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On a class of partial fractional integro-differential inclusions

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Abstract. A Darboux problem associated to a fractional hyperbolic integro-differential inclusion defined by a Caputo type fractional derivative is studied. We obtain an existence result for this problem in the situation when the values of the set-valued map are not convex by employing a method originally introduced by Filippov. Also, we provide the existence of solutions continuously depending on a parameter for the problem studied. This second result allows to deduce a continuous selection of the solution set of the problem considered.

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1. Introduction

In the last years one may see a strong development of the theory of differential equations and inclusions of fractional order ([4, 11, 15, 17] etc.). The main reason is that fractional differential equations are very useful tools in order to model many physical phenomena ([5, 16, 18, 19, 20] etc.). In the fractional calculus there are several fractional derivatives. From them, the fractional derivative introduced by Caputo in [7] allows to use Cauchy conditions which have physical meanings.

A Caputo type fractional derivative of a function with respect to another function ([15]) that extends and unifies several fractional derivatives existing in the literature like Caputo, Caputo-Hadamard, Caputo-Katugampola was intensively studied in recent years [1, 2, 3] etc.. In particular, existence results and qualitative properties of the solutions for fractional differential equations defined by this fractional derivative are obtained in [2, 3].

In the present paper we study Darboux problem inclusions of the following form

$$D_C^{\alpha,\psi}u(x,y) \in F(x,y,u(x,y),(I^{\alpha,\psi}u)(x,y)) \quad a.e.\ (x,y) \in \Pi, \tag{1.1}$$

$$u(x,0) = \varphi_1(x), \quad u(0,y) = \varphi_2(y) \quad (x,y) \in \Pi, \tag{1.2}$$

where $\Pi = I_1 \times I_2$, $I_1 = [0,T_1]$, $I_2 = [0,T_2]$, $\varphi_1(.) : I_1 \to \mathbf{R}$, $\varphi_2(.) : I_2 \to \mathbf{R}$ with $\varphi_1(0) = \varphi_2(0)$, $F(.,.) : \Pi \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $I^{\alpha,\psi}$ is

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the generalized left-sided mixed integral and $D_C^{\alpha,\psi}$ is the fractional derivative mentioned above, $\alpha = (\alpha_1, \alpha_2) \in [0,1) \times [0,1)$ and $\psi(.) = (\psi_1(.), \psi_2(.)) \in C^1(I_1, \mathbf{R}) \times C^1(I_2, \mathbf{R})$.

The goal of the present paper is twofold. First, we show that Filippov's ideas ([12]) can be suitably adapted in order to obtain the existence of a solution of problem (1.1)-(1.2). We recall that for an "ordinary" differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem ([12]) provides the existence of a solution starting from a given "quasi" solution. At the same time, the result gives an estimate between the "quasi" solution and the solution of the differential inclusion. Secondly, we obtain the existence of solutions continuously depending on a parameter for problem (1.1)-(1.2). This result is, in fact, a continuous version of our first result. In the proof of this second theorem we essentially use a result of Bressan and Colombo ([6]) concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values. Our second theorem allows us to deduce a continuous selection of the solution set of the problem considered.

The results in the present paper extend and unify similar results obtained for partial fractional integro-differential inclusions defined by Caputo fractional derivative ([8]), by Hadamard fractional derivative ([9]) and by Caputo-Katugampola fractional derivative ([10]).

The paper is organized as follows: in Section 2 we recall some preliminary results that we will use in the sequel and in Section 3 we prove the main results of the paper.

2. Preliminaries

Consider $\beta > 0$, $f(.) \in L^1([0,T], \mathbf{R})$ and $\psi(.) \in C^n([0,T], \mathbf{R})$ such that $\psi'(t) > 0 \ \forall\, t \in [0,T]$.

Definition 2.1 ([1, 15])**.** a) The ψ - Riemann-Liouville fractional integral of f of order β is defined by

$$I^{\beta,\psi} f(t) = \frac{1}{\Gamma(\beta)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\beta-1} f(s)ds,$$

where Γ is the (Euler's) Gamma function defined by $\Gamma(\beta) = \int_0^\infty t^{\beta-1}e^{-t}dt$.

b) The ψ - Riemann-Liouville fractional derivative of f of order β is defined by

$$D^{\beta,\psi} f(t) = \frac{1}{\Gamma(n-\beta)} (\frac{1}{\psi'(t)} \frac{d}{dt})^n \int_0^t \psi'(s)(\psi(t) - \psi(s))^{n-\beta-1} f(s)ds,$$

where $n = [\beta] + 1$.

c) The ψ - Caputo fractional derivative of f of order β is defined by

$$D_C^{\beta,\psi} f(t) = D^{\beta,\psi}[f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(0)}{k!} (\psi(t) - \psi(0))^k],$$

where $f_\psi^{[k]}(t) = (\frac{1}{\psi'(t)}\frac{d}{dt})^k x(t)$, $n = \beta$ if $\alpha \in \mathbf{N}$ and $n = [\beta]+1$, otherwise.

We note that if $\beta = m \in \mathbf{N}$ then $D_C^{\beta,\psi} f(t) = f_\psi^{[m]}(t)$ and if $n = [\beta]+1$ then $D_C^{\beta,\psi} f(t) = \frac{1}{\Gamma(n-\beta)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-\alpha-1} f_\psi^{[n]}(s)ds$. Also, if $\psi(t) \equiv t$ one obtains Caputo's fractional derivative ([7]), if $\psi(t) \equiv \ln(t)$ one obtains Caputo-Hadamard's fractional derivative ([13]) and, finally, if $\psi(t) \equiv t^\sigma$ one obtains Caputo-Katugampola's fractional derivative ([14]).

Consider $I_1 = [0, T_1]$, $I_2 = [0, T_2]$ and $\Pi = [0, T_1] \times [0, T_2]$. Denote by $\mathcal{L}(\Pi)$ the σ- algebra of the Lebesgue measurable subsets of Π and by $\mathcal{B}(\mathbf{R})$ the family of all Borel subsets of $\mathbf{R}$.

Let $C(\Pi, \mathbf{R})$ be the Banach space of all continuous functions from Π to $\mathbf{R}$ with the norm $||u||_C = \sup\{|u(x, y)|; (x, y) \in \Pi\}$ and $L^1(\Pi, \mathbf{R})$ be the Banach space of functions $u(\cdot, \cdot) : \Pi \to \mathbf{R}$ which are integrable, normed by $\|u\|_{L^1} = \int_0^{T_1}\int_0^{T_2} |u(x, y)|dxdy$.

Next, $\alpha = (\alpha_1, \alpha_2) \in [0, 1) \times [0, 1)$ and $\psi(.) = (\psi_1(.), \psi_2(.)) \in C^1(I_1, \mathbf{R}) \times C^1(I_2, \mathbf{R})$ such that $\psi_1'(t) > 0, \psi_2'(t) > 0 \ \forall\, t \in I$.

Definition 2.2. a) The ψ mixed integral of order α of $f(.,.) \in L^1(\Pi, \mathbf{R})$ is defined by

$$(I^{\alpha,\psi} f)(x, y) = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1-1}(\psi_2(y) - \psi_2(t))^{\alpha_2-1} f(s, t)dsdt.$$

b) The ψ mixed fractional-order derivative of order α of $f(.,.) \in L^1(\Pi, \mathbf{R})$ is defined by

$$(D_C^{\alpha,\psi} f)(x, y) = (I^{1-\alpha,\psi} \frac{\partial^2 f}{\partial x \partial y})(x, y) = \frac{1}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)\cdot (\psi_1(x) - \psi_1(s))^{-\alpha_1}(\psi_2(y) - \psi_2(t))^{-\alpha_2} \frac{\partial^2 f}{\partial s \partial t}(s, t)dsdt.$$

In the definition above by $1-\alpha$ we mean $(1-\alpha_1, 1-\alpha_2) \in (0, 1] \times (0, 1]$.

Definition 2.3. A function $u(.,.) \in C(\Pi, \mathbf{R})$ is said to be a solution of problem (1.1)-(1.2) if there exists $f(.,.) \in L^1(\Pi, \mathbf{R})$ such that

$$f(x, y) \in F(x, y, u(x, y), (I^{\alpha,\psi} u)(x, y)) \quad a.e.\ (\Pi), \tag{2.1}$$

$$u(x, y) = \nu(x, y) + (I^{\alpha,\psi} f)(x, y), \quad (x, y) \in \Pi, \tag{2.2}$$

where $\nu(x, y) = \varphi_1(x) + \varphi_2(y) - \varphi_1(0)$.

The pair $(u(.,.), f(.,.))$ is called a *trajectory-selection* pair of problem (1.1)-(1.2).

The previous definition is justified by the fact that a simple computation shows that $u(.,.)$ satisfies $D_C^{\alpha,\psi} u(x, y) \equiv f(x, y)$, $u(x, 0) \equiv \varphi_1(x)$, $u(0, y) \equiv \varphi_2(y)$, $(x, y) \in \Pi$ if and only if (2.2) is verified.

Let (X, d) be a metric space. The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$,

$d^*(A,B) = \sup\{d(a,B);\, a \in A\}$, where $d(x,B) = \inf\{d(x,y); y \in B\}$. With $cl(A)$ we denote the closure of the set $A \subset X$.

Recall that a subset $D \subset L^1(\Pi, \mathbf{R})$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(\Pi)$ one has $u\chi_A + v\chi_B \in D$, where $B = I\backslash A$. We denote by $\mathcal{D}$ the family of all decomposable closed subsets of $L^1(\Pi, \mathbf{R})$.

Let $G(.,.) : \Pi \times \mathbf{R}^m \to \mathcal{P}(\mathbf{R}^n)$ be a set-valued map. Recall that $G(.,.)$ is called $\mathcal{L}(\Pi) \otimes \mathcal{B}(\mathbf{R}^m)$ measurable if for any closed subset $C \subset \mathbf{R}^n$ we have $\{(x,y,z) \in \Pi \times \mathbf{R}^m; F(x,y,z) \cap C\} \neq \emptyset\} \in \mathcal{L}(\Pi) \otimes \mathcal{B}(\mathbf{R}^m)$.

Consider the Banach space $\mathbf{S} := \{(\varphi,\psi) \in C(I_1,\mathbf{R}) \times C(I_2,\mathbf{R}); \varphi(0) = \psi(0)\}$ endowed with the norm $||(\varphi,\psi)|| = ||\varphi||_C + ||\psi||_C$ and for $(\varphi,\psi) \in \mathbf{S}$ denote $\mathcal{S}(\varphi,\psi)$ the set of all solutions of problem (1.1)-(1.2).

We recall now some results that we are going to use in the next section.

Lemma 2.4 ([21])**.** *Let $G(\cdot,\cdot) : \Pi \to \mathcal{P}(\mathbf{R}^n)$ be a compact valued measurable multifunction and $h(\cdot,\cdot) : \Pi \to \mathbf{R}^n$ a measurable function.*

Then there exists a measurable selection $g(\cdot,\cdot)$ of $G(\cdot,\cdot)$ such that

$$|g(x,y) - h(x,y)| = d(h(x,y), G(x,y)), \quad a.e.\ (\Pi).$$

Next (S,d) is a separable metric space and X is a Banach space. We recall that a multifunction $G(\cdot) : S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed in S.

Lemma 2.5 ([6])**.** *Let $G^*(.,.) : \Pi \times S \to \mathcal{P}(\mathbf{R}^n)$ be a closed valued $\mathcal{L}(\Pi) \otimes \mathcal{B}(S)$ measurable multifunction such that $G^*((x,y),.)$ is l.s.c. for any $(x,y) \in \Pi$.*

Then the set-valued map $H(.)$ defined by

$$H(s) = \{g \in L^1(\Pi, \mathbf{R}^n); \quad g(x,y) \in G^*(x,y,s) \quad a.e.\ (\Pi)\}$$

is l.s.c. with nonempty decomposable closed values if and only if there exists a continuous mapping $q(.) : S \to L^1(\Pi, \mathbf{R})$ such that

$$d(0, G^*(x,y,s)) \le q(s)(x,y) \quad a.e.\ (\Pi),\ \forall s \in S.$$

Lemma 2.6 ([6])**.** *Let $H(.) : S \to \mathcal{D}$ be a l.s.c. set-valued map with closed decomposable values and let $f(.) : S \to L^1(\Pi, \mathbf{R}^n)$, $p(.) : S \to L^1(\Pi, \mathbf{R})$ be continuous such that the multifunction $G(.) : S \to \mathcal{D}$ defined by*

$$G(s) = cl\{h \in H(s); \quad ||h(x,y) - f(s)(x,y)|| < p(s)(x,y) \quad a.e.\ (\Pi)\}$$

has nonempty values.

Then $G(.)$ has a continuous selection.

3. The main results

In what follows we work under the following hypotheses.

Hypothesis H1. $F(.,.) : \Pi \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map with non-empty, compact values that verifies:
i) For all $u, v \in \mathbf{R}$, $F(.,., u, v)$ is measurable.
ii) There exists $L_1, L_2 \geq 0$ such that for almost all $(x, y) \in \Pi$,

$$d_H(F(x, y, u_1, v_1), F(x, y, u_2, v_2)) \leq L_1|u_1 - u_2| + L_2|v_1 - v_2|,$$

$\forall u_1, v_1, u_2, v_2 \in \mathbf{R}$.

In what follows $g(.,.) \in L^1(\Pi, \mathbf{R})$ is given and there exists $\xi(.,.) \in L^1(\Pi, \mathbf{R}_+)$ with $M := \sup_{(x,y)\in\Pi}(I^{\alpha,\psi}\xi)(x, y) < +\infty$ which satisfies

$$d(g(x, y), F(x, y, w(x, y), (I^{\alpha,\psi}w)(x, y))) \leq \xi(x, y) \quad a.e.\ (\Pi),$$

where $w(.,.)$ is a solution of the fractional hyperbolic differential equation

$$D_C^{\alpha,\psi}w(x, y) = g(x, y) \quad (x, y) \in \Pi, \tag{3.1}$$

$$w(x, 0) = \theta_1(x), \quad w(0, y) = \theta_2(y) \quad (x, y) \in \Pi, \tag{3.2}$$

with $(\theta_1, \theta_2) \in \mathbf{S}$.

Set $\nu_1(x, y) = \theta_1(x) + \theta_2(y) - \theta_1(0)$, $(x, y) \in \Pi$, $K_1 = \frac{(\psi_1(T_1))^{\alpha_1}(\psi_2(T_2))^{\alpha_2}}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}$ and $K = K_1(L_1 + K_1L_2)$.

Theorem 3.1. *Let Hypothesis H1 be satisfied, $K < 1$ and consider $g(.,.)$, $\xi(.,.)$, $w(.,.)$ as above, $(\varphi_1, \varphi_2) \in \mathbf{S}$ and $\nu(x, y) = \varphi_1(x) + \varphi_2(y) - \varphi_1(0)$, $(x, y) \in \Pi$.*

Then there exists $(v(.,.), f(.,.))$ a trajectory-selection pair of problem (1.1)-(1.2) *such that*

$$|v(x, y) - w(x, y)| \leq \frac{||\nu - \nu_1||_C + M}{1 - K}, \quad \forall (x, y) \in \Pi, \tag{3.3}$$

$$|f(x, y) - g(x, y)| \leq \frac{(L_1 + K_1L_2)(||\nu - \nu_1||_C + M)}{1 - K} + \xi(x, y), \quad a.e.\ (\Pi). \tag{3.4}$$

Proof. We define $f_0(.,.) = g(.,.)$, $v_0(.,.) = w(.,.)$. By Lemma 2.4 there exists a measurable function $f_1(.,.)$ such that $f_1(x, y) \in F(x, y, v_0(x, y),$
$(I^{\alpha,\psi}v_0)(x, y))$ a.e. (Π) and for almost all $(x, y) \in \Pi$

$$|f_0(x, y) - f_1(x, y)| = d(g(x, y), F(x, y, v_0(x, y), (I^{\alpha,\psi}v_0)(x, y))) \leq \xi(x, y).$$

Define, for $(x, y) \in \Pi$

$$v_1(x, y) = \nu(x, y) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1-1}(\psi_2(y) - \psi_2(t))^{\alpha_2-1}f_1(s, t)dsdt.$$

Since

$$w(x, y) = \nu_1(x, y) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1-1}(\psi_2(y) - \psi_2(t))^{\alpha_2-1}f_0(s, t)dsdt.$$

one has

$$|v_1(x,y) - v_0(x,y)| \leq |\nu(x,y) - \nu_1(x,y)| + \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)\cdot (\psi_1(x) - \psi_1(s))^{\alpha_1-1}(\psi_2(y) - \psi_2(t))^{\alpha_2-1}||f_1(s,t) - f_0(s,t)||dsdt \leq ||\nu - \nu_1||_C + \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1-1}(\psi_2(y) - \psi_2(t))^{\alpha_2-1}\xi(s,t)dsdt. \leq ||\nu - \nu_1||_C + M.$$

From Lemma 2.4 we deduce the existence of a measurable function $f_2(.,.)$ such that $f_2(x,y) \in F(x,y,v_1(x,y),(I^{\alpha,\psi}v_1)(x,y))$ a.e. (Π) and for almost all $(x,y) \in \Pi$

$$|f_2(x,y) - f_1(x,y)| \leq d(f_1(x,y), F(x,y,v_1(x,y),(I^{\alpha,\psi}v_1)(x,y))) \leq d_H(F(x,y,v_0(x,y),(I_0^{\alpha,\psi}v_0)(x,y)), F(x,y,v_1(x,y),(I^{\alpha,\psi}v_1)(x,y))) \leq L_1|v_1(x,y) - v_0(x,y)| + L_2|(I^{\alpha,\psi}v_0)(x,y) - (I^{\alpha,\psi}v_1)(x,y)| \leq L_1(||\nu - \nu_1||_C + M) + L_2\tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1-1}\cdot (\psi_2(y) - \psi_2(t))^{\alpha_2-1}(||\nu - \nu_1||_C + M)dsdt = (L_1 + K_1L_2)(||\nu - \nu_1||_C + M).$$

Define, for $(x,y) \in \Pi$

$$v_2(x,y) = \nu(x,y) + \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1-1}(\psi_2(y) - \psi_2(t))^{\alpha_2-1}f_2(s,t)dsdt$$

and one has

$$|v_2(x,y) - v_1(x,y)| \leq \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1-1}(\psi_2(y) -\psi_2(t))^{\alpha_2-1}|f_2(s,t) - f_1(s,t)|dsdt \leq \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1-1}(\psi_2(y) - \psi_2(t))^{\alpha_2-1}(L_1 + K_1L_2)(||\nu - \nu_1||_C + M)dsdt = K(||\nu - \nu_1||_C + M).$$

Assuming that for some $p \geq 2$ we have already constructed the sequences $(v_i(.,.))_{i=1}^p$, $(f_i(.,.))_{i=1}^p$ satisfying

$$|v_p(x,y) - v_{p-1}(x,y)| \leq K^{p-1}(||\nu - \nu_1||_C + M) \quad (x,y) \in \Pi, \tag{3.5}$$

$$|f_p(x,y) - f_{p-1}(x,y)| \leq (L_1 + K_1L_2)K^{p-2}(||\nu - \nu_1||_C + M) \quad a.e.\ (\Pi). \tag{3.6}$$

We apply Lemma 2.4 and we find a measurable function $f_{p+1}(.,.)$ such that $f_{p+1}(x,y) \in F(x,y,v_p(x,y),(I^{\alpha,\psi}v_p)(x,y))$ a.e. (Π) and for almost all $(x,y) \in \Pi$

$$|f_{p+1}(x,y) - f_p(x,y)| \leq d(f_{p+1}(x,y), F(x,y,v_{p-1}(x,y),(I^{\alpha,\psi}v_{p-1})(x,y))) \leq d_H(F(x,y,v_p(x,y),(I^{\alpha,\psi}v_p)(x,y)), F(x,y,v_{p-1}(x,y),(I^{\alpha,\psi}v_{p-1})(x,y))) \leq L_1|v_p(x,y) - v_{p-1}(x,y)| + L_2|(I^{\alpha,\psi}v_p)(x,y) - (I^{\alpha,\psi}v_{p-1})(x,y)| \leq L_1[K^{p-2}(||\nu - \nu_1||_C + M)] + L_2K_1K^{p-2}(||\nu - \nu_1||_C + M) = (L_1 + K_1L_2)K^{p-1}(||\nu - \nu_1||_C + M).$$

Define, for $(x,y) \in \Pi$

$$v_{p+1}(x,y) = \nu(x,y) + \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1-1}(\psi_2(y) -\psi_2(t))^{\alpha_2-1}f_{p+1}(s,t)dsdt \tag{3.7}$$

We have

$$|v_{p+1}(x,y) - v_p(x,y)| \leq \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1 - 1} (\psi_2(y) - \psi_2(t))^{\alpha_2 - 1} |f_{p+1}(s,t) - f_p(s,t)| dsdt \leq \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t) (\psi_1(x) - \psi_1(s))^{\alpha_1 - 1} (\psi_2(y) - \psi_2(t))^{\alpha_2 - 1} K^{p-1}(||\nu - \nu_1||_C + M)(L_1 + K_1 L_2) dsdt = K^{p-1}(||\nu - \nu_1||_C + M) K_1 (L_1 + K_1 L_2) = K^p(||\nu - \nu_1||_C + M).$$

From (3.5) we deduce that the sequence $(v_p(.,.))_{p\geq 0}$ is Cauchy in $C(\Pi, \mathbf{R})$, so it converges to $v(.,.) \in C(\Pi, \mathbf{R})$. Taking into account (3.6) we infer that the sequence $(f_p(.,.))_{p\geq 0}$ is Cauchy in $L^1(\Pi, \mathbf{R})$, thus it converges to $f(.,.) \in L^1(\Pi, \mathbf{R})$.

Using the fact that the values of $F(.,.)$ are closed we get that $f(x,y) \in F(x,y,v(x,y),(I^{\alpha,\psi}v)(x,y))$ a.e. (Π).

One may write successively,

$$\left| \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1 - 1}(\psi_2(y) - \psi_2(t))^{\alpha_2 - 1} f_p(s,t) dsdt - \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1 - 1}(\psi_2(y) - \psi_2(t))^{\alpha_2 - 1} f(s,t) dsdt \right| \leq \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t)(\psi_1(x) - \psi_1(s))^{\alpha_1 - 1} (\psi_2(y) - \psi_2(t))^{\alpha_2 - 1} |f_p(s,t) - f(s,t)| dsdt \leq \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi_1'(s)\psi_2'(t) (\psi_1(x) - \psi_1(s))^{\alpha_1 - 1}(\psi_2(y) - \psi_2(t))^{\alpha_2 - 1}(L_1 + K_1 L_2)|u_{p-1}(s,t) - u(s,t)| dsdt \leq K||u_{p-1}(.,.) - u(.,.)||_C.$$

Thus, we pass to the limit in (3.2) and we get that $v(.,.)$ is a solution of problem (1.1)- (1.2). At the same time, by adding inequalities (3.5) for any $(x,y) \in \Pi$ we have

(3.8)

$$|v_p(x,y) - w(x,y)| \leq |v_p(x,y) - v_{p-1}(x,y)| + |v_{p-1}(x,y) - v_{p-2}(x,y)| + \ldots + |v_2(x,y) - v_1(x,y)| + |v_1(x,y) - v_0(x,y)| \leq (K^{p-1} + K^{p-2} + ... + K + 1)(||\nu - \nu_1||_C + M) \leq \frac{||\nu - \nu_1||_C + M}{1-K}.$$

Similarly, by adding inequalities (3.6) for almost all $(x,y) \in \Pi$ we have

$$|f_p(x,y) - g(x,y)| \leq |f_p(x,y) - f_{p-1}(x,y)| + |f_{p-1}(x,y) - f_{p-2}(x,y)| + \ldots + |f_2(x,y) - f_1(x,y)| + |f_1(x,y) - f_0(x,y)| \leq (L_1 + K_1 L_2)(K^{p-2} + ... + K + 1)(||\nu - \nu_1||_C + M) + \xi(x,y) \leq (L_1 + K_1 L_2)\frac{||\nu - \nu_1||_C + M}{1-K} + \xi(x,y). \tag{3.9}$$

Finally we pass to the limit with $p \to \infty$ in (3.8) and (3.9) and we get (3.3) and (3.4), respectively, which completes the proof. □

Remark 3.2. If in Theorem 3.1 $\psi(t) \equiv t$ we obtain Theorem 3.2 in [8]; if in Theorem 3.1 $\psi(t) \equiv \ln(t)$ we get Theorem 4 in [9] and if in Theorem 3.1 $\psi(t) \equiv t^\sigma$ we cover Theorem 3.1 in [10].

If in Theorem 3.1 we take $g = 0$, $w = 0$, $\theta_1 = 0$, $\theta_2 = 0$, then we obtain the following existence result for solutions of problem (1.1)-(1.2).

Corollary 3.3. *Let Hypothesis H1 be satisfied, $K<1$ and assume that there exists $\xi(.,.)\in L^1(\Pi,\mathbf{R}_+)$ with $M:=\sup_{(x,y)\in\Pi}(I^{\alpha,\psi}\xi)(x,y)<+\infty$ such that $d(0,F(x,y,0,0))\leq \xi(x,y)\ \forall (x,y)\in\Pi$.*

Then there exists $v(.,.)\in C(\Pi,\mathbf{R})$ a solution of problem (1.1)-(1.2) such that

$$|v(x,y)|\leq \frac{||\nu||_C+M}{1-K},\quad \forall (x,y)\in\Pi.$$

In the last part of this paper we obtain a continuous version of Theorem 3.1.

Hypothesis H2. i) S is a separable metric space, $\varphi_1(.)\to C(I_1,\mathbf{R}),\varphi_2(.):S\to C(I_2,\mathbf{R})$ and $\varepsilon(.):S\to(0,\infty)$ are continuous mappings and $\varphi_1(s)(0)\equiv\varphi_2(s)(0)$.

ii) There exists the continuous mappings $\theta_1(.)\to C(I_1,\mathbf{R}),\theta_2(.):S\to C(I_2,\mathbf{R})$ $g(.):S\to L^1(\Pi,\mathbf{R})$, $\xi(.):S\to L^1(\Pi,\mathbf{R})$ and $w(.):S\to C(\Pi,\mathbf{R})$ such that $\theta_1(s)(0)\equiv\theta_2(s)(0)$,

$$(Dw(s))_C^{\alpha,\psi}(x,y)=g(s)(x,y)\quad a.e.\ (\Pi),\quad \forall s\in S,$$

$$w(s)(x,0)=\theta_1(s)(x),\quad w(s)(0,y)=\theta_2(s)(y)\quad (x,y)\in\Pi,\quad \forall s\in S,$$

$$d(g(s)(x,y),F(x,y,w(s)(x,y),(I^{\alpha,\psi}w(s))(x,y)))\leq\xi(s)(x,y)\,a.e.\ (\Pi),\forall s\in S$$

and the mapping $s\to M(s):=\sup_{(x,y)\in\Pi}(I^{\alpha,\psi}\xi(s))(x,y)$ is continuous.

We use next the following notations $\nu(s)(x,y)=\varphi_1(s)(x)+\varphi_2(s)(y)-\varphi_1(s)(0)$, $\nu_1(s)(x,y)=\theta_1(s)(x)+\theta_2(s)(y)-\theta_1(s)(0)$, $(x,y)\in\Pi$, $a(s)=\sup_{(x,y)\in\Pi}|\nu(s)(x,y)-\nu_1(s)(x,y)|$, $s\in S$.

Theorem 3.4. *Assume that Hypotheses H1 and H2 are satisfied and $K<1$.*

Then there exist a continuous mapping $v(.):S\to C(\Pi,\mathbf{R})$ such that for any $s\in S$, $v(s)(.,.)$ is a solution of problem (1.1) which satisfies $v(s)(x,0)=\varphi_1(s)(x)$, $v(s)(0,y)=\varphi_2(s)(y)$ $(x,y)\in\Pi,s\in S$ and

$$|v(s)(x,y)-w(s)(x,y)|\leq\frac{a(s)+\varepsilon(s)+M(s)}{1-K}\quad \forall (x,y)\in\Pi,\forall s\in S.$$

Proof. We make the following notations

$$v_0(.,.)=w(.,.),\quad \xi_p(s):=K^{p-1}(a(s)+\varepsilon(s)+M(s)),\quad p\geq 1.$$

We consider the set-valued maps $G_0(.),H_0(.)$ defined, respectively, by

$$G_0(s)=\{h\in L^1(\Pi,\mathbf{R});h(x,y)\in F(x,y,w(s)(x,y),(I^{\alpha,\psi}w(s))(x,y))a.e.(\Pi)\}$$

$$H_0(s)=cl\{h\in G_0(s);|h(x,y)-g(s)(x,y)|<\xi(s)(x,y)+\frac{1}{K_1}\varepsilon(s)\}.$$

Taking into account that $d(g(s)(x,y),F(x,y,w(s)(x,y),(I^{\alpha,\psi}w(s))(x,y))\leq\xi(s)(x,y)<\xi(s)(x,y)+\frac{1}{K_1}\varepsilon(s)$ the set $H_0(s)$ is not empty.

Set $F_0^*(x,y,s) = F(x,y,w(s)(x,y),(I^{\alpha,\psi}w(s))(x,y))$ and note that

$$d(0,F_0^*(x,y,s)) \leq |g(s)(x,y)| + \xi(s)(x,y) =: \xi^*(s)(x,y)$$

and $\xi^*(.) : S \to L^1(I,\mathbf{R})$ is continuous.

Applying now Lemma 2.5 and Lemma 2.6 we obtain the existence of a continuous selection f_0 of H_0 such that $\forall s \in S,\ (x,y) \in \Pi$,

$$f_0(s)(x,y) \in F(x,y,w(s)(x,y),(I^{\alpha,\psi}w(s))(x,y)) \quad a.e.\,(\Pi),\ \forall s \in S,$$

$$|f_0(s)(x,y) - g(s)(x,y)| \leq \xi_0(s)(x,y) = \xi(s)(x,y) + \frac{1}{K_1}\varepsilon(s).$$

We define

$$v_1(s)(x,y) = \nu(s)(x,y) + \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y \psi_1'(z)\psi_2'(t)(\psi_1(x)-\psi_1(z))^{\alpha_1-1}\cdot (\psi_2(y)-\psi_2(t))^{\alpha_2-1}f_0(s)(z,t)dzdt$$

and one has

$$|v_1(s)(x,y) - v_0(s)(x,y)| \leq a(s) + \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y \psi_1'(z)\psi_2'(t)(\psi_1(x)- \psi_1(z))^{\alpha_1-1}(\psi_2(y)-\psi_2(t))^{\alpha_2-1}|f_0(s)(z,t) - g(s)(z,t)|dzdt \leq a(s)+ \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y \psi_1'(z)\psi_2'(t)(\psi_1(x)-\psi_1(z))^{\alpha_1-1}(\psi_2(y)-\psi_2(t))^{\alpha_2-1} (\xi(s)(z,t) + \tfrac{1}{K_1}\varepsilon(s))dzdt \leq a(s) + M(s) + \varepsilon(s) =: \xi_1(s), \quad (x,y) \in \Pi, s \in S.$$

We construct the sequences of approximations $f_p(.,.) : S \to L^1(\Pi,\mathbf{R})$, $v_p(.,.) : S \to C(\Pi,\mathbf{R})$ with the following properties:

a) $f_p(.,.) : S \to L^1(\Pi,\mathbf{R})$, $v_p(.,.) : S \to C(\Pi,\mathbf{R})$ are continuous,

b) $f_p(s)(x,y) \in F(x,y,v_p(s)(x,y),(I^{\alpha,\psi}v_p(s))(x,y))$, a.e. (Π), $s \in S$,

c) $|f_p(s)(x,y) - f_{p-1}(s)(x,y)| \leq (L_1 + K_1L_2)\xi_p(s)$, a.e. (Π), $s \in S$.

d) $v_{p+1}(s)(x,y) = \nu(s)(x,y) + \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y \psi_1'(z)\psi_2'(t)(\psi_1(x) - \psi_1(z))^{\alpha_1-1}(\psi_2(y)-\psi_2(t))^{\alpha_2-1}f_p(s)(z,t)dzdt$, $(x,y) \in \Pi, s \in S$.

Assume that we have already constructed $f_i(.), v_i(.)$ satisfying a)-c) and define $v_{p+1}(.)$ as in d). From c) and d) one has

$$|v_{p+1}(s)(x,y) - v_p(s)(x,y)| \leq \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y \psi_1'(z)\psi_2'(t)(\psi_1(x)- \psi_1(z))^{\alpha_1-1}(\psi_2(y)-\psi_2(t))^{\alpha_2-1}|f_p(s)(z,t) - f_{p-1}(s)(z,t)|dzdt \leq \tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y \psi_1'(z)\psi_2'(t)(\psi_1(x)-\psi_1(z))^{\alpha_1-1}(\psi_2(y)-\psi_2(t))^{\alpha_2-1}\cdot (L_1 + K_1L_2)\xi_p(s)dzdt = K_1(L_1+K_1L_2)\xi_p(s) = \xi_{p+1}(s). \tag{3.10}$$

On the other hand,

$$d(f_p(s)(x,y), F(x,y,v_{p+1}(s)(x,y),(I^{\alpha,\psi}v_{p+1}(s))(x,y))) \leq L_1|v_{p+1}(s)(x,y) - v_p(s)(x,y)| + L_2\tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y \psi_1'(z)\psi_2'(t)(\psi_1(x) -\psi_1(z))^{\alpha_1-1}(\psi_2(y)-\psi_2(t))^{\alpha_2-1}|v_{p+1}(s)(z,t) - v_p(s)(z,t)|dzdt \leq (L_1 + K_1L_2)\xi_{p+1}(s). \tag{3.11}$$

For any $s \in S$ we define the set-valued maps $G_{p+1}(s) = \{u \in L^1(\Pi,\mathbf{R});\ u(x,y) \in F(x,y,v_{p+1}(s)(x,y),(I^{\alpha,\psi}v_{p+1}(s))(x,y))\ a.e.\,(\Pi)\}$ and

$$H_{p+1}(s) = cl\{u \in G_{p+1}(s);\ |u(x,y) - f_p(s)(x,y)| < (L_1 + K_1L_2)\xi_{p+1}(s)\}.$$

We note that from (3.11) the set $H_{p+1}(s)$ is not empty.

Set $F^*_{p+1}(x,y,s)=F(x,y,v_{p+1}(s)(x,y),(I^{\alpha,\psi}v_{p+1}(s))(x,y))$ and note that

$$d(0,F^*_{p+1}(x,y,s))\leq |f_p(s)(x,y)|+(L_1+K_1L_3)\xi_{p+1}(s)=:\xi^*_{p+1}(s)(x,y)$$

and $\xi^*_{p+1}(.):S\to L^1(I,\mathbf{R})$ is continuous.

By Lemma 2.5 and Lemma 2.6 we obtain the existence of a continuous function $f_{p+1}(.):S\to L^1(\Pi,\mathbf{R})$ such that

$$f_{p+1}(s)(x,y)\in F(x,y,v_{p+1}(s)(x,y),(I^{\alpha,\psi}v_{p+1}(s))(x,y)) \quad a.e.\,(\Pi),\ \forall s\in S,$$

$$|f_{p+1}(s)(x,y)-f_p(s)(x,y)|\leq (L_1+K_1L_2)\xi_{p+1}(s) \quad \forall s\in S,\ (x,y)\in\Pi.$$

From (3.10), c) and d) we obtain

$$|v_{p+1}(s)(.,.)-v_p(s)(.,.)|_C\leq \xi_{p+1}(s)=K^p(a(s)+\varepsilon(s)+M(s)), \tag{3.12}$$

$$|f_{p+1}(s)(.,.)-f_p(s)(.,.)|_1\leq K^{p-1}(L_1+K_1L_2)T_1T_2(a(s)+\varepsilon(s)+M(s)). \tag{3.13}$$

Thus, $f_p(s)(.,.)$, $v_p(s)(.,.)$ are Cauchy sequences in the Banach spaces $L^1(\Pi,\mathbf{R})$ and $C(\Pi,\mathbf{R})$, respectively. Consider $f(.):S\to L^1(\Pi,\mathbf{R})$, $v(.):S\to C(\Pi,\mathbf{R})$ their limits. The function $s\to a(s)+\varepsilon(s)+M(s)$ is continuous, hence locally bounded. Therefore, (3.13) implies that for every $s'\in S$ the sequence $f_p(s')(.,.)$ satisfies the Cauchy condition uniformly with respect to s' on some neighborhood of s. Therefore, $s\to f(s)(.,.)$ is continuous from S into $L^1(\Pi,\mathbf{R})$.

As before, from (3.12), $v_p(s)(.,.)$ is Cauchy in $C(\Pi,\mathbf{R})$ locally uniformly with respect to s. Hence $s\to v(s)(.,.)$ is continuous from S into $C(\Pi,\mathbf{R})$. At the same time, since $v_p(s)(.,.)$ converges uniformly to $v(s)(.,.)$ and

$$\begin{aligned}&d(f_p(s)(x,y),F(x,y,v(s)(x,y),(I^{\alpha,\psi}v(s))(x,y))\leq\\ &(L_1+K_1L_2)|v_p(s)(x,y)-v(s)(x,y)| \quad a.e.\,(\Pi),\quad \forall s\in S\end{aligned}$$

passing to the limit along a subsequence of $f_p(s)(.,.)$ converging pointwise to $f(s)(.,.)$ we obtain

$$f(s)(x,y)\in F(x,y,v(s)(x,y),(I^{\alpha,\psi}v(s))(x,y)) \quad a.e.\ (\Pi),\ \forall s\in S.$$

One may write successively,

$$\begin{aligned}&|\tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y\psi_1'(z)\psi_2'(t)(\psi_1(x)-\psi_1(z))^{\alpha_1-1}(\psi_2(y)-\psi_2(t))^{\alpha_2-1}f_p(s)(z,\\ &t)dzdt-\tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y\psi_1'(z)\psi_2'(t)(\psi_1(x)-\psi_1(z))^{\alpha_1-1}(\psi_2(y)-\psi_2(t))^{\alpha_2-1}\\ &f(s)(z,t)dzdt|\leq\tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y\psi_1'(z)\psi_2'(t)(\psi_1(x)-\psi_1(z))^{\alpha_1-1}(\psi_2(y)-\\ &\psi_2(t))^{\alpha_2-1}f_0(s)(z,t)dzdt|f_p(s)(z,t)-f(s)(z,t)|dzdt\leq\tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y\\ &\psi_1'(z)\psi_2'(t)(\psi_1(x)-\psi_1(z))^{\alpha_1-1}(\psi_2(y)-\psi_2(t))^{\alpha_2-1}(L_1+K_1L_2)|v_{p-1}(s)(z,\\ &t)-v(s)(z,t)|dzdt\leq K||v_{p-1}(s)(.,.)-v(s)(.,.)||_C.\end{aligned}$$

So, we pass to the limit in d) and we get $\forall(x,y)\in\Pi,s\in S$

$$\begin{aligned}&v(s)(x,y)=\nu(s)(x,y)+\tfrac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y\psi_1'(z)\psi_2'(t)(\psi_1(x)-\psi_1(z))^{\alpha_1-1}\cdot\\ &(\psi_2(y)-\psi_2(t))^{\alpha_2-1}f(s)(z,t)dzdt,\end{aligned}$$

i.e., $v(s)(.,.)$ is the required solution.

Finally, by adding inequalities (3.10) for all $p \geq 1$ we get

$$|v_{p+1}(s)(x,y) - w(s)(x,y)| \leq \sum_{l=1}^{p+1} \xi_l(s) \leq \frac{a(s)+\varepsilon(s)+M(s)}{1-K}. \tag{3.14}$$

Passing to the limit in (3.14) we obtain the conclusion of the theorem. □

Remark 3.5. If in Theorem 3.4 $\psi(t) \equiv t$ we obtain Theorem 3.7 in [8]; if in Theorem 3.4 $\psi(t) \equiv \ln(t)$ we find Theorem 6 in [9] and if in Theorem 3.4 $\psi(t) \equiv t^\sigma$ we get Theorem 3.2 in [10].

Theorem 3.4 allows to provide a continuous selection of the solution set of problem (1.1)-(1.2).

Hypothesis H3. Hypothesis H1 is satisfied, $K < 1$ and there exists $q(.,.) \in L^1(\Pi, \mathbf{R}_+)$ with $\sup_{(x,y)\in\Pi}(I^{\alpha,\psi}q)(x,y) < \infty$ such that $d(0, F(x,y,0,0)) \leq q(x,y)$ a.e. (Π).

Corollary 3.6. *Assume that Hypothesis H3 is satisfied.*

Then there exists a function $v(.,.) : \Pi \times \mathbf{S} \to \mathbf{R}$ such that

a) $v(.,(\xi,\eta)) \in \mathcal{S}(\xi,\eta)$, $\forall(\xi,\eta) \in \mathbf{S}$.

b) $(\xi,\eta) \to v(.,(\xi,\eta))$ is continuous from $\mathbf{S}$ into $C(\Pi,\mathbf{R})$.

Proof. We take $S = \mathbf{S}$, $\varphi_1(\mu,\eta) = \mu$, $\varphi_2(\mu,\eta) = \eta$ $\forall(\mu,\eta) \in \mathbf{S}$, $\varepsilon(.) : \mathbf{S} \to (0,\infty)$ an arbitrary continuous function, $g(.) = 0$, $w(.) = 0$, $\xi(s)(x,y) \equiv q(x,y)$ $\forall s = (\mu,\eta) \in \mathbf{S}$, $(x,y) \in \Pi$ and we apply Theorem 3.4 in order to obtain the conclusion of the corollary. □

Example 3.7. Consider the following problem

$$D_C^{(\frac{1}{2},\frac{1}{2}),(1,1)}u(x,y) = \frac{a}{3e^{x+y+2}(1+|u(x,y)|)} \quad a.e.\ (x,y) \in [0,1]\times[0,1],$$

$$u(x,0) = x, \quad u(0,y) = y^2 \quad (x,y) \in [0,1]\times[0,1],$$

where $a \in (0, 3e^2\Gamma(\frac{3}{2})^2)$. In this case $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\psi_1(t) = \psi_2(t) \equiv t$, $\varphi_1(x) = x$, $\varphi_2(y) = y^2$, $F(x,y,u,v) = \{\frac{a}{3e^{x+y+2}(1+|u|)}\}$, $T_1 = T_2 = 1$. A straightforward computation shows that the Lipschitz constant of F is $L_1 = \frac{a}{3e^2}$, $d(0,F(x,y,0,0)) = \frac{a}{3e^{x+y+2}} \leq \frac{a}{3e^2}$, $K_1 = \frac{1}{\Gamma(\frac{3}{2})^2}$, $K = K_1L_1 \leq \frac{a}{3e^2\Gamma(\frac{3}{2})^2} < 1$.

Therefore, we can apply Corollary 3.3 and we obtain the existence of a solution $v(.,.)$ which satisfies

$$||u(x,y)|| \leq \frac{6e^2+a}{3e^2\Gamma(\frac{3}{2})^2 - a}, \quad \forall(x,y) \in [0,1]\times[0,1].$$

Example 3.8. Consider $\alpha_1 = \alpha_2 = \frac{1}{2}$, $\psi_1(t) = \psi_2(t) \equiv t$, $\varphi_1(x) = x^2$, $\varphi_2(y) = y$, $T_1 = T_2 = 1$ and $b \in (0, \frac{\Gamma(\frac{3}{2})^4}{1+\Gamma(\frac{3}{2})^2})$ Define $F(.,.) : [0,1]\times[0,1]\times\mathbf{R}\times\mathbf{R} \to \mathcal{P}(\mathbf{R})$ by

$$F(x,y,u,v) = [-b\frac{|u|}{1+|u|}, 0] \cup [0, b\frac{|v|}{1+|v|}].$$

Since

$$d(0, F(x, y, 0, 0)) \leq b \quad \forall\, x, y \in [0,1],$$

$$\mathrm{d}_H(F(x, y, u_1, v_1), F(x, y, u_2, v_2)) \leq b|u_1 - u_2| + b|v_1 - v_2| \quad \forall\; u_1, u_2, v_1, v_2 \in \mathbf{R},$$

in this case $K_1 = \frac{1}{\Gamma(\frac{3}{2})^2}$, $K = K_1(1+K_1)b = \frac{1}{\Gamma(\frac{3}{2})^2}(1 + \frac{1}{\Gamma(\frac{3}{2})^2})b$ and, taking into account the choice of b, $K < 1$. Therefore, applying Corrollary 3.3 to the problem

$$D_C^{(\frac{1}{2},\frac{1}{2}),(1,1)} u(x,y) \in F(x, y, u(x,y), \frac{1}{\Gamma(\frac{1}{2})^2} \int_0^x \int_0^y \frac{1}{\sqrt{(x-s)(y-t)}} u(s,t)dsdt)$$

$$u(x,0) = x^2, \quad u(0,y) = y \quad (x,y) \in [0,1] \times [0,1]$$

we obtain a solution, which satifies the following a priori bound

$$||u(x,y)|| \leq \frac{(b+2)\Gamma(\frac{3}{2})^4}{\Gamma(\frac{3}{2})^4 - (1 + \Gamma(\frac{3}{2})^2)b}, \quad \forall (x,y) \in [0,1] \times [0,1].$$

4. Conclusions

In this paper we studied a Darboux problem associated to a fractional hyperbolic integro-differential inclusion defined by a Caputo type fractional derivative and by a set-valued map with non-convex values. We obtained the existence of solutions using a method originally introduced by Filippov. Also, we provide the existence of solutions continuously depending on a parameter for the problem studied. The last result allowed us to deduce a continuous selection of the solution set of the problem considered.

The results in the present paper extend and unify similar results obtained for partial fractional integro-differential inclusions defined by Caputo fractional derivative, by Hadamard fractional derivative and by Caputo-Katugampola fractional derivative.

Afterwards, such results are essential in order to obtain qualitative results concerning the solutions of fractional differential inclusions defined by the Caputo type fractional derivative considered such as: controllability along a reference trajectory, differentiability of solutions with respect to the initial conditions of the problem considered.

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