u\left(x,t\right)\right)G\left(u\left(x,t\right)\right)\leq 0. \tag{3.2}$$

3.2. Kinetic solution

To find the kinetic formulation of (3.1) or of (3.2), I apply the inequality (3.2) to the entropies $\eta_\xi(u)=|u-\xi|-|\xi|$ and the entropy fluxes $\phi_\xi(u)=\operatorname{sgn}(u-\xi)\left(A(u)-A(\xi)\right)-\operatorname{sgn}(\xi)A(\xi)$:

$$\begin{aligned}&\partial_t\big(|u\left(x,t\right)-\xi|-|\xi|\big)\\&\quad+\operatorname{div}_x\big(\operatorname{sgn}(u(x,t)-\xi)\left(A(u(x,t))-A(\xi)\right)-\operatorname{sgn}(\xi)A(\xi)\big)\\&\quad-\operatorname{sgn}(u(x,t)-\xi)G\left(u\left(x,t\right)\right)\leq 0.\end{aligned} \tag{3.3}$$

I denote the left-hand side of (3.3) $-2m(x,t,\xi)$ and can write the entropy inequality

$$m(x,t,\xi)\geq 0.$$

The kinetic formulation of the entropy inequality (3.2) or simply of the hyperbolic scalar conservation law (3.1) is the equation giving the relation between $\partial_\xi m(x,t,\xi)$ and the partial derivatives (in the weak sense) of $\chi(\xi,u(x,t))$. From the equality

$$\begin{aligned}-2m(x,t,\xi)=&\,\partial_t\big(|u\left(x,t\right)-\xi|-|\xi|\big)\\&+\operatorname{div}_x\big(\operatorname{sgn}(u(x,t)-\xi)\left(A(u(x,t))-A(\xi)\right)-\operatorname{sgn}(\xi)A(\xi)\big)\\&-\operatorname{sgn}(u(x,t)-\xi)G\left(u\left(x,t\right)\right)\end{aligned}$$

and the weak derivatives

- $\partial_\xi\big(|u\left(x,t\right)-\xi|-|\xi|\big)=-2\chi(\xi,u(x,t))$
- $\partial_\xi\big(\operatorname{sgn}(u(x,t)-\xi)\left(A_i(u(x,t))-A_i(\xi)\right)-\operatorname{sgn}(\xi)A_i(\xi)\big)$
  $=-2A_i^{'}(\xi)\chi(\xi,u(x,t))$
- $\partial_\xi\big(\operatorname{sgn}(u(x,t)-\xi)G\left(u(x,t)\right)\big)=-2G\left(\xi\right)\delta_{u(x,t)}(d\xi)$

I obtain the kinetic formulation :

$$\partial_t \left(\chi\left(\xi, u\left(x,t\right)\right)\right) + A^{'}\left(\xi\right).\nabla_x \left(\chi\left(\xi, u\left(x,t\right)\right)\right) - G\left(\xi\right)\delta_{u(x,t)}(d\xi) = \partial_\xi m(x,t,\xi).$$

Remark 3.1. The inequality

$$\partial_\xi \big(\chi\left(\xi, u\left(x,t\right)\right)\big) = \delta_0(d\xi) - \delta_{u(x,t)}(d\xi)$$

allows me to write the kinetic formulation

$$\begin{aligned}&\partial_t \left(\chi\left(\xi, u\left(x,t\right)\right)\right) + A^{'}\left(\xi\right).\nabla_x \left(\chi\left(\xi, u\left(x,t\right)\right)\right)\\ &\quad + G\left(\xi\right)\partial_\xi\big(\chi\left(\xi, u\left(x,t\right)\right)\big) - G\left(\xi\right)\delta_0(d\xi) = \partial_\xi m(x,t,\xi).\end{aligned}$$

4. Case of the hyperbolic scalar conservation law with a deterministic source term depending on space and time variables

4.1. Entropy solution

Now, I can write the consequences of the Clausius theorem for the conservation law with source term

$$\partial_t \left(u\left(x,t\right)\right) + \mathrm{div}_x \left(A\left(x,t,u\left(x,t\right)\right)\right) = G\left(x,t,u\left(x,t\right)\right) \tag{4.1}$$

with $A : \mathbb{R}_x^d \times \mathbb{R}_t^+ \times \mathbb{R}_\xi \to \mathbb{R}^d$ and $G : \mathbb{R}_x^d \times \mathbb{R}_t^+ \times \mathbb{R}_\xi \to \mathbb{R}$.
When the physical phenomenon corresponding to (4.1) is reversible, that is when u is regular, the entropy quantity $\eta\left(u\left(x,t\right)\right)$ is conserved thus

$$\partial_t \left(\eta\left(u\left(x,t\right)\right)\right) + \mathrm{div}_x \left(\phi\left(x,t,u\left(x,t\right)\right)\right) = H\left(x,t,u\left(x,t\right)\right),$$

$H\left(x,t,u\left(x,t\right)\right)$ being the entropy source. With A, η and ϕ being regular functions, those two conservation laws can be written

$$\begin{aligned}&\partial_t \left(u\left(x,t\right)\right) + \left(\mathrm{div}_x A\right)\left(x,t,u(x,t)\right)\\ &\quad + \partial_\xi A\left(x,t,u\left(x,t\right)\right).\nabla_x \left(u\left(x,t\right)\right) = G\left(x,t,u\left(x,t\right)\right)\end{aligned}$$

and

$$\begin{aligned}&\eta^{'}\left(u(x,t)\right)\partial_t \left(u\left(x,t\right)\right) + \left(\mathrm{div}_x \phi\right)\left(x,t,u(x,t)\right)\\ &\quad + \partial_\xi \phi\left(x,t,u\left(x,t\right)\right).\nabla_x \left(u\left(x,t\right)\right) = H\left(x,t,u\left(x,t\right)\right).\end{aligned}$$

By identification of these two conservation laws, I have only one possibility for the definition of the entropy flux:

$$\partial_\xi \phi(x,t,\xi) = \eta^{'}(\xi)\partial_\xi A\left(x,t,\xi\right).$$

Multiplying the first conservation law by $\eta^{'}(u(x,t))$ and subtracting the second conservation law, I obtain

$$\begin{aligned}&\eta^{'}(u(x,t))\left(\mathrm{div}_x A\right)\left(x,t,u(x,t)\right) - \left(\mathrm{div}_x \phi\right)\left(x,t,u(x,t)\right)\\ &\quad = \eta^{'}(u(x,t))G\left(x,t,u\left(x,t\right)\right) - H\left(x,t,u\left(x,t\right)\right).\end{aligned}$$

That gives the definition of the entropy source:

$$H\left(x,t,\xi\right)=\eta^{'}(\xi)G\left(x,t,\xi\right)-\eta^{'}(\xi)\left(\mathrm{div}_{x}A\right)\left(x,t,\xi\right)+\left(\mathrm{div}_{x}\phi\right)\left(x,t,\xi\right).$$

In the case where $\eta(\xi)=|\xi-\kappa|$ is a Kruzkov entropy, the corresponding entropy flux can be written

$$\phi(x,t,\xi)=\mathrm{sgn}(\xi-\kappa)\left(A(x,t,\xi)-A(x,t,\kappa)\right),$$

that gives (in that particular case), the definition of the entropy source

$$H\left(x,t,\xi\right)=\eta^{'}(\xi)G\left(x,t,\xi\right)-\eta^{'}(\xi)\left(\mathrm{div}_{x}A\right)\left(x,t,\kappa\right).$$

But when the physical phenomenon corresponding to (4.1) is irreversible, that is when u is not regular, an entropy defect is created over time which can be written

$$\partial_t\left(\eta\left(u\left(x,t\right)\right)\right)+\mathrm{div}_x\left(\phi\left(x,t,u\left(x,t\right)\right)\right)-H\left(x,t,u\left(x,t\right)\right)\leq 0 \tag{4.2}$$

or, with the test functions $(x,t)\mapsto\varphi(x,t)\in C_c^\infty(\mathbb{R}^d\times\mathbb{R}^+\ ;\ \mathbb{R}^+)$:

$$\begin{aligned}-\int_{\mathbb{R}^d}\eta(u(x,0))\varphi(x,0)dx-\int_{\mathbb{R}^d\times\mathbb{R}^+}\eta(u(x,t))\partial_t\varphi(x,t)dxdt\\ -\int_{\mathbb{R}^d\times\mathbb{R}^+}\phi\left(x,t,u\left(x,t\right)\right).\nabla_x\varphi(x,t)dxdt\\ -\int_{\mathbb{R}^d\times\mathbb{R}^+}\eta^{'}(u(x,t))\big(G-\left(\mathrm{div}_xA\right)\big)\left(x,t,u(x,t)\right)\varphi(x,t)dxdt\\ -\int_{\mathbb{R}^d\times\mathbb{R}^+}\left(\mathrm{div}_x\phi\right)\left(x,t,u(x,t)\right)\varphi(x,t)dxdt\leq 0.\end{aligned}$$

4.2. Kinetic solution

To give the kinetic formulation of (4.1) or of (4.2), I apply the inequality (4.2) to the entropies $\eta_\xi(u)=|u-\xi|-|\xi|$ and to the entropy fluxes $\phi_\xi(x,t,u)=\mathrm{sgn}(u-\xi)\left(A(x,t,u)-A(x,t,\xi)\right)-\mathrm{sgn}(\xi)A(x,t,\xi)$:

$$\begin{aligned}&\partial_t\big(|u\left(x,t\right)-\xi|-|\xi|\big)\\ &\quad+\mathrm{div}_x\big(\mathrm{sgn}(u(x,t)-\xi)\left(A(x,t,u(x,t))-A(x,t,\xi)\right)-\mathrm{sgn}(\xi)A(x,t,\xi)\big)\\ &\quad-H_\xi\left(x,t,u(x,t)\right)\leq 0\end{aligned} \tag{4.3}$$

with

$$\begin{aligned}H_\xi(x,t,u)&=\mathrm{sgn}(u-\xi)\big(G(x,t,u)-(\mathrm{div}_xA)(x,t,\xi)\big)-\mathrm{sgn}(\xi)\left(\mathrm{div}_xA\right)(x,t,\xi)\\ &=\mathrm{sgn}(u-\xi)G(x,t,u)-2\chi(\xi,u)\left(\mathrm{div}_xA\right)(x,t,\xi),\end{aligned}$$

which gives

$$\begin{aligned}&\partial_t\big(|u\left(x,t\right)-\xi|-|\xi|\big)\\ &\quad+\mathrm{div}_x\big(\mathrm{sgn}(u(x,t)-\xi)\left(A(x,t,u(x,t))-A(x,t,\xi)\right)-\mathrm{sgn}(\xi)A(x,t,\xi)\big)\\ &\quad-\mathrm{sgn}(u(x,t)-\xi)G(x,t,u(x,t))+2\chi(\xi,u(x,t))\left(\mathrm{div}_xA\right)(x,t,\xi)\leq 0.\end{aligned}$$

I denote the left-hand side of (4.3) $-2m(x,t,\xi)$ and can write the entropy inequality

$$m(x,t,\xi) \geq 0.$$

The kinetic formulation of the entropy inequality (4.2) or simply of the hyperbolic scalar conservation law (4.1) is the equation giving the link between $\partial_\xi m(x,t,\xi)$ and the weak partial derivatives of $\chi(\xi,u(x,t))$.
From the equality

$$\begin{aligned}-2m(x,t,\xi) &= \partial_t\big(|u\,(x,t)-\xi|-|\xi|\big)\\ &\quad+\operatorname{div}_x\big(\operatorname{sgn}(u(x,t)-\xi)\,(A(x,t,u(x,t))-A(x,t,\xi))-\operatorname{sgn}(\xi)A(x,t,\xi)\big)\\ &\quad-\operatorname{sgn}(u(x,t)-\xi)G(x,t,u(x,t))+2\chi(\xi,u(x,t))\,(\operatorname{div}_x A)\,(x,t,\xi)\end{aligned}$$

and the weak derivatives

- $\partial_\xi\big(|u-\xi|-|\xi|\big)=-2\chi(\xi,u)$
- $\partial_\xi\big(\phi_{\xi,i}(x,t,u)\big)=-2\partial_\xi\big(A_i(x,t,\xi)\big)\chi(\xi,u)$
- $\partial_\xi\big(\operatorname{sgn}(u-\xi)G(x,t,u)\big)=-2G\,(x,t,\xi)\,\delta_u(d\xi)$
- $\partial_\xi\big(\chi(\xi,u)\operatorname{div}_x A(x,t,\xi)\big)$
  $=\partial_\xi\big(\chi(\xi,u)\big)\operatorname{div}_x A(x,t,\xi)+\chi(\xi,u)\partial_\xi\big(\operatorname{div}_x A(x,t,\xi)\big),$

I obtain the kinetic formulation:

$$\begin{aligned}&\partial_t\,(\chi\,(\xi,u\,(x,t)))+\partial_\xi A\,(x,t,\xi)\,.\nabla_x\,(\chi\,(\xi,u\,(x,t)))\\ &\quad-G\,(x,t,\xi)\,\delta_{u(x,t)}(d\xi)-\partial_\xi\big(\chi(\xi,u(x,t))\big)\operatorname{div}_x A(x,t,\xi)=\partial_\xi m(x,t,\xi).\end{aligned}$$

Remark 4.1. The equality

$$\partial_\xi\big(\chi\,(\xi,u\,(x,t))\,\big)=\delta_0(d\xi)-\delta_{u(x,t)}(d\xi)$$

allows me to write the kinetic formulation

$$\begin{aligned}\partial_t\,(\chi\,(\xi,u\,(x,t)))&+\partial_\xi A\,(x,t,\xi)\,.\nabla_x\,(\chi\,(\xi,u\,(x,t)))\\ &+\big(G\,(x,t,\xi)-\operatorname{div}_x A(x,t,\xi)\big)\partial_\xi\big(\chi\,(\xi,u\,(x,t))\,\big)\\ &-G\,(x,t,\xi)\,\delta_0(d\xi)=\partial_\xi m(x,t,\xi).\end{aligned}$$

5. Case of the hyperbolic scalar conservation law with a source term which is a multiplicative brownian noise

5.1. Entropy solution

Let $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t))$ be a filtered probability space satisfying the usual hypothesis of right-continuity and completeness. Let $T>0$ and $d\in\mathbb{N}^*$. Let us study the first-order hyperbolic scalar conservation law with a continuous in time stochastic source term

$$d\,(u\,(x,t,\omega))+\operatorname{div}_x\,(A\,(u\,(x,t,\omega)))\,dt=\Phi\,(x,u\,(x,t,\omega))\,dW\,(t,\omega)\,,\tag{5.1}$$

with W a cylindrical Wiener process defined on a separable Hilbert space H and $\Phi:\ \mathbb{R}^d\times\mathbb{R}\to L_2(H,\mathbb{R})$ a continuous function (see [9] for details on assumptions).
To simplify the calculations, one can think of W as a real brownian motion and $\Phi:\ \mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ continuous, the ideas and results are similar. To simplify the notations, I will omit the variable ω. Here are the consequences of the Clausius theorem:
If the physical phenomenon associated with the conservation law was reversible, that is if the diffusion process u was regular in the space variable $x\in\mathbb{R}^d$, the quantity of entropy $\eta\left(u\left(x,t\right)\right)$ would be conserved thus

$$d\left(\eta\left(u\left(x,t\right)\right)\right)+\mathrm{div}_x\left(\phi\left(u\left(x,t\right)\right)\right)dt=d\left(H\left(u\left(x,t\right)\right)\right)$$

where $H\left(u\left(x,t\right)\right)$ is the entropy source. If A, η and ϕ were regular functions, the two conservation laws could be written

$$d\left(u\left(x,t\right)\right)+A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)dt=\Phi\left(x,u\left(x,t\right)\right)dW\left(t\right)$$

and

$$d\left(\eta\left(u\left(x,t\right)\right)\right)+\phi^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)dt=d\left(H\left(u\left(x,t\right)\right)\right).$$

To identify the two conservation laws, I have to notice that the time 'chain rule' is not true for diffusion processes. Instead, I use the Itô Lemma. In other words, I don't have

$$d\left(\eta\left(u\left(x,t\right)\right)\right)=\eta^{'}\left(u\left(x,t\right)\right)d\left(u\left(x,t\right)\right)$$

but I have

$$d\left(\eta\left(u\left(x,t\right)\right)\right)=\eta^{'}\left(u\left(x,t\right)\right)d\left(u\left(x,t\right)\right)+\frac{1}{2}\eta^{''}\left(u\left(x,t\right)\right)\|\Phi\left(x,u\left(x,t\right)\right)\|^2_{L_2(H,\mathbb{R})}dt$$

for any convex function $\eta\in C^2\left(\mathbb{R}\right)$. Hence, the conservation law of the entropy would be written

$$\begin{aligned}\eta^{'}\left(u\left(x,t\right)\right)d\left(u\left(x,t\right)\right)&+\frac{1}{2}\eta^{''}\left(u\left(x,t\right)\right)\|\Phi\left(x,u\left(x,t\right)\right)\|^2_{L_2(H,\mathbb{R})}dt\\&+\phi^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)dt=d\left(H\left(u\left(x,t\right)\right)\right)\end{aligned}$$

By identification of the two conservation laws, I obtain the definition of the entropy source

$$\begin{aligned}d\left(H\left(u\left(x,t\right)\right)\right)=&\frac{1}{2}\eta^{''}\left(u\left(x,t\right)\right)\|\Phi\left(x,u\left(x,t\right)\right)\|^2_{L_2(H,\mathbb{R})}dt\\&+\eta^{'}\left(u\left(x,t\right)\right)\Phi\left(x,u\left(x,t\right)\right)dW(t).\end{aligned}$$

If the physical phenomenon associated with the conservation law was irreversible, that is if the diffusion process u was not regular in the space variable $x\in\mathbb{R}^d$, that would create an entropy defect which could be written

$$\begin{aligned}&d\left(\eta\left(u\left(x,t\right)\right)\right)-\frac{1}{2}\eta^{''}\left(u\left(x,t\right)\right)\|\Phi\left(x,u\left(x,t\right)\right)\|^2_{L_2(H,\mathbb{R})}dt\\&+\mathrm{div}_x\left(\phi\left(u\left(x,t\right)\right)\right)dt-\eta^{'}\left(u\left(x,t\right)\right)\Phi\left(x,u\left(x,t\right)\right)dW(t)\le 0.\end{aligned}\tag{5.2}$$

This inequality is a weak in space formulation of the conservation law with a source term which is a multiplicative brownian noise. It is written with the test functions $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^+)$, for all fixed $t \in \mathbb{R}^+$, almost surely, in the following way:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \varphi(x)\big(\eta\left(u\left(x,t\right)\right)-\eta\left(u\left(x,0\right)\right)\big)dx - \int_{\mathbb{R}^d}\int_0^t \phi\left(u(x,s)\right)ds.\nabla_x\varphi(x)dx\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^d}\varphi(x)\int_0^t \eta^{''}\left(u\left(x,s\right)\right)\|\Phi\left(x,u\left(x,s\right)\right)\|^2_{L_2(H,\mathbb{R})}dsdx\\
&\quad -\int_{\mathbb{R}^d}\varphi(x)\int_0^t \eta^{'}\left(u\left(x,s\right)\right)\Phi\left(x,u\left(x,s\right)\right)dW\left(s\right)dx \leq 0.
\end{aligned}$$

5.2. Kinetic solution

To get the kinetic formulation of

$$d\left(u\left(x,t,\omega\right)\right)+\operatorname{div}_x\left(A\left(u\left(x,t,\omega\right)\right)\right)dt = \Phi\left(x,u\left(x,t,\omega\right)\right)dW\left(t,\omega\right), \tag{5.3}$$

I perform the operation $dx \otimes \delta_{u(x,t,\omega)}\left(d\xi\right)$ to both sides of the equality and apply the Clausius theorem method (the variable ω is now omitted).
If u was regular in the space variable $x \in \mathbb{R}^d$, that is if the physical phenomenon associated with (5.3) was reversible, I would obtain

$$\begin{aligned}
&\left(d\left(u\left(x,t\right)dx\right)+A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)dtdx\right)\otimes\delta_{u(x,t)}\left(d\xi\right)\\
&\qquad = \Phi\left(x,u\left(x,t\right)\right)dW\left(t\right)dx\otimes\delta_{u(x,t)}\left(d\xi\right).
\end{aligned}\tag{5.4}$$

The 'chain rule' is untrue for Itô processes, instead I use a weak in x,ξ Itô Lemma:

$$\begin{aligned}
d\left(\chi\left(\xi,u\left(x,t\right)\right)d\xi dx\right) =& d\left(u\left(x,t\right)dx\right)\otimes\delta_{u(x,t)}\left(d\xi\right)\\
&-\frac{1}{2}\partial_\xi\left(\|\Phi\left(x,\xi\right)\|^2_{L_2(H,\mathbb{R})}dtdx\otimes\delta_{u(x,t)}\left(d\xi\right)\right).
\end{aligned}$$

Remark 5.1. Digression on the weak in x,ξ Itô Lemma for diffusion processes. If $\varphi \in C_c^\infty\left(\mathbb{R}\right)$, then the Itô Lemma applied to the process u gives

$$d\varphi\left(u\left(x,t\right)\right)=\varphi^{'}\left(u\left(x,t\right)\right)du\left(x,t\right)+\frac{1}{2}\varphi^{''}\left(u\left(x,t\right)\right)\|\Phi\left(x,u\left(x,t\right)\right)\|^2_{L_2(H,\mathbb{R})}dt.$$

I can multiply by $h(x)$ for any $h \in C_c^\infty\left(\mathbb{R}^d\right)$, and integrate over $\mathbb{R}^d$ against dx to obtain:

$$\begin{aligned}
d\left(\int_{\mathbb{R}^d}h(x)\varphi\left(u\left(x,t\right)\right)dx\right) =& \int_{\mathbb{R}^d}h(x)\varphi^{'}\left(u\left(x,t\right)\right)du\left(x,t\right)dx\\
&+\frac{1}{2}\int_{\mathbb{R}^d}h(x)\varphi^{''}\left(u\left(x,t\right)\right)\|\Phi\left(x,u\left(x,t\right)\right)\|^2_{L_2(H,\mathbb{R})}dtdx
\end{aligned}$$

which can be written

$$
\begin{aligned}
& d\left(\int_{\mathbb{R}^d} h(x) \int_{\mathbb{R}} \varphi^{'}(\xi)\, \chi\left(\xi, u\left(x,t\right)\right) d\xi dx\right) \\
& \quad = \int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}(\xi)\, du\left(x,t\right) dx \otimes \delta_{u(x,t)}\left(d\xi\right) \\
& \quad + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{''}(\xi)\, \|\Phi\left(x,\xi\right)\|^2_{L_2(H,\mathbb{R})} dt dx \otimes \delta_{u(x,t)}\left(d\xi\right)
\end{aligned}
$$

that is

$$
\begin{aligned}
& d\left(\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}(\xi)\, \chi\left(\xi, u\left(x,t\right)\right) d\xi dx\right) \\
& \quad = \int_{\mathbb{R}^d\times\mathbb{R}} \varphi^{'}(\xi)\, du\left(x,t\right) dx \otimes \delta_{u(x,t)}\left(d\xi\right) \\
& \quad - \frac{1}{2}\left\langle \partial_\xi \left(\|\Phi\left(x,\xi\right)\|^2_{L_2(H,\mathbb{R})} dt dx \otimes \delta_{u(x,t)}\left(d\xi\right)\right), \varphi^{'}(\xi)\, h(x)\right\rangle
\end{aligned}
$$

which gives the following weak in x, ξ Itô formula

$$
\begin{aligned}
d\left(\chi\left(\xi, u\left(x,t\right)\right) d\xi dx\right) = & du\left(x,t\right) dx \ \otimes \delta_{u(x,t)}\left(d\xi\right) \\
& - \frac{1}{2}\partial_\xi \left(\|\Phi\left(x,\xi\right)\|^2_{L_2(H,\mathbb{R})} dt dx \otimes \delta_{u(x,t)}\left(d\xi\right)\right).
\end{aligned}
$$

Using the weak in x, ξ Itô Lemma and the following formula

$$
A^{'}(\xi).\nabla_x\big(\chi\left(\xi, u(x,t)\right)\big) = A^{'}\left(u(x,t)\right).\nabla_x\left(u(x,t)\right) dx \otimes \delta_{u(x,t)}(d\xi),
$$

the equation (5.4) would be written

$$
\begin{aligned}
d\big(\chi\left(\xi, u\left(x,t\right)\right) d\xi dx\big) + A^{'}(\xi)\,.\nabla_x\left(\chi\left(\xi, u\left(x,t\right)\right)\right) dt & \\
- \Phi\left(x,\xi\right) dx \otimes \delta_{u(x,t)}\left(d\xi\right) dW\left(t\right) & \\
+ \frac{1}{2}\partial_\xi \left(\|\Phi\left(x,\xi\right)\|^2_{L_2(H,\mathbb{R})} dt dx \otimes \delta_{u(x,t)}\left(d\xi\right)\right) & = 0.
\end{aligned}
$$

If the physical phenomenon associated with the conservation law was irreversible, that is if u was not regular in $x \in \mathbb{R}^d$, the entropy defect would be written

$$
\begin{aligned}
& d\big(\chi(\xi, u\left(x,t,\omega\right)) d\xi dx\big) \\
& \quad + A^{'}(\xi)\,.\nabla_x\left(\chi\left(\xi, u\left(x,t,\omega\right)\right)\right) dt - \Phi\left(x,\xi\right) dx \otimes \delta_{u(x,t,\omega)}\left(d\xi\right) dW\left(t\right) \\
& \quad + \frac{1}{2}\partial_\xi \left(\|\Phi\left(x,\xi\right)\|^2_{L_2(H,\mathbb{R})} dx \otimes \delta_{u(x,t,\omega)}\left(d\xi\right)\right) dt = \partial_\xi m_\omega\left(dx, dt, d\xi\right)
\end{aligned}
$$

with $m : \Omega \to \mathcal{M}_b^+\left(\mathbb{R}^d \times [0,+\infty) \times \mathbb{R}\right)$ a non-negative entropy defect random measure where $\mathcal{M}_b^+$ denotes the set of all finite Borel non-negative measures.
Conclusion : The kinetic formulation weak in the space variable x and in ξ (but not in the time variable t) of the hyperbolic scalar conservation law

with a source term which is a multiplicative brownian noise, can be written $\forall g \in C_c^{\infty}\left(\mathbb{R}^d \times \mathbb{R}_\xi\right)$, $\forall 0 \leq s \leq t < +\infty$, almost surely,

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \mathbb{R}_\xi} \chi\left(\xi, u\left(x,t\right)\right) g\left(x,\xi\right) dx d\xi - \int_{\mathbb{R}^d \times \mathbb{R}_\xi} \chi\left(\xi, u\left(x,s\right)\right) g\left(x,\xi\right) dx d\xi \\
&\quad = \int_s^t \int_{\mathbb{R}^d \times \mathbb{R}_\xi} A^{'}\left(\xi\right).\nabla_x\left(g\left(x,\xi\right)\right) \chi\left(\xi, u\left(x,r\right)\right) dx d\xi dr \\
&\quad + \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} g\left(x,\xi\right) \Phi\left(x,\xi\right) \delta_{u(x,r)}\left(d\xi\right) dx dW\left(r\right) \\
&\quad + \frac{1}{2} \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} \partial_\xi g\left(x,\xi\right) \left\|\Phi\left(x,\xi\right)\right\|^2_{L_2(H,\mathbb{R})} \delta_{u(x,r)}\left(d\xi\right) dx dr \\
&\quad - \int_s^t \int_{\mathbb{R}^d \times \mathbb{R}_\xi} \partial_\xi g\left(x,\xi\right) m\left(dx, dr, d\xi\right)
\end{aligned}$$

or can be written

$$\begin{aligned}
&\int_{\mathbb{R}^d \times \mathbb{R}_\xi} \mathbf{1}_{u(x,t)>\xi} \times g\left(x,\xi\right) dx d\xi - \int_{\mathbb{R}^d \times \mathbb{R}_\xi} \mathbf{1}_{u(x,s)>\xi} \times g\left(x,\xi\right) dx d\xi \\
&\quad = \int_s^t \int_{\mathbb{R}^d \times \mathbb{R}_\xi} \mathbf{1}_{u(x,r)>\xi} \times A^{'}\left(\xi\right).\nabla_x\left(g\left(x,\xi\right)\right) dx d\xi dr \\
&\quad + \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} g\left(x,\xi\right) \Phi\left(x,\xi\right) \delta_{u(x,r)}\left(d\xi\right) dx dW\left(r\right) \\
&\quad + \frac{1}{2} \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} \partial_\xi g\left(x,\xi\right) \left\|\Phi\left(x,\xi\right)\right\|^2_{L_2(H,\mathbb{R})} \delta_{u(x,r)}\left(d\xi\right) dx dr \\
&\quad - \int_s^t \int_{\mathbb{R}^d \times \mathbb{R}_\xi} \partial_\xi g\left(x,\xi\right) m\left(dx, dr, d\xi\right).
\end{aligned}$$

6. Case of the hyperbolic scalar conservation law with a source term which is a pure jump Lévy noise

6.1. Entropy solution

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ be a filtered probability space satisfying the usual hypothesis of right-continuity and completeness. Let $T > 0$ and $d \in \mathbb{N}^*$. Let us study the first-order hyperbolic scalar conservation law with a discontinuous in time stochastic source term

(6.1)

$$d\left(u\left(x,t,\omega\right)\right) + \mathrm{div}_x\left(A\left(u\left(x,t,\omega\right)\right)\right) dt = \int_{|z|>0} \Phi\left(x, u\left(x,t,\omega\right); z\right) \tilde{N}(dz, dt, \omega),$$

where $\tilde{N}$ is the compensated Poisson random measure associated to the Poisson random measure N defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and the intensity measure $\mu = \nu(dz)dt$

defined on $\Big(\mathbb{R}\times[0,+\infty),\mathcal{B}\left(\mathbb{R}\times[0,+\infty)\right)\Big)$:

$$\forall B\in\mathcal{B}\left(\mathbb{R}\times[0,+\infty)\right) \text{ such that } \mu(B)<+\infty, \quad \tilde{N}(B)=N(B)-\mu(B)$$

with ν a Levy measure on $\mathbb{R}$ that is a Borel measure on $\mathbb{R}$ verifying

$$\nu\left(\{0\}\right)=0 \text{ and } \int_{\mathbb{R}\setminus\{0\}}\left(1\wedge z^2\right)\nu(dz)<+\infty.$$

For each $C\in\mathcal{B}(\mathbb{R})$, I assume that the stochastic process

$$N_C:(\omega,t)\in\Omega\times[0,+\infty)\mapsto N_C(\omega,t):=N(\omega)\left(C\times[0,t]\right)\in\mathbb{N}\cup\{+\infty\}$$

is adapted to the filtration $(\mathcal{F}_t)$. Moreover, I assume that for each $(s,t,C)\in[0,+\infty)^2\times\mathcal{B}(\mathbb{R})$ such that $0\le s<t$, the random variable $N(.)(C\times(s,t])$ is independent of $\mathcal{F}_s$. The variable ω is omitted in the sequel of the section.
Here, I follow the assumptions of [4] for the existence and uniqueness of the solution of the Cauchy problem. The nonlinear flux function A in (6.1) is supposed to be of class C^2: $A\in C^2(\mathbb{R};\mathbb{R}^d)$. I assume that A and its derivatives have at most polynomial growth.
I assume that the coefficient $\Phi:\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous with respect to the first variable, a contraction with respect to the second variable, and bounded with respect to the third variable in the following sense: there exists two constants $K\in(0;+\infty)$ and $\lambda^*\in(0;1)$ such that for all $x,y\in\mathbb{R}^d, u,v\in\mathbb{R}, z\in\mathbb{R}$:

$$\left|\Phi(x,u;z)-\Phi(y,v;z)\right|^2\le\left(K|x-y|^2+\lambda^*|u-v|^2\right)(|z|^2\wedge 1). \tag{6.2}$$

Moreover, I assume that the growth at infinity of the coefficient Φ is limited in the following sense: there exists $g\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ such that

$$\left|\Phi(x,u;z)\right|^2\le\left|g(x)\right|^2\left(|u|^2+1\right)(|z|^2\wedge 1), \quad \forall x\in\mathbb{R}^d, u\in\mathbb{R}, z\in\mathbb{R}. \tag{6.3}$$

Here are the consequences of the Clausius theorem:
If the physical phenomenon associated with the conservation law was reversible, that is if the Lévy-type process u was regular in the space variable $x\in\mathbb{R}^d$, the quantity of entropy $\eta\left(u\left(x,t\right)\right)$ would be conserved thus

$$d\left(\eta\left(u\left(x,t\right)\right)\right)+\operatorname{div}_x\left(\phi\left(u\left(x,t\right)\right)\right)dt=d\left(H\left(u\left(x,t\right)\right)\right)$$

where $H\left(u\left(x,t\right)\right)$ is the entropy source. If A, η and ϕ were regular functions, the two conservation laws could be written

$$d\left(u\left(x,t\right)\right)+A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)dt=\int_{|z|>0}\Phi\left(x,u\left(x,t\right);z\right)\tilde{N}(dz,dt) \tag{6.4}$$

and

$$d\left(\eta\left(u\left(x,t\right)\right)\right)+\phi^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)dt=d\left(H\left(u\left(x,t\right)\right)\right).$$

I can notice that the first one implies that u would have a càdlàg modification.

To find the entropy source, I use the Itô Lemma found in [1] or in [2], for any convex function $\eta \in C^2(\mathbb{R})$ and a process u solution of (6.4):

$$
\begin{aligned}
\eta(u(x,t)) - \eta(u(x,0)) &= -\int_0^t \eta^{'}(u(x,s-))\mathrm{div}_x\left(A(u(x,s))\right)ds \\
&- \int_0^t \eta^{'}(u(x,s-))\int_{|z|>0} \Phi(x,u(x,s);z)\nu(dz)ds \\
&+ \int_0^t \int_{|z|>0} \eta\Big(u(x,s-) + \Phi(x,u(x,s);z)\Big) - \eta\left(u(x,s-)\right)\nu(dz)ds \\
&+ \int_0^t \int_{|z|>0} \left(\eta\Big(u(x,s-) + \Phi(x,u(x,s);z)\Big) - \eta\left(u(x,s-)\right)\right)\tilde{N}(dz,ds)
\end{aligned}
$$

Using the càdlàg modification of u, it can be written in the differential form:

(6.5)

$$
\begin{aligned}
d\eta(u(x,t)) &= -\eta^{'}(u(x,t))\left[\mathrm{div}_x\left(A(u(x,t)) + \int_{|z|>0}\Phi(x,u(x,t);z)\nu(dz)\right]dt \\
&+ \int_{|z|>0}\left(\eta\Big(u(x,t) + \Phi(x,u(x,t);z)\Big) - \eta\left(u(x,t)\right)\right)\nu(dz)dt \\
&+ \int_{|z|>0}\left(\eta\Big(u(x,t-) + \Phi(x,u(x,t);z)\Big) - \eta\left(u(x,t-)\right)\right)\tilde{N}(dz,dt)
\end{aligned}
$$

Using the definition of the entropy flux which is $\phi'(\xi) = \eta'(\xi)A'(\xi), \forall \xi \in \mathbb{R}$, I get the definition of the entropy source

$$
\begin{aligned}
dH(u(x,t)) &= -\eta^{'}(u(x,t))\int_{|z|>0}\Phi(x,u(x,t);z)\nu(dz)dt \\
&+ \int_{|z|>0}\left(\eta\Big(u(x,t) + \Phi(x,u(x,t);z)\Big) - \eta\left(u(x,t)\right)\right)\nu(dz)dt \\
&+ \int_{|z|>0}\left(\eta\Big(u(x,t-) + \Phi(x,u(x,t);z)\Big) - \eta\left(u(x,t-)\right)\right)\tilde{N}(dz,dt).
\end{aligned}
$$

If the physical phenomenon associated with the conservation law was irreversible, that is if the Lévy-type process u was not regular in the space variable

$x \in \mathbb{R}^d$, that would create an entropy defect which could be written

(6.6)
$$d\big(\eta\left(u\left(x,t\right)\right)\big) + \text{div}_x\left(\phi\left(u\left(x,t\right)\right)\right)dt + \eta^{'}(u(x,t))\int_{|z|>0}\Phi(x,u(x,t);z)\nu(dz)dt$$
$$-\int_{|z|>0}\left(\eta\Big(u(x,t)+\Phi(x,u(x,t);z)\Big) - \eta\left(u(x,t)\right)\right)\nu(dz)dt$$
$$-\int_{|z|>0}\left(\eta\Big(u(x,t-)+\Phi(x,u(x,t);z)\Big) - \eta\left(u(x,t-)\right)\right)\tilde{N}(dz,dt) \le 0.$$

This inequality is a weak in space formulation of the conservation law with a source term which is a pure jump Lévy noise. It is written with the test functions $\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^+)$, for all fixed $t \in \mathbb{R}^+$, almost surely, in the following way:

$$\int_{\mathbb{R}^d}\varphi(x)\big(\eta\left(u\left(x,t\right)\right) - \eta\left(u\left(x,0\right)\right)\big)dx - \int_{\mathbb{R}^d}\int_0^t \phi\left(u(x,s)\right)ds.\nabla_x\varphi(x)dx$$
$$+\int_{\mathbb{R}^d}\varphi(x)\int_0^t \eta^{'}(u(x,s))\int_{|z|>0}\Phi(x,u(x,s);z)\nu(dz)dsdx$$
$$-\int_{\mathbb{R}^d}\varphi(x)\int_0^t\int_{|z|>0}\left(\eta\Big(u(x,s)+\Phi(x,u(x,s);z)\Big) - \eta\left(u(x,s)\right)\right)\nu(dz)dsdx$$
$$-\int_{\mathbb{R}^d}\varphi(x)\int_0^t\int_{|z|>0}\Bigg(\eta\Big(u(x,s-)+\Phi(x,u(x,s);z)\Big)$$
$$- \eta\left(u(x,s-)\right)\Bigg)\tilde{N}(dz,ds)dx \le 0.$$

6.2. Kinetic solution

To get the kinetic formulation of
(6.7)
$$d\left(u\left(x,t,\omega\right)\right) + \text{div}_x\left(A\left(u\left(x,t,\omega\right)\right)\right)dt = \int_{|z|>0}\Phi\left(x,u\left(x,t,\omega\right);z\right)\tilde{N}(dz,dt,\omega),$$

I perform the operation $dx \otimes \delta_{u(x,t,\omega)}(d\xi)$ to both sides of the equality and then I apply the Clausius theorem method (the variable ω is now omitted):
If u was regular in the space variable $x \in \mathbb{R}^d$, that is if the physical phenomenon associated with (6.7) was reversible, I would obtain

$$\left(d\left(u\left(x,t\right)dx\right) + A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)dtdx\right)\otimes\delta_{u(x,t)}(d\xi)$$
$$= \int_{|z|>0}\Phi\left(x,u\left(x,t\right);z\right)\tilde{N}(dz,dt)dx\otimes\delta_{u(x,t)}(d\xi)\,. \tag{6.8}$$

The 'chain rule' is untrue for Lévy-type processes, instead I use a weak in x, ξ Itô Lemma:

$$
\begin{aligned}
&d\left(\chi\left(\xi, u\left(x,t\right)\right) d\xi dx\right) = -\int_{|z|>0} \Phi(x,\xi;z)\nu(dz)dx \otimes \delta_{u(x,t)}\left(d\xi\right) dt \\
&+ du\left(x,t\right) dx \otimes \delta_{u(x,t)}\left(d\xi\right) - \int_{|z|>0} \Phi\left(x, u\left(x,t\right); z\right) dx \otimes \delta_{u(x,t)}\left(d\xi\right) \tilde{N}(dz,dt) \\
&+ \int_{|z|>0} \left(\chi\Big(\xi, u(x,t) + \Phi(x,u(x,t);z)\Big) - \chi\left(\xi, u(x,t)\right)\right) d\xi dx\ \nu(dz)dt \\
&+ \int_{|z|>0} \left(\chi\Big(\xi, u(x,t-) + \Phi(x,u(x,t);z)\Big) - \chi\left(\xi, u(x,t-)\right)\right) d\xi dx\ \tilde{N}(dz,dt).
\end{aligned}
$$

Remark 6.1. Digression on the weak in x, ξ Itô Lemma for Lévy-type processes. If $\varphi \in C_c^\infty\left(\mathbb{R}\right)$, then the Itô Lemma applied to the process u gives (see equality (6.5)):

$$
\begin{aligned}
&d\varphi(u(x,t)) = \\
&\quad - \varphi^{'}(u(x,t)) \left[A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right) + \int_{|z|>0} \Phi(x,u(x,t);z)\nu(dz)\right] dt \\
&\quad + \int_{|z|>0} \left(\varphi\Big(u(x,t) + \Phi(x,u(x,t);z)\Big) - \varphi\left(u(x,t)\right)\right)\nu(dz)dt \\
&\quad + \int_{|z|>0} \left(\varphi\Big(u(x,t-) + \Phi(x,u(x,t);z)\Big) - \varphi\left(u(x,t-)\right)\right)\tilde{N}(dz,dt).
\end{aligned}
$$

that I can write

$$
\begin{aligned}
d\varphi(u(x,t)) &= \varphi^{'}(u(x,t)) \left[d\left(u\left(x,t\right)\right) - \int_{|z|>0} \Phi\left(x,u\left(x,t\right);z\right)\tilde{N}(dz,dt)\right] \\
&- \varphi^{'}(u(x,t)) \int_{|z|>0} \Phi(x,u(x,t);z)\nu(dz)dt \\
&+ \int_{|z|>0} \left(\varphi\Big(u(x,t) + \Phi(x,u(x,t);z)\Big) - \varphi\left(u(x,t)\right)\right)\nu(dz)dt \\
&+ \int_{|z|>0} \left(\varphi\Big(u(x,t-) + \Phi(x,u(x,t);z)\Big) - \varphi\left(u(x,t-)\right)\right)\tilde{N}(dz,dt).
\end{aligned}
$$

I can multiply by $h(x)$ for any $h \in C_c^\infty\left(\mathbb{R}^d\right)$, and integrate over $\mathbb{R}^d$ against dx

to obtain:

$$
\begin{aligned}
&d\left(\int_{\mathbb{R}^d} h(x)\varphi(u(x,t))dx\right) = \\
&\quad -\int_{\mathbb{R}^d} h(x)\varphi^{'}(u(x,t))\int_{|z|>0}\Phi(x,u(x,t);z)\nu(dz)dxdt \\
&\quad +\int_{\mathbb{R}^d} h(x)\varphi^{'}(u(x,t))\left[d\left(u\left(x,t\right)\right)-\int_{|z|>0}\Phi\left(x,u\left(x,t\right);z\right)\tilde{N}(dz,dt)\right]dx \\
&\quad +\int_{\mathbb{R}^d} h(x)\int_{|z|>0}\left(\varphi\Big(u(x,t)+\Phi(x,u(x,t);z)\Big)-\varphi\left(u(x,t)\right)\right)\nu(dz)dxdt \\
&\quad +\int_{|z|>0}\int_{\mathbb{R}^d} h(x)\left(\varphi\Big(u(x,t-)+\Phi(x,u(x,t);z)\Big)-\varphi\left(u(x,t-)\right)\right)dx\tilde{N}(dz,dt)
\end{aligned}
$$

which can be written

$$
\begin{aligned}
&d\left(\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}\left(\xi\right)\chi\left(\xi,u\left(x,t\right)\right)d\xi dx\right) \\
&\quad =\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}\left(\xi\right)du\left(x,t\right)dx\otimes\delta_{u(x,t)}\left(d\xi\right) \\
&\qquad -\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}\left(\xi\right)\int_{|z|>0}\Phi(x,\xi;z)\nu(dz)dx\otimes\delta_{u(x,t)}\left(d\xi\right)dt \\
&\qquad -\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}\left(\xi\right)\int_{|z|>0}\Phi\left(x,u\left(x,t\right);z\right)\tilde{N}(dz,dt)dx\otimes\delta_{u(x,t)}\left(d\xi\right) \\
&\qquad +\int_{|z|>0}\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}(\xi)\Bigg(\chi\Big(\xi,u(x,t)+\Phi(x,u(x,t);z)\Big) \\
&\qquad\qquad -\chi\left(\xi,u(x,t)\right)\Bigg)d\xi dx\,\nu(dz)dt \\
&\qquad +\int_{|z|>0}\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}(\xi)\Bigg(\chi\Big(\xi,u(x,t-)+\Phi(x,u(x,t);z)\Big) \\
&\qquad\qquad -\chi\left(\xi,u(x,t-)\right)\Bigg)d\xi dx\,\tilde{N}(dz,dt).
\end{aligned}
$$

I obtain the weak in x, ξ Itô formula

$$d\left(\chi\left(\xi, u\left(x,t\right)\right)d\xi dx\right) = -\int_{|z|>0} \Phi(x,\xi;z)\nu(dz)dx \otimes \delta_{u(x,t)}\left(d\xi\right)dt$$
$$+ du\left(x,t\right)dx \otimes \delta_{u(x,t)}\left(d\xi\right) - \int_{|z|>0} \Phi\left(x, u\left(x,t\right);z\right)dx \otimes \delta_{u(x,t)}\left(d\xi\right)\tilde{N}(dz,dt)$$
$$+ \int_{|z|>0}\left(\chi\Big(\xi, u(x,t) + \Phi(x,u(x,t);z)\Big) - \chi\left(\xi, u(x,t)\right)\right)d\xi dx\, \nu(dz)dt$$
$$+ \int_{|z|>0}\left(\chi\Big(\xi, u(x,t-) + \Phi(x,u(x,t);z)\Big) - \chi\left(\xi, u(x,t-)\right)\right)d\xi dx\, \tilde{N}(dz,dt).$$

Then, using the following formula

$$A^{'}(\xi).\nabla_x\big(\chi\left(\xi,u(x,t)\right)\big) = A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u(x,t)\right)dx \otimes \delta_{u(x,t)}(d\xi),$$

and the weak in x, ξ Itô Lemma, the equation (6.8) would become

$$d\left(\chi\left(\xi, u\left(x,t\right)\right)d\xi dx\right) + A^{'}(\xi).\nabla_x\big(\chi\left(\xi,u(x,t)\right)\big)dt$$
$$+ \int_{|z|>0} \Phi(x,\xi;z)\nu(dz)dx \otimes \delta_{u(x,t)}\left(d\xi\right)dt$$
$$- \int_{|z|>0}\left(\chi\Big(\xi, u(x,t) + \Phi(x,u(x,t);z)\Big) - \chi\left(\xi, u(x,t)\right)\right)d\xi dx\, \nu(dz)dt$$
$$- \int_{|z|>0}\left(\chi\Big(\xi, u(x,t-) + \Phi(x,u(x,t);z)\Big) - \chi\left(\xi, u(x,t-)\right)\right)d\xi dx\, \tilde{N}(dz,dt)$$
$$= 0.$$

If the physical phenomenon associated with the conservation law was irreversible, that is if u was not regular in $x \in \mathbb{R}^d$, the entropy defect created over time would be written

$$d\left(\chi\left(\xi, u\left(x,t\right)\right)d\xi dx\right) + A^{'}(\xi).\nabla_x\big(\chi\left(\xi,u(x,t)\right)\big)dt$$
$$+ \int_{|z|>0} \Phi(x,\xi;z)\nu(dz)dx \otimes \delta_{u(x,t)}\left(d\xi\right)dt$$
$$- \int_{|z|>0}\left(\chi\Big(\xi, u(x,t) + \Phi(x,u(x,t);z)\Big) - \chi\left(\xi, u(x,t)\right)\right)d\xi dx\, \nu(dz)dt$$
$$- \int_{|z|>0}\left(\chi\Big(\xi, u(x,t-) + \Phi(x,u(x,t);z)\Big) - \chi\left(\xi, u(x,t-)\right)\right)d\xi dx\, \tilde{N}(dz,dt)$$
$$= \partial_\xi m\left(dx,dt,d\xi\right)$$

with $m : \Omega \to \mathcal{M}_b^+\left(\mathbb{R}^d \times [0,+\infty) \times \mathbb{R}\right)$ a non-negative entropy defect random measure where $\mathcal{M}_b^+$ denotes the set of all finite Borel non-negative measures.

<u>Conclusion</u> : The kinetic formulation weak in the space variable x and in ξ (but not in the time variable) of the hyperbolic scalar conservation law with a source term which is a pure jump Lévy noise, can be written $\forall g \in C_c^\infty\left(\mathbb{R}^d \times \mathbb{R}_\xi\right)$, $\forall 0 \leq s \leq t < +\infty$, almost surely,

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}_\xi} \chi\left(\xi, u\left(x,t\right)\right) g\left(x,\xi\right) dxd\xi - \int_{\mathbb{R}^d\times\mathbb{R}_\xi} \chi\left(\xi, u\left(x,s\right)\right) g\left(x,\xi\right) dxd\xi \\
&\quad = \int_s^t \int_{\mathbb{R}^d\times\mathbb{R}_\xi} \chi\left(\xi, u\left(x,r\right)\right) A^{'}\left(\xi\right).\nabla_x\left(g\left(x,\xi\right)\right) dxd\xi dr \\
&\qquad + \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} g\left(x,\xi\right) \int_{|z|>0} \Phi(x,\xi;z)\nu(dz)\delta_{u(x,r)}\left(d\xi\right) dxdr \\
&\qquad - \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} g\left(x,\xi\right) \int_{|z|>0} \Bigg(\chi\Big(\xi, u(x,r) + \Phi(x,u(x,r);z)\Big) \\
&\qquad\qquad\qquad\qquad - \chi\left(\xi, u(x,r)\right)\Bigg)\nu(dz)d\xi dx\, dr \\
&\qquad - \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} \int_{|z|>0} g\left(x,\xi\right) \Bigg(\chi\Big(\xi, u(x,r-) + \Phi(x,u(x,r);z)\Big) \\
&\qquad\qquad\qquad\qquad - \chi\left(\xi, u(x,r-)\right)\Bigg) d\xi dx\, \tilde{N}(dz,dr) \\
&\qquad - \int_s^t \int_{\mathbb{R}^d\times\mathbb{R}_\xi} \partial_\xi g\left(x,\xi\right) m\left(dx,dr,d\xi\right)
\end{aligned}$$

or can be written

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{R}_\xi} g\left(x,\xi\right) \mathbf{1}_{u(x,t)>\xi} dxd\xi - \int_{\mathbb{R}^d\times\mathbb{R}_\xi} g\left(x,\xi\right) \mathbf{1}_{u(x,s)>\xi} dxd\xi \\
&= \int_s^t \int_{\mathbb{R}^d\times\mathbb{R}_\xi} \mathbf{1}_{u(x,r)>\xi} \times A^{'}\left(\xi\right).\nabla_x\left(g\left(x,\xi\right)\right) dxd\xi dr \\
&+ \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} g\left(x,\xi\right) \int_{|z|>0} \Phi(x,\xi;z)\nu(dz)\delta_{u(x,r)}\left(d\xi\right) dxdr \\
&- \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} g\left(x,\xi\right) \int_{|z|>0} \Big(\mathbf{1}_{u(x,r)+\Phi(x,u(x,r);z)>\xi} - \mathbf{1}_{u(x,r)>\xi}\Big)\nu(dz)d\xi dx\, dr \\
&- \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} \int_{|z|>0} g\left(x,\xi\right) \Big(\mathbf{1}_{u(x,r-)+\Phi(x,u(x,r);z)>\xi} \\
&\qquad\qquad\qquad - \mathbf{1}_{u(x,r-)>\xi}\Big) d\xi dx \tilde{N}(dz,dr) \\
&- \int_s^t \int_{\mathbb{R}^d\times\mathbb{R}_\xi} \partial_\xi g\left(x,\xi\right) m\left(dx,dr,d\xi\right)
\end{aligned}$$

7. Conclusion: case of the hyperbolic scalar conservation law with a source term which is a general Lévy noise

7.1. Entropy solution

Gathering the assumptions of sections 3.1, 5.1 and 6.1, I will apply the Clausius theorem method to

$$
\begin{aligned}
&d\left(u\left(x,t,\omega\right)\right)+\operatorname{div}_x\left(A\left(u\left(x,t,\omega\right)\right)\right)dt=G\left(u(x,t,\omega)\right)dt\\
&\quad+\Phi_1\left(x,u\left(x,t,\omega\right)\right)dW\left(t,\omega\right)+\int_{|z|>0}\Phi_2\left(x,u\left(x,t,\omega\right);z\right)\tilde{N}(dz,dt,\omega),
\end{aligned}
\tag{7.1}
$$

to get the definition of entropy solutions. Here are the consequences of the Clausius theorem (the variable ω being omitted).

If the physical phenomenon associated with the conservation law was reversible, that is if the Lévy-type process u was regular in the space variable $x\in\mathbb{R}^d$, the quantity of entropy $\eta\left(u\left(x,t\right)\right)$ would be conserved thus

$$
d\left(\eta\left(u\left(x,t\right)\right)\right)+\operatorname{div}_x\left(\phi\left(u\left(x,t\right)\right)\right)dt=d\left(H\left(u\left(x,t\right)\right)\right)
$$

where $H\left(u\left(x,t\right)\right)$ is the entropy source. If A, η and ϕ were regular functions, the two conservation laws could be written

$$
d\left(u\left(x,t\right)\right)+A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)dt=\int_{|z|>0}\Phi_2\left(x,u\left(x,t\right);z\right)\tilde{N}(dz,dt)
\tag{7.2}
$$

and

$$
d\left(\eta\left(u\left(x,t\right)\right)\right)+\phi^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)dt=d\left(H\left(u\left(x,t\right)\right)\right).
$$

I can notice that the first one implies that u would have a càdlàg modification. To find the entropy source, I use the Itô Lemma found in [1] (theorem 4.4.7 page 251), for any convex function $\eta\in C^2\left(\mathbb{R}\right)$ and a process u solution of (7.2):

$$
\begin{aligned}
d\eta(u(x,t))&=-\eta^{'}(u(x,t))\Big[\operatorname{div}_x\left(A(u(x,t))+\int_{|z|>0}\Phi_2(x,u(x,t);z)\nu(dz)\right]dt\\
&\quad+\eta^{'}(u(x,t))\Big[G\left(u(x,t)\right)dt+\Phi_1\left(x,u(x,t)\right)dW(t)\Big]\\
&\quad+\frac{1}{2}\eta^{''}\left(u(x,t)\right)\left\|\Phi_1\left(x,u(x,t)\right)\right\|^2_{L_2(H;\mathbb{R})}dt\\
&\quad+\int_{|z|>0}\left(\eta\Big(u(x,t)+\Phi_2(x,u(x,t);z)\Big)-\eta\left(u(x,t)\right)\right)\nu(dz)dt\\
&\quad+\int_{|z|>0}\left(\eta\Big(u(x,t-)+\Phi_2(x,u(x,t);z)\Big)-\eta\left(u(x,t-)\right)\right)\tilde{N}(dz,dt)
\end{aligned}
\tag{7.3}
$$

Using the definition of the entropy flux which is $\phi'(\xi) = \eta'(\xi)A'(\xi), \forall \xi \in \mathbb{R}$, I get the definition of the entropy source:

$$
\begin{aligned}
dH(u(x,t)) = &-\eta^{'}(u(x,t)) \int_{|z|>0} \Phi_2(x,u(x,t);z)\nu(dz)dt \\
&+ \eta^{'}(u(x,t)) \Big[G(u(x,t))\,dt + \Phi_1(x,u(x,t))\,dW(t) \Big] \\
&+ \frac{1}{2}\eta^{''}(u(x,t)) \left\| \Phi_1(x,u(x,t)) \right\|^2_{L_2(H;\mathbb{R})} dt \\
&+ \int_{|z|>0} \left(\eta\Big(u(x,t) + \Phi_2(x,u(x,t);z) \Big) - \eta(u(x,t)) \right) \nu(dz)dt \\
&+ \int_{|z|>0} \left(\eta\Big(u(x,t-) + \Phi_2(x,u(x,t);z) \Big) - \eta(u(x,t-)) \right) \tilde{N}(dz,dt).
\end{aligned}
$$

If the physical phenomenon associated with the conservation law was irreversible, that is if the Lévy-type process u was not regular in the space variable $x \in \mathbb{R}^d$, that would create an entropy defect which could be written

(7.4)

$$
\begin{aligned}
& d\big(\eta(u(x,t))\big) + \operatorname{div}_x(\phi_2(u(x,t)))\,dt \\
&\quad + \eta^{'}(u(x,t)) \int_{|z|>0} \Phi_2(x,u(x,t);z)\nu(dz)dt \\
&\quad - \eta^{'}(u(x,t)) \Big[G(u(x,t))\,dt + \Phi_1(x,u(x,t))\,dW(t) \Big] \\
&\quad - \frac{1}{2}\eta^{''}(u(x,t)) \left\| \Phi_1(x,u(x,t)) \right\|^2_{L_2(H;\mathbb{R})} dt \\
&\quad - \int_{|z|>0} \left(\eta\Big(u(x,t) + \Phi_2(x,u(x,t);z) \Big) - \eta(u(x,t)) \right) \nu(dz)dt \\
&\quad - \int_{|z|>0} \left(\eta\Big(u(x,t-) + \Phi_2(x,u(x,t);z) \Big) - \eta(u(x,t-)) \right) \tilde{N}(dz,dt) \leq 0.
\end{aligned}
$$

This inequality is a weak in space formulation of the conservation law with a source term which is a general Lévy noise. It is written with the test functions

$\varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R}^+)$, for all fixed $t \in \mathbb{R}^+$, almost surely, in the following way :

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \varphi(x)\big(\eta\left(u\left(x,t\right)\right)-\eta\left(u\left(x,0\right)\right)\big)dx-\int_{\mathbb{R}^d}\int_0^t \phi\left(u(x,s)\right)ds.\nabla_x\varphi(x)dx\\
&-\frac{1}{2}\int_{\mathbb{R}^d}\varphi(x)\int_0^t \eta^{''}\left(u\left(x,s\right)\right)\left\|\Phi_1\left(x,u\left(x,s\right)\right)\right\|^2_{L_2(H,\mathbb{R})}dsdx\\
&-\int_{\mathbb{R}^d}\varphi(x)\int_0^t \eta^{'}\left(u\left(x,s\right)\right)\Phi_1\left(x,u\left(x,s\right)\right)dW\left(s\right)dx\\
&+\int_{\mathbb{R}^d}\varphi(x)\int_0^t \eta^{'}(u(x,s))\left[\int_{|z|>0}\Phi_2(x,u(x,s);z)\nu(dz)-G\left(u(x,s)\right)\right]dsdx\\
&-\int_{\mathbb{R}^d}\varphi(x)\int_0^t\int_{|z|>0}\left(\eta\Big(u(x,s)+\Phi_2(x,u(x,s);z)\Big)-\eta\left(u(x,s)\right)\right)\nu(dz)dsdx\\
&-\int_{\mathbb{R}^d}\varphi(x)\int_0^t\int_{|z|>0}\Bigg(\eta\Big(u(x,s-)\\
&\qquad\qquad+\Phi_2(x,u(x,s);z)\Big)-\eta\left(u(x,s-)\right)\Bigg)\tilde{N}(dz,ds)dx\le 0.
\end{aligned}
$$

7.2. Kinetic solution

To get the kinetic formulation of

$$
\begin{aligned}
&d\left(u\left(x,t,\omega\right)\right)+\mathrm{div}_x\left(A\left(u\left(x,t,\omega\right)\right)\right)dt=G\left(u(x,t,\omega)\right)dt\\
&\quad+\Phi_1\left(x,u\left(x,t,\omega\right)\right)dW\left(t,\omega\right)+\int_{|z|>0}\Phi_2\left(x,u\left(x,t,\omega\right);z\right)\tilde{N}(dz,dt,\omega),
\end{aligned}
\tag{7.5}
$$

I perform the operation $dx\otimes\delta_{u(x,t,\omega)}\left(d\xi\right)$ to both sides of the equality and then I apply the Clausius theorem method (the variable ω is now omitted):

If u was regular in the space variable $x\in\mathbb{R}^d$, that is if the physical phenomenon associated with (7.5) was reversible, I would obtain

$$
\begin{aligned}
&\Big(d\left(u\left(x,t\right)dx\right)+A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)dtdx\Big)\otimes\delta_{u(x,t)}\left(d\xi\right)\\
&\quad=\left(G\left(u(x,t)\right)dt+\Phi_1\left(x,u\left(x,t\right)\right)dW\left(t\right)\right)dx\otimes\delta_{u(x,t)}\left(d\xi\right)\\
&\qquad+\int_{|z|>0}\Phi_2\left(x,u\left(x,t\right);z\right)\tilde{N}(dz,dt)dx\otimes\delta_{u(x,t)}\left(d\xi\right).
\end{aligned}
\tag{7.6}
$$

The 'chain rule' is untrue for Lévy-type processes, instead I use a weak in x,ξ

Itô Lemma:

$$d\left(\chi\left(\xi,u\left(x,t\right)\right)d\xi dx\right)=-\int_{|z|>0}\Phi_2(x,\xi;z)\nu(dz)dx\otimes\delta_{u(x,t)}\left(d\xi\right)dt$$

$$+du\left(x,t\right)dx\otimes\delta_{u(x,t)}\left(d\xi\right)-\int_{|z|>0}\Phi_2\left(x,u\left(x,t\right);z\right)dx\otimes\delta_{u(x,t)}\left(d\xi\right)\tilde{N}(dz,dt)$$

$$+\int_{|z|>0}\left(\chi\Big(\xi,u(x,t)+\Phi_2(x,u(x,t);z)\Big)-\chi\left(\xi,u(x,t)\right)\right)d\xi dx\;\nu(dz)dt$$

$$+\int_{|z|>0}\left(\chi\Big(\xi,u(x,t-)+\Phi_2(x,u(x,t);z)\Big)-\chi\left(\xi,u(x,t-)\right)\right)d\xi dx\;\tilde{N}(dz,dt)$$

$$-\frac{1}{2}\partial_\xi\left(\|\Phi_1\left(x,\xi\right)\|^2_{L_2(H,\mathbb{R})}dx\otimes\delta_{u(x,t)}\left(d\xi\right)\right)dt.$$

Remark 7.1. Digression on the weak in x,ξ Itô Lemma for general Lévy-type processes.
If $\varphi\in C^\infty_c\left(\mathbb{R}\right)$, then the Itô Lemma applied to the process u gives (see equality (7.3)):

$$d\varphi(u(x,t))=$$

$$-\varphi^{'}\left(u(x,t)\right)\left[A^{'}\left(u\left(x,t\right)\right).\nabla_x\left(u\left(x,t\right)\right)+\int_{|z|>0}\Phi_2(x,u(x,t);z)\nu(dz)\right]dt$$

$$+\varphi^{'}\left(u(x,t)\right)\Big[G\left(u(x,t)\right)dt+\Phi_1\left(x,u(x,t)\right)dW(t)\Big]$$

$$+\frac{1}{2}\varphi^{''}\left(u(x,t)\right)\|\Phi_1\left(x,u(x,t)\right)\|^2_{L_2(H;\mathbb{R})}dt$$

$$+\int_{|z|>0}\left(\varphi\Big(u(x,t)+\Phi_2(x,u(x,t);z)\Big)-\varphi\left(u(x,t)\right)\right)\nu(dz)dt$$

$$+\int_{|z|>0}\left(\varphi\Big(u(x,t-)+\Phi_2(x,u(x,t);z)\Big)-\varphi\left(u(x,t-)\right)\right)\tilde{N}(dz,dt)$$

that I can write

$$d\varphi(u(x,t))=\varphi^{'}\left(u(x,t)\right)\left[d\left(u\left(x,t\right)\right)-\int_{|z|>0}\Phi_2\left(x,u\left(x,t\right);z\right)\tilde{N}(dz,dt)\right]$$

$$+\frac{1}{2}\varphi^{''}\left(u(x,t)\right)\|\Phi_1\left(x,u(x,t)\right)\|^2_{L_2(H;\mathbb{R})}dt$$

$$-\varphi^{'}\left(u(x,t)\right)\int_{|z|>0}\Phi_2(x,u(x,t);z)\nu(dz)dt$$

$$+\int_{|z|>0}\left(\varphi\Big(u(x,t)+\Phi_2(x,u(x,t);z)\Big)-\varphi\left(u(x,t)\right)\right)\nu(dz)dt$$

$$+\int_{|z|>0}\left(\varphi\Big(u(x,t-)+\Phi_2(x,u(x,t);z)\Big)-\varphi\left(u(x,t-)\right)\right)\tilde{N}(dz,dt).$$

I can multiply by $h(x)$ for any $h \in C_c^\infty\left(\mathbb{R}^d\right)$, and integrate over $\mathbb{R}^d$ against dx to obtain:

$$
\begin{aligned}
& d\left(\int_{\mathbb{R}^d} h(x)\varphi(u(x,t))dx\right) \\
& = -\int_{\mathbb{R}^d} h(x)\varphi^{'}(u(x,t)) \int_{|z|>0} \Phi_2(x,u(x,t);z)\nu(dz)dxdt \\
& + \int_{\mathbb{R}^d} h(x)\varphi^{'}(u(x,t)) \left[d\left(u\left(x,t\right)\right) - \int_{|z|>0} \Phi_2\left(x,u\left(x,t\right);z\right)\tilde{N}(dz,dt)\right] dx \\
& + \frac{1}{2}\int_{\mathbb{R}^d} h(x)\varphi^{''}\left(u\left(x,t\right)\right) \left\|\Phi_1\left(x,u\left(x,t\right)\right)\right\|^2_{L_2(H,\mathbb{R})} dtdx \\
& + \int_{\mathbb{R}^d} h(x) \int_{|z|>0} \left(\varphi\Big(u(x,t)+\Phi_2(x,u(x,t);z)\Big) - \varphi\left(u(x,t)\right)\right)\nu(dz)dxdt \\
& + \int_{|z|>0}\int_{\mathbb{R}^d} h(x)\left(\varphi\Big(u(x,t-)+\Phi_2(x,u(x,t);z)\Big) - \varphi\left(u(x,t-)\right)\right)dx\tilde{N}(dz,dt)
\end{aligned}
$$

which can be written

$$
\begin{aligned}
& d\Big(\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}\left(\xi\right)\chi\left(\xi,u\left(x,t\right)\right)d\xi dx\Big) \\
& = -\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}\left(\xi\right)\int_{|z|>0}\Phi_2(x,\xi;z)\nu(dz)dx\otimes\delta_{u(x,t)}\left(d\xi\right)dt \\
& \quad + \int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}\left(\xi\right)du\left(x,t\right)dx\otimes\delta_{u(x,t)}\left(d\xi\right) \\
& \quad + \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{''}\left(\xi\right)\left\|\Phi_1\left(x,\xi\right)\right\|^2_{L_2(H,\mathbb{R})} dx\otimes\delta_{u(x,t)}\left(d\xi\right)dt \\
& \quad - \int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}\left(\xi\right)\int_{|z|>0}\Phi_2\left(x,u\left(x,t\right);z\right)\tilde{N}(dz,dt)dx\otimes\delta_{u(x,t)}\left(d\xi\right) \\
& \quad + \int_{|z|>0}\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}(\xi)\Bigg(\chi\Big(\xi,u(x,t)+\Phi_2(x,u(x,t);z)\Big) \\
& \qquad\qquad - \chi\left(\xi,u(x,t)\right)\Bigg)d\xi dx\;\nu(dz)dt \\
& \quad + \int_{|z|>0}\int_{\mathbb{R}^d\times\mathbb{R}} h(x)\varphi^{'}(\xi)\Bigg(\chi\Big(\xi,u(x,t-)+\Phi_2(x,u(x,t);z)\Big) \\
& \qquad\qquad - \chi\left(\xi,u(x,t-)\right)\Bigg)d\xi dx\;\tilde{N}(dz,dt).
\end{aligned}
$$

I obtain the weak in x, ξ Itô formula

$$
\begin{aligned}
& d\left(\chi\left(\xi, u\left(x, t\right)\right) d\xi dx\right) = - \int_{|z|>0} \Phi_2(x, \xi; z) \nu(dz) dx \otimes \delta_{u(x,t)}\left(d\xi\right) dt \\
& + du\left(x, t\right) dx \otimes \delta_{u(x,t)}\left(d\xi\right) - \int_{|z|>0} \Phi_2\left(x, u\left(x, t\right); z\right) dx \otimes \delta_{u(x,t)}\left(d\xi\right) \tilde{N}(dz, dt) \\
& + \int_{|z|>0} \left(\chi\Big(\xi, u(x,t) + \Phi_2(x, u(x,t); z)\Big) - \chi\left(\xi, u(x,t)\right)\right) d\xi dx\ \nu(dz) dt \\
& + \int_{|z|>0} \left(\chi\Big(\xi, u(x,t-) + \Phi_2(x, u(x,t); z)\Big) - \chi\left(\xi, u(x,t-)\right)\right) d\xi dx\ \tilde{N}(dz, dt) \\
& - \frac{1}{2} \partial_\xi \left(\|\Phi_1\left(x, \xi\right)\|^2_{L_2(H,\mathbb{R})} dx \otimes \delta_{u(x,t)}\left(d\xi\right)\right) dt.
\end{aligned}
$$

Then, using the following formula

$$
A^{'}(\xi).\nabla_x\big(\chi\left(\xi, u(x,t)\right)\big) = A^{'}\left(u\left(x, t\right)\right).\nabla_x\left(u(x,t)\right) dx \otimes \delta_{u(x,t)}(d\xi),
$$

and the weak in x, ξ Itô Lemma, equation (7.6) would become

$$
\begin{aligned}
& d\left(\chi\left(\xi, u\left(x, t\right)\right) d\xi dx\right) + A^{'}(\xi).\nabla_x\big(\chi\left(\xi, u(x,t)\right)\big) dt \\
& + \int_{|z|>0} \Phi_2(x, \xi; z) \nu(dz) dx \otimes \delta_{u(x,t)}\left(d\xi\right) dt \\
& - \int_{|z|>0} \left(\chi\Big(\xi, u(x,t) + \Phi_2(x, u(x,t); z)\Big) - \chi\left(\xi, u(x,t)\right)\right) d\xi dx\ \nu(dz) dt \\
& - \int_{|z|>0} \left(\chi\Big(\xi, u(x,t-) + \Phi_2(x, u(x,t); z)\Big) - \chi\left(\xi, u(x,t-)\right)\right) d\xi dx\ \tilde{N}(dz, dt) \\
& - \Phi_1\left(x, \xi\right) dx \otimes \delta_{u(x,t)}\left(d\xi\right) dW\left(t\right) \\
& + \frac{1}{2} \partial_\xi \left(\|\Phi_1\left(x, \xi\right)\|^2_{L_2(H,\mathbb{R})} dx \otimes \delta_{u(x,t)}\left(d\xi\right)\right) dt \\
& - G(\xi) dx \otimes \delta_{u(x,t)}\left(d\xi\right) dt = 0.
\end{aligned}
$$

If the physical phenomenon associated with the conservation law was irreversible, that is if u was not regular in $x \in \mathbb{R}^d$, the entropy defect created over time would be written

$$
\begin{aligned}
& d\left(\chi\left(\xi, u\left(x,t\right)\right) d\xi dx\right) + A^{'}\left(\xi\right).\nabla_x \big(\chi\left(\xi, u(x,t)\right)\big) dt \\
& + \int_{|z|>0} \Phi_2(x,\xi;z)\nu(dz)dx \otimes \delta_{u(x,t)}\left(d\xi\right) dt \\
& - \int_{|z|>0} \Bigg(\chi\Big(\xi, u(x,t) + \Phi_2(x,u(x,t);z)\Big) - \chi\left(\xi, u(x,t)\right)\Bigg) d\xi dx \ \nu(dz)dt \\
& - \int_{|z|>0} \Bigg(\chi\Big(\xi, u(x,t-) + \Phi_2(x,u(x,t);z)\Big) - \chi\left(\xi, u(x,t-)\right)\Bigg) d\xi dx \ \tilde{N}(dz,dt) \\
& - \Phi_1\left(x,\xi\right) dx \otimes \delta_{u(x,t)}\left(d\xi\right) dW\left(t\right) + \frac{1}{2}\partial_\xi \left(\|\Phi_1\left(x,\xi\right)\|^2_{L_2(H,\mathbb{R})} dx \otimes \delta_{u(x,t)}\left(d\xi\right)\right) dt \\
& - G(\xi)dx \otimes \delta_{u(x,t)}\left(d\xi\right) dt = \partial_\xi m\left(dx, dt, d\xi\right)
\end{aligned}
$$

with $m : \Omega \to \mathcal{M}_b^+\left(\mathbb{R}^d \times [0,+\infty) \times \mathbb{R}\right)$ a non-negative entropy defect random measure where $\mathcal{M}_b^+$ denotes the set of all finite Borel non-negative measures.
<u>Conclusion</u> : The kinetic formulation weak in the space variable x and in ξ (but not in the time variable) of the hyperbolic scalar conservation law with a source term which is a general Lévy noise, can be written $\forall g \in C_c^\infty\left(\mathbb{R}^d \times \mathbb{R}_\xi\right)$, $\forall 0 \le s \le t < +\infty$, almost surely,

$$
\begin{aligned}
& \int_{\mathbb{R}^d \times \mathbb{R}_\xi} g\left(x,\xi\right) \mathbf{1}_{u(x,t)>\xi} dx d\xi - \int_{\mathbb{R}^d \times \mathbb{R}_\xi} g\left(x,\xi\right) \mathbf{1}_{u(x,s)>\xi} dx d\xi \\
& = \int_s^t \int_{\mathbb{R}^d \times \mathbb{R}_\xi} \mathbf{1}_{u(x,r)>\xi} \times A^{'}\left(\xi\right).\nabla_x\left(g\left(x,\xi\right)\right) dx d\xi dr \\
& + \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} g\left(x,\xi\right) \int_{|z|>0} \Phi_2(x,\xi;z)\nu(dz)\delta_{u(x,r)}\left(d\xi\right) dx dr \\
& - \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} g\left(x,\xi\right) \int_{|z|>0} \Bigg(\mathbf{1}_{u(x,r)+\Phi_2(x,u(x,r);z)>\xi} - \mathbf{1}_{u(x,r)>\xi}\Bigg) \nu(dz) d\xi dx \ dr \\
& - \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} \int_{|z|>0} g\left(x,\xi\right) \Bigg(\mathbf{1}_{u(x,r-)+\Phi_2(x,u(x,r);z)>\xi} \\
& \qquad\qquad\qquad - \mathbf{1}_{u(x,r-)>\xi}\Bigg) d\xi dx \ \tilde{N}(dz,dr) \\
& + \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} g\left(x,\xi\right) \Phi_1\left(x,\xi\right) \delta_{u(x,r,\omega)}\left(d\xi\right) dx dW\left(r\right) \\
& + \frac{1}{2} \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} \partial_\xi g\left(x,\xi\right) \|\Phi_1\left(x,\xi\right)\|^2_{L_2(H,\mathbb{R})} \delta_{u(x,t)}\left(d\xi\right) dx dt \\
& + \int_s^t \int_{\mathbb{R}^d} \int_{\mathbb{R}_\xi} g\left(x,\xi\right) G(\xi)\delta_{u(x,r)}\left(d\xi\right) dx dr - \int_s^t \int_{\mathbb{R}^d \times \mathbb{R}_\xi} \partial_\xi g\left(x,\xi\right) m\left(dx,dr,d\xi\right).
\end{aligned}
$$

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