Novi Sad J. Math. Vol. 53, No. 1, 2023, 97-116
https://doi.org/10.30755/NSJOM.12544

A study on convergence of sequences of functions in asymmetric metric spaces using ideals

Argha Ghosh[1]

Abstract. We introduce and study the notions of backward and forward $\mathcal{I}(\alpha)$-convergence and $\mathcal{I}$-exhaustiveness of sequences of functions between asymmetric metric spaces. We establish a relation between backward (resp. forward) $\mathcal{I}(\alpha)$-convergent and backward (resp. forward) $\mathcal{I}$-exhaustiveness. Also, we introduce and study ideal versions of some classical notions (Alexandroff and strong uniform) of convergence of sequences of functions in this context. We give some examples to ensure the alternation of basic results from the metric case.

AMS Mathematics Subject Classification (2010):40A35, 26A03, 26A15

Key words and phrases: Ideal convergence, asymmetric metric spaces, backward and forward $\mathcal{I}(\alpha)$-convergence, backward and forward $\mathcal{I}$-exhaustiveness, sequences of functions

1. Introduction

The need to find necessary and sufficient conditions to preserve continuity of the pointwise limit function of a sequence of continuous functions led to the discovery of several notions of convergence of sequences of functions. In 1883/1884, Arzelà [2] gave a necessary and sufficient condition under which the pointwise limit of a sequence of real-valued continuous functions on a compact interval is continuous, and in [1], Alexandroff carried out a similar investigation for a sequence of continuous functions from a topological space to a metric space. Recently, in 2009, Beer and Levi [5] gave another such condition via the notion of strong uniform convergence on the bornology of all finite subsets of a metric space. Later, Caserta et al. [7] proved, the Beer-Levi condition is equivalent to that of Arzelà and Alexandroff conditions.

In this direction, the notion of α-convergence (also known as continuous convergence) of real-valued sequences of functions was introduced in [6] (see also [19, 26]). The notion of α-convergence is stronger than the notion of pointwise convergence, and the notion of α-convergence is equivalent to the notion of uniform convergence on the compact domain if and only if the limit function is continuous. In [18], Gregoriades and Papanastassiou introduced the notion of exhaustiveness of sequences of functions from a metric space to a metric space, and they proved a sequence of functions is α-convergent to a function if and only if it is exhaustive and pointwise convergent to the same function.

[1]Department of Mathematics, Brainware University, Barasat, Kolkata-700125, India. E-mail: buagbu@yahoo.co.in

Recently, in 2011, Caserta et al. [8] introduced and studied some statistical versions of Arzelà convergence, Alexandroff convergence and strong uniform convergence. Further, in 2012, Caserta et al. [9] introduced and studied statistical α-convergence and statistical exhaustiveness. As a generalization of the notion of usual convergence of sequences of real numbers, Fast [16] and Steinhaus [25] individually introduced the idea of statistical convergence of sequences of real numbers. Since then the researchers have done several works in this context (see [17, 14, 28] for instance).

One of the most important generalization of statistical convergence is the notion of $\mathcal{I}$-convergence (ideal convergence) introduced by Kostyrko et al. [20], where $\mathcal{I}$ is an ideal of $\mathbb{N}$. Like statistical convergence, ideal convergence has applications in different fields of mathematics. In 2005, Lahiri et al. [21] studied ideal convergence in topological spaces. And in 2007, Balcerzak et al. [4] introduced the notions of statistical and ideal convergence for the sequences of functions. Recently, the notions of $\mathcal{I}(\alpha)$-convergence and $\mathcal{I}$-exhaustiveness were introduced by Papachristodoulos et al. [24], and these notions were a generalization of the notions of statistical α-convergence and statistical exhaustiveness, respectively. Further, Athanassiadou et al. [3] studied ideal versions of Arzelà convergence, Alexandroff convergence and strong uniform convergence. For more applications of $\mathcal{I}$-convergence in Functional Analysis and Topology, see the book edited by Dutta and Rhoades [15] (see also [12, 13] and references therein).

On the other hand, in 1931, Wilson [29] introduced the idea of asymmetric metric spaces in the name of quasi metric spaces. If one drops the axiom of symmetry in the definition of metric spaces, then one gets the definition of asymmetric metric spaces. Different studies revealed that the notions of convergence and compactness in asymmetric metric spaces differ from that of metric spaces.

It is not a surprise that asymmetric metric spaces have many applications in applied mathematics and materials science (see [22, 23] and references therein). To provide a more general framework for applications, Collins et al. [10] studied compactness and completeness in asymmetric metric spaces, and thus established an asymmetric Arzelà-Ascoli theorem. Further, in 2015, Toyganözü and Pehlivan [27] introduced the notions of forward and backward exhaustiveness and statistical exhaustiveness. But one may notice the absence of some aforementioned classical notions of convergence of sequences of functions in the same paper. So one may ask for an investigation on that direction. And we do the same in this paper via the notion of ideals.

In Section 3, we introduce the notions of backward and forward $\mathcal{I}(\alpha)$-convergence and $\mathcal{I}$-exhaustiveness of sequences of functions from an asymmetric metric space to an asymmetric metric space. And in Section 4, we introduce uniform $\mathcal{I}$-convergence and weak $\mathcal{I}$-exhaustiveness in the context of asymmetric metric spaces. Moreover, we establish a connection between all these new concepts.

In Section 5, we introduce the notions of backward and forward $\mathcal{I}$-strong uniform convergence of sequences of functions from an asymmetric metric space to an asymmetric metric space. We establish a connection between backward and forward $\mathcal{I}$-strong uniform convergence, pointwise $\mathcal{I}$-convergence and weak $\mathcal{I}$-exhaustiveness.

In Section 6, we introduce the notions of backward and forward $\mathcal{I}$-Alexandroff convergence of sequences of functions from an asymmetric metric space to an asymmetric metric space. We establish a connection between backward and forward $\mathcal{I}$-strong uniform convergence and $\mathcal{I}$-Alexandroff convergence.

2. Basic Definitions and Notations

Throughout the article, we write $\mathbb{N}$ and $\mathbb{R}$ to denote the set of all natural numbers and the set of all real numbers, respectively. In this section, we recall some definitions and notations, which we need later.

2.1. Ideals and Filters

Definition 2.1. [17] A subset K of $\mathbb{N}$ is said to have natural density (or asymptotic density) $d(K)$ if

$$d(K) = \lim_{n\to\infty} \frac{|K(n)|}{n},$$

where $K(n) = \{j \in K : j \leq n\}$ and $|K(n)|$ represents the number of elements in $K(n)$.

Definition 2.2. [20] If X is a non-empty set, then a family $\mathcal{I} \subset 2^X$ is said to be an ideal of X if
(a) $\emptyset \in \mathcal{I}$,
(b) $A\ ;\ B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, and
(c) $A \in \mathcal{I}\ ;\ B \subset A$ implies $B \in \mathcal{I}$.
The ideal $\mathcal{I}$ is said to be a non-trivial ideal if $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$.

Definition 2.3. [20] If X is a non-empty set, then a family $\mathcal{F} \subset 2^X$ is said to be a filter of X if
(a) $\emptyset \notin \mathcal{F}$,
(b) $A\ ;\ B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$, and
(c) $A \in \mathcal{F}\ ;\ A \subset B$ implies $B \in \mathcal{F}$.

Clearly, if $\mathcal{I} \subset 2^X$ is a non-trivial ideal of X, then

$$\mathcal{F}(\mathcal{I}) = \{A \subset X \ :\ X \setminus A \in \mathcal{I}\}$$

is a filter of X, called the filter associated with $\mathcal{I}$.

A non-trivial ideal of $X(\neq \emptyset)$ is said to be admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Example 2.4. (a) Let $\mathcal{I}_{fin}$ be the collection of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_{fin}$ is an ideal of $\mathbb{N}$.
(b) Let $\mathcal{I}_d$ be the collection of all subsets of $\mathbb{N}$ with asymptotic density zero. Then $\mathcal{I}_d$ is an ideal of $\mathbb{N}$.

Definition 2.5. [24] An admissible ideal $\mathcal{I}$ is said to be *good* if for every sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets such that $A_n \notin \mathcal{I}$ there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of pairwise disjoint sets such that $B_n \subset A_n$, $B_n \in \mathcal{I}$ and $\cup_{n=1}^{\infty} B_n \notin \mathcal{I}$.

From now on throughout the paper, we assume $\mathcal{I}$ as an admissible ideal of $\mathbb{N}$ unless otherwise mentioned.

2.2. Asymmetric metric spaces

We now recall some definitions and concepts on asymmetric metric spaces (see [10, 11, 27]).

Definition 2.6. An asymmetric metric space is an ordered pair (X, d) where X is a non-empty set and d is an asymmetric metric on X, that is, a function $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$, the following hold:

(a) $d(x, y) \geq 0$

(b) $d(x, y) = 0$ if and only if $x = y$

(c) $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 2.7. The forward topology τ^+ induced by d is the topology generated by the forward open balls $B^+(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ for $x \in X$, $\varepsilon > 0$.

Similarly, the backward topology τ^- induced by d is the topology generated by the backward open balls $B^-(x, \varepsilon) = \{y \in X : d(y, x) < \varepsilon\}$ for $x \in X$, $\varepsilon > 0$.

Definition 2.8. An asymmetric metric space (X, d) is said to satisfy approximate metric axiom (or (AMA)) if there exists a function $c : X \times X \to \mathbb{R}$ such that for every $z, y \in X$,

$$d(y, z) \leq c(z, y)d(z, y),$$

where c satisfies the following condition:

$\forall z \ \ \exists \delta_z > 0$ such that $y \in B^+(z, \delta_z) \implies c(z, y) \leq C(z)$, where $C(z) > 0$ is a real number.

Note 2.9. Observe that every metric space satisfies (AMA), however, there is an asymmetric metric space which satisfies (AMA) without being a metric space (see Example 2.1, [11]).

Definition 2.10. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in an asymmetric metric space X is said to be backward $\mathcal{I}$-convergent to $x \in X$ if, for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : d(x_k, x) \geq \varepsilon\} \in \mathcal{I}.$$

In this case, we write $x_k \xrightarrow{\mathcal{I}^-} x$.

Similarly, a sequence $\{x_k\}_{k\in\mathbb{N}}$ in an asymmetric metric space X is said to be forward $\mathcal{I}$-convergent to $x \in X$ if, for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : d(x, x_k) \geq \varepsilon\} \in \mathcal{I}.$$

In this case, we write $x_k \xrightarrow{\mathcal{I}^+} x$.

Note 2.11. (a) When $\mathcal{I} = \mathcal{I}_{fin}$, the concept of backward (resp. forward) $\mathcal{I}$-convergence coincides with that of backward (resp. forward) convergence of sequences in the asymmetric metric space (X, d) introduced in [10].
(b) When $\mathcal{I} = \mathcal{I}_d$, the concept of backward (resp. forward) $\mathcal{I}$-convergence coincides with that of backward (resp. forward) statistical convergence of sequences in the asymmetric metric space (X, d) introduced in [27].
(c) If $\mathcal{I}$ is an admissible ideal, then the notion of backward (resp. forward) convergence implies the notion of backward (resp. forward) $\mathcal{I}$-convergence of sequences in the asymmetric metric space (X, d).

The above definitions of backward and forward $\mathcal{I}$-convergence can be restated in the following form:

Definition 2.12. A sequence $\{x_k\}_{k\in\mathbb{N}}$ in an asymmetric metric space X is said to be backward $\mathcal{I}$-convergent to $x \in X$ if, for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : d(x_k, x) < \varepsilon\} \in \mathcal{F}(\mathcal{I}).$$

Similarly, a sequence $\{x_k\}_{k\in\mathbb{N}}$ in an asymmetric metric space X is said to be forward $\mathcal{I}$-convergent to $x \in X$ if, for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : d(x, x_k) \geq \varepsilon\} \in \mathcal{F}(\mathcal{I}).$$

In [11], Das et al. proved the following proposition, which states a relation between the forward $\mathcal{I}$-convergence and backward $\mathcal{I}$-convergence.

Proposition 2.13. *Let (X, d) be an asymmetric metric space satisfying the property (AMA). Then forward $\mathcal{I}$-convergence of a sequence implies the backward $\mathcal{I}$-convergence and the limits are same.*

Definition 2.14. Let (X, d) and (Y, ρ) be asymmetric metric spaces. A function $f : X \to Y$ is said to be backward continuous (or, f^- continuous) at $x \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$y \in B^-(x, \delta) \implies \rho(f(y), f(x)) < \varepsilon.$$

Similarly, a function $f : X \to Y$ is said to be forward continuous (or, f^+ continuous) at $x \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$y \in B^+(x, \delta) \implies \rho(f(x), f(y)) < \varepsilon.$$

Definition 2.15. (Sequential version of continuity) A function $f : X \to Y$ is said to be f^- continuous at $x \in X$, if whenever a sequence $\{x_k\}_{k\in\mathbb{N}}$ backward converges to x in (X, d), the sequence $\{f(x_k)\}_{k\in\mathbb{N}}$ backward converges to $f(x)$ in (Y, ρ).

Similarly, a function $f : X \to Y$ is said to be f^+ continuous at $x \in X$, if whenever a sequence $\{x_k\}_{k\in\mathbb{N}}$ forward converges to x in (X, d), the sequence $\{f(x_k)\}_{k\in\mathbb{N}}$ forward converges to $f(x)$ in (Y, ρ).

Definition 2.16. Let (X, d) be an asymmetric metric space. Let $M \subset X$. Then M is said to be backward compact if every open cover of M in the backward topology has a finite subcover.

The statement holds analogously for the other types, that is, M is said to be forward compact if every open cover of M in the forward topology has a finite subcover.

3. $\mathcal{I}(\alpha)$-convergence and $\mathcal{I}$-exhaustiveness in asymmetric metric spaces

From now on throughout the article, we write $X = (X, d)$ and $Y = (Y, \rho)$ to denote asymmetric metric spaces. Given spaces X and Y, we write Y^X to denote the set of all functions from X to Y.

First, we introduce the concepts of forward and backward $\mathcal{I}(\alpha)$-convergence of sequences of functions from X to Y.

Definition 3.1. A sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is said to be backward $\mathcal{I}(\alpha)$-convergent to $f \in Y^X$ at $x \in X$ if for every sequence $\{x_k\}_{k\in\mathbb{N}}$ in X backward $\mathcal{I}$-converging to $x \in X$, the sequence $\{f_k(x_k)\}_{k\in\mathbb{N}}$ backward $\mathcal{I}$-converges to $f(x)$. A sequence $\{f_k\}_{k\in\mathbb{N}}$ is said to be backward $\mathcal{I}(\alpha)$-convergent to f if it is backward $\mathcal{I}(\alpha)$-convergent to f at every $x \in X$.

Similarly, a sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is said to be forward $\mathcal{I}(\alpha)$-convergent to $f \in Y^X$ at $x \in X$ if for every sequence $\{x_k\}_{k\in\mathbb{N}}$ in X forward $\mathcal{I}$-converging to $x \in X$, the sequence $\{f_k(x_k)\}_{k\in\mathbb{N}}$ forward $\mathcal{I}$-converges to $f(x)$. A sequence $\{f_k\}_{k\in\mathbb{N}}$ is said to be forward $\mathcal{I}(\alpha)$-convergent to f if it is forward $\mathcal{I}(\alpha)$-convergent to f at every $x \in X$.

Note 3.2. (a) When $\mathcal{I} = \mathcal{I}_{fin}$, the backward (resp. forward) $\mathcal{I}(\alpha)$-convergence is said to be backward (resp. forward) α-convergence of sequences of functions (see [18] when X and Y are metric spaces).
(b) When $\mathcal{I} = \mathcal{I}_d$, the backward (resp. forward) $\mathcal{I}(\alpha)$-convergence is said to be backward (resp. forward) statistically α-convergence of sequences of functions (see [9] when X and Y are metric spaces).
(c) If $\mathcal{I}$ is an admissible ideal, then the notion of backward (resp. forward) α-convergence implies the notion of backward (resp. forward) $\mathcal{I}(\alpha)$-convergence of sequences of functions.

Remark 3.3. The notion of backward $\mathcal{I}(\alpha)$-convergence and the notion of forward $\mathcal{I}(\alpha)$-convergence are not equivalent. To show this, we cite the following example.

Example 3.4. Consider the Sorgenfrey asymmetric metric space $(\mathbb{R}, d)$, where $\mathbb{R}$ is the set of all real numbers and $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is a function defined by

$$d(x, y) = \begin{cases} y - x & \text{if } y \geq x \\ 1 & \text{if } y < x. \end{cases}$$

Here $B^+(x,\delta) = [x, x+\delta)$ and $B^-(x,\delta) = (x-\delta, x]$ provided $\delta \leq 1$. Let $\mathcal{I}$ be any admissible ideal of $\mathbb{N}$. Now for each $k \in \mathbb{N}$, consider the function $f_k : \mathbb{R} \to \mathbb{R}$ defined by

$$f_k(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ k & \text{if } x < 0 \end{cases}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ for all $x \in \mathbb{R}$. Let $\{x_k\}_{k\in\mathbb{N}}$ forward $\mathcal{I}$-converge to 0. We show $A = \{k \in \mathbb{N} : x_k < 0\} \in \mathcal{I}$. If possible, let $A \notin \mathcal{I}$. Then $\{k \in \mathbb{N} : d(0, x_k) \geq \frac{1}{2}\}$ contains the set A and thus $\{k \in \mathbb{N} : d(0, x_k) \geq \frac{1}{2}\} \notin \mathcal{I}$, which contradicts the fact that $\{x_k\}_{k\in\mathbb{N}}$ forward $\mathcal{I}$-converges to 0. Hence $A = \{k \in \mathbb{N} : x_k < 0\} \in \mathcal{I}$. Now, $\{k \in \mathbb{N} : d(f(0), f_k(x_k)) \geq \varepsilon\} \subset A$ for every $\varepsilon > 0$. Since $A \in \mathcal{I}$, thus $\{k \in \mathbb{N} : d(f(0), f_k(x_k)) \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$. Thus $\{f_k(x_k)\}_{k\in\mathbb{N}}$ forward $\mathcal{I}$-converges to $f(0)$. Hence $\{f_k\}_{k\in\mathbb{N}}$ is forward $\mathcal{I}(\alpha)$-convergent to f at $x = 0$.

Now the sequence $\{-\frac{1}{k}\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-convergent to 0. But

$$\{k \in \mathbb{N} : d(f_k(-\frac{1}{k}), f(0)) \geq \frac{1}{2}\} = \mathbb{N} \notin \mathcal{I}.$$

Thus $\{f_k(-\frac{1}{k})\}_{k\in\mathbb{N}}$ is not backward $\mathcal{I}$-convergent to $f(0)$. Consequently, $\{f_k\}_{k\in\mathbb{N}}$ is not backward $\mathcal{I}(\alpha)$-convergent to f at $x = 0$.

Definition 3.5. A sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is said to be backward $\mathcal{I}$-exhaustive at $x \in X$ if, for each $\varepsilon > 0$ there are $\delta > 0$ and $A \in \mathcal{F}(\mathcal{I})$ such that for each $y \in B^-(x,\delta)$ we have $\rho(f_k(y), f_k(x)) < \varepsilon$ for each $k \in A$. A sequence $\{f_k\}_{k\in\mathbb{N}}$ is said to be backward $\mathcal{I}$-exhaustive if it is backward $\mathcal{I}$-exhaustive at every $x \in X$.

Similarly, a sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is said to be forward $\mathcal{I}$-exhaustive at $x \in X$ if, for each $\varepsilon > 0$ there are $\delta > 0$ and $A \in \mathcal{F}(\mathcal{I})$ such that for each $y \in B^+(x,\delta)$ we have $\rho(f_k(x), f_k(y)) < \varepsilon$ for each $k \in A$. A sequence $\{f_k\}_{k\in\mathbb{N}}$ is said to be forward $\mathcal{I}$-exhaustive if it is forward $\mathcal{I}$-exhaustive at every $x \in X$.

Note 3.6. (a) When $\mathcal{I} = \mathcal{I}_{fin}$, the above notion of backward (resp. forward) $\mathcal{I}$-exhaustiveness coincides with the notion of backward (resp. forward) exhaustiveness of sequences of functions introduced in [27].
(b) When $\mathcal{I} = \mathcal{I}_d$, the above notion of backward (resp. forward) $\mathcal{I}$-exhaustiveness coincides with the notion of backward (resp. forward) statistical exhaustiveness of sequences of functions introduced in [27].
(c) If $\mathcal{I}$ is an admissible ideal, then the notion of backward (resp. forward) exhaustiveness implies the notion of backward (resp. forward) $\mathcal{I}$-exhaustiveness of sequences of functions.

Definition 3.7. A sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is said to be backward (resp. forward) pointwise $\mathcal{I}$-convergent to $f \in Y^X$ if $f_k(x) \xrightarrow{\mathcal{I}^-} f(x)$ (resp. $f_k(x) \xrightarrow{\mathcal{I}^+} f(x)$) for each $x \in X$.

Note 3.8. (*a*) When $\mathcal{I} = \mathcal{I}_{fin}$, the above notion of backward (resp. forward) pointwise $\mathcal{I}$-convergence coincides with the notion of backward (resp. forward) pointwise convergence of sequences of functions introduced in [27].
(*b*) When $\mathcal{I} = \mathcal{I}_d$, the above notion of backward (resp. forward) pointwise $\mathcal{I}$-convergence coincides with the notion of backward (resp. forward) statistical pointwise convergence of sequences of functions introduced in [27].
(*c*) If $\mathcal{I}$ is an admissible ideal, then the notion of backward (resp. forward) pointwise convergence implies the notion of backward (resp. forward) pointwise $\mathcal{I}$-convergence of sequences of functions.

Compare the next theorem to [24, Proposition 2.3]. In this context, we need both the concepts of backward and forward $\mathcal{I}$-convergence to get the analogous result.

Theorem 3.9. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Let $x \in X$ and $\{f_k\}_{k\in\mathbb{N}}$ be backward pointwise $\mathcal{I}$-convergent to f at x. If $\{f_k\}_{k\in\mathbb{N}}$ is forward pointwise $\mathcal{I}$-convergent to f at every $z \in X \setminus \{x\}$ and $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-exhaustive at x, then f is f^--continuous at x.*

Proof. Let $\varepsilon > 0$ be given. Since $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-exhaustive at x, there are $\delta > 0$ and $A \in \mathcal{F}(\mathcal{I})$ such that for every $y \in B^-(x, \delta)$ we have $\rho(f_k(y), f_k(x)) < \frac{\varepsilon}{3}$ for all $k \in A$. Now let $y \in B^-(x, \delta) \setminus \{x\}$. Since $\{f_k\}_{k\in\mathbb{N}}$ is forward pointwise $\mathcal{I}$-convergent to f at every $z \in X \setminus \{x\}$, so $f_k(y) \xrightarrow{\mathcal{I}^+} f(y)$. Then there exists $A_y \in \mathcal{F}(\mathcal{I})$ such that $\rho(f(y), f_k(y)) < \frac{\varepsilon}{3}$ for all $k \in A_y$. Again, since $\{f_k\}_{k\in\mathbb{N}}$ is backward pointwise $\mathcal{I}$-convergent to f at x, there exists $A_x \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(x), f(x)) < \frac{\varepsilon}{3}$ for all $k \in A_x$. Now since $A \cap A_x \cap A_y \in \mathcal{F}(\mathcal{I})$, so $A \cap A_x \cap A_y \neq \emptyset$. Choose $j \in A \cap A_x \cap A_y$. Then, for $y \in B^-(x, \delta) \setminus \{x\}$, we have

$$\rho(f(y), f(x)) \leq \rho(f(y), f_j(y)) + \rho(f_j(y), f_j(x)) + \rho(f_j(x), f(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Also, $\rho(f(x), f(x)) < \varepsilon$. Hence f is f^--continuous at x. □

Compare the next two theorems respectively to [24, Theorem 2.5] and [24, Theorem 2.7], which state relations between backward $\mathcal{I}$-exhaustiveness and backward $\mathcal{I}(\alpha)$-convergence.

Theorem 3.10. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. If $\{f_k\}_{k\in\mathbb{N}}$ is backward pointwise $\mathcal{I}$-convergent to f at $x \in X$ and $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-exhaustive at x, then $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}(\alpha)$-convergent to f at x.*

Proof. Let $\varepsilon > 0$ be given. Then there are $\delta > 0$ and $A_1 \in \mathcal{F}(\mathcal{I})$ such that whenever $d(y, x) < \delta$ we have $\rho(f_k(y), f_k(x)) < \frac{\varepsilon}{2}$ for all $k \in A_1$. And $f_k(x) \xrightarrow{\mathcal{I}^-} f(x)$ implies the existence of a set $A_2 \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(x), f(x)) < \frac{\varepsilon}{2}$ for all $k \in A_2$. Let $x_k \xrightarrow{\mathcal{I}^-} x$. Then there exists $A_3 \in \mathcal{F}(\mathcal{I})$ such that $d(x_k, x) < \delta$ for all $k \in A_3$. Now $A_1 \cap A_2 \cap A_3 \in \mathcal{F}(\mathcal{I})$. Then, for $k \in A_1 \cap A_2 \cap A_3$, we have

$$\rho(f_k(x_k), f(x)) \leq \rho(f_k(x_k), f_k(x)) + \rho(f_k(x), f(x)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}(\alpha)$-convergent to f at x.

□

Theorem 3.11. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Assume that backward $\mathcal{I}$-convergence implies forward $\mathcal{I}$-convergence in Y. If the ideal $\mathcal{I}$ is good and $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}(\alpha)$-convergent to f at $x \in X$, then $\{f_k\}_{k\in\mathbb{N}}$ is backward pointwise $\mathcal{I}$-convergent to f at $x \in X$ and $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-exhaustive at x.*

Proof. Clearly, $\{f_k\}_{k\in\mathbb{N}}$ is backward pointwise $\mathcal{I}$-convergent to f at x. If possible, let $\{f_k\}_{k\in\mathbb{N}}$ be not backward $\mathcal{I}$-exhaustive at x. Then there exists $\varepsilon' > 0$ such that for all $\delta > 0$ and $A \in \mathcal{F}(\mathcal{I})$ there exist $z \in B^-(x, \delta)$ and $k \in A$ such that $\rho(f_k(z), f_k(x)) \geq \varepsilon'$. Consider $A = \mathbb{N}$ and $\delta = \frac{1}{k}$. Then there exists $n_k \in \mathbb{N}$ such that $\rho(f_{n_k}(x_k), f_{n_k}(x)) \geq \varepsilon'$ for some $x_k \in B^-(x, \frac{1}{k})$. We consider only one such x_k corresponding to each such n_k. Let A_k denote all such $n_k \in \mathbb{N}$ satisfying the above inequality and B_k denote the collection of corresponding unique x_k's. We claim that $A_k \notin \mathcal{I}$. Suppose $A_k \in \mathcal{I}$. Then $\mathbb{N} \setminus A_k \in \mathcal{F}(\mathcal{I})$. Thus there exists $n_0^k \in \mathbb{N} \setminus A_k$ such that $\rho(f_{n_0^k}(x_0^k), f_{n_0^k}(x)) \geq \varepsilon'$ for some $x_0^k \in B^-(x, \frac{1}{k})$, which contradicts the definition of A_k. Thus $A_k \notin \mathcal{I}$ for each $k \in \mathbb{N}$. Since $\mathcal{I}$ is *good*, there exists a countable sequence $\{P_k\}_{k\in\mathbb{N}}$ of pairwise disjoint sets such that $P_k \subset A_k$, $P_k \in \mathcal{I}$ for each $k \in \mathbb{N}$ and $\cup_{k=1}^{\infty} P_k \notin \mathcal{I}$.

Now let $P_k = \{p_1^k < p_2^k < ...\}$. Consider a sequence $\{z_n\}_{n\in\mathbb{N}}$ as follows: $z_n = x$ if $n \notin \cup_{k=1}^{\infty} P_k$ and $z_n = x_j^k$ if $n \in P_k$ and $n = p_j^k$, where $x_j^k \in B_k$ corresponds to the natural number $p_j^k \in A_k$.

Let $\varepsilon > 0$ be given. Then there exists a least $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \varepsilon$. Now

$$\{n \in \mathbb{N} : d(z_n, x) \geq \varepsilon\} \subset \cup_{k=1}^{k_0-1} P_k.$$

Since $\cup_{k=1}^{k_0-1} P_k \in \mathcal{I}$, thus $\{n \in \mathbb{N} : d(z_n, x_0) \geq \varepsilon\} \in \mathcal{I}$. Thus $\mathcal{I}^- - lim z_n = x$.

On the other hand, since $\{n \in \mathbb{N} : \rho(f_n(z_n), f_n(x)) \geq \varepsilon'\} = \cup_{k=1}^{\infty} P_k \notin \mathcal{I}$, therefore $\{f_n(z_n)\}$ does not backward $\mathcal{I}$-converge to $f(x)$, which is a contradiction. Hence $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-exhaustive at x. □

One can get the above results analogously for the other types.

4. Weak $\mathcal{I}$-exhaustiveness and Uniform $\mathcal{I}$-convergence

In [24, Definition 2.13], Papachristodoulos et al. introduced the notion of weak $\mathcal{I}$-exhaustiveness of sequences of functions from a metric space to a metric space. They proved the $\mathcal{I}$-pointwise limit function of a sequence of continuous functions is continuous if and only if the sequence of functions is weakly $\mathcal{I}$-exhaustive. We give a similar result in the realm of asymmetric metric spaces.

Definition 4.1. A sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is said to be backward weakly $\mathcal{I}$-exhaustive at $x \in X$ if, for each $\varepsilon > 0$, there is $\delta > 0$ such that for each $y \in B^-(x,\delta)$ there exists $A_y \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(y), f_k(x)) < \varepsilon$ for each $k \in A_y$. A sequence $\{f_k\}_{k\in\mathbb{N}}$ is said to be backward weakly $\mathcal{I}$-exhaustive if it is backward weakly $\mathcal{I}$-exhaustive at every $x \in X$.

Similarly, a sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is said to be forward weakly $\mathcal{I}$-exhaustive at $x \in X$ if, for each $\varepsilon > 0$, there is $\delta > 0$ such that for each $y \in B^+(x,\delta)$ there exists $A_y \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(x), f_k(y)) < \varepsilon$ for each $k \in A_y$. A sequence $\{f_k\}_{k\in\mathbb{N}}$ is said to be forward weakly $\mathcal{I}$-exhaustive if it is forward weakly $\mathcal{I}$-exhaustive at every $x \in X$.

Compare the following theorem to [24, Proposition 2.14], which states that the backward pointwise $\mathcal{I}$-limit function of a sequence of backward continuous functions is backward continuous if and only if the sequence of functions is backward weakly $\mathcal{I}$-exhaustive under some assumptions.

Theorem 4.2. *Let (X,d) and (Y,ρ) be asymmetric metric spaces. Assume that backward $\mathcal{I}$-convergence implies forward $\mathcal{I}$-convergence in Y and $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ be backward pointwise $\mathcal{I}$-convergent to $f \in Y^X$. Then $\{f_k\}_{k\in\mathbb{N}}$ is backward weakly $\mathcal{I}$-exhaustive at $x \in X$ if and only if f is f^--continuous at x.*

Proof. Let $\varepsilon > 0$ be given and f be f^--continuous at x. Then there exists $\delta > 0$ such that $\rho(f(y), f(x)) < \frac{\varepsilon}{3}$ whenever $y \in B^-(x,\delta)$. Let $y \in B^-(x,\delta)$. Since $\{f_k\}_{k\in\mathbb{N}}$ is backward pointwise $\mathcal{I}$-convergent to f, thus $f_k(x) \xrightarrow{\mathcal{I}^-} f(x)$ and $f_k(y) \xrightarrow{\mathcal{I}^-} f(y)$. Now $f_k(y) \xrightarrow{\mathcal{I}^-} f(y)$ implies the existence of a set $B_y \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(y), f(y)) < \frac{\varepsilon}{3}$, $\forall k \in B_y$. Again by the assumption $f_k(x) \xrightarrow{\mathcal{I}^-} f(x)$ implies $f_k(x) \xrightarrow{\mathcal{I}^+} f(x)$. And $f_k(x) \xrightarrow{\mathcal{I}^+} f(x)$ implies the existence of a set $A_x \in \mathcal{F}(\mathcal{I})$ such that $\rho(f(x), f_k(x)) < \frac{\varepsilon}{3}$, $\forall k \in A_x$. Now $A_x \cap B_y \in \mathcal{F}(\mathcal{I})$. Set $A_y = A_x \cap B_y$. Then, for all $k \in A_y$, we have

$$\rho(f_k(y), f_k(x)) \le \rho(f_k(y), f(y)) + \rho(f(y), f(x)) + \rho(f(x), f_k(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence $\{f_k\}_{k\in\mathbb{N}}$ is backward weakly $\mathcal{I}$-exhaustive at x.

Conversely, let $\{f_k\}_{k\in\mathbb{N}}$ be backward weakly $\mathcal{I}$-exhaustive at x. Let $\varepsilon > 0$ be given. Then there exists a $\delta > 0$ such that for each $y \in B^-(x,\delta)$ there exists $A_y \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(y), f_k(x)) < \frac{\varepsilon}{3}$ for each $k \in A_y$. Since $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is backward pointwise $\mathcal{I}$-convergent to f, thus $f_k(x) \xrightarrow{\mathcal{I}^-} f(x)$ and $f_k(y) \xrightarrow{\mathcal{I}^-} f(y)$. Now $f_k(x) \xrightarrow{\mathcal{I}^-} f(x)$ implies the existence of a set $B_x \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(x), f(x)) < \frac{\varepsilon}{3}$, $\forall k \in B_x$. Again by the assumption $f_k(y) \xrightarrow{\mathcal{I}^-} f(y)$ implies $f_k(y) \xrightarrow{\mathcal{I}^+} f(y)$. And $f_k(y) \xrightarrow{\mathcal{I}^+} f(y)$ implies the existence of a set $B_y \in \mathcal{F}(\mathcal{I})$ such that $\rho(f(y), f_k(y)) < \frac{\varepsilon}{3}$, $\forall k \in B_y$. Pick an arbitrary $z \in B^-(x,\delta)$. Then

$B_x \cap B_z \cap A_z \in \mathcal{F}(\mathcal{I})$. Choose $j \in B_x \cap B_z \cap A_z$. Thus, we have

$$\rho(f(z), f(x)) \leq \rho(f(z), f_j(z)) + \rho(f_j(z), f_j(x)) + \rho(f_j(x), f(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

$\square$

Similarly, we have the following result.

Corollary 4.3. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Assume that forward $\mathcal{I}$-convergence implies backward $\mathcal{I}$-convergence in Y and $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is forward pointwise $\mathcal{I}$-convergent to $f \in Y^X$. Then $\{f_k\}_{k\in\mathbb{N}}$ is forward weakly $\mathcal{I}$-exhaustive at $x \in X$ if and only if f is f^+-continuous at x.*

We show in the very next example that if backward $\mathcal{I}$-convergence does not imply forward $\mathcal{I}$-convergence in Y, then the above Theorem 4.2 may not hold.

Example 4.4. Consider the Sorgenfrey asymmetric metric space $(\mathbb{R}, d)$, where $\mathbb{R}$ is the set of all real numbers and $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is a function defined by

$$d(x, y) = \begin{cases} y - x & \text{if } y \geq x \\ 1 & \text{if } y < x. \end{cases}$$

Let $\mathcal{I}$ be any admissible ideal of $\mathbb{N}$. In this asymmetric metric space backward $\mathcal{I}$-convergence does not imply forward $\mathcal{I}$-convergence. Indeed, the sequence $\{x_k\}_{k\in\mathbb{N}}$ where $x_k = 2(1 - \frac{1}{k})$ is backward $\mathcal{I}$-convergent to 2, but $\{x_k\}_{k\in\mathbb{N}}$ is not forward $\mathcal{I}$-convergent to 2.

Now consider a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$, where $f_k : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_k(x) = \begin{cases} -\frac{1}{2k} & \text{if } x < 0 \\ -\frac{1}{k} & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Then $\{f_k\}_{k\in\mathbb{N}}$ is backward pointwise $\mathcal{I}$-convergent to

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Clearly, f is f^--continuous at $x = 0$, however, $\{f_k\}_{k\in\mathbb{N}}$ is not backward weakly $\mathcal{I}$-exhaustive at $x = 0$.

Remark 4.5. If a sequence of functions is backward (resp. forward) $\mathcal{I}$-exhaustive, then the sequence of functions is backward (resp. forward) weakly $\mathcal{I}$-exhaustive. In the very next example, we show that there is a sequence of functions which is forward weakly $\mathcal{I}$-exhaustive without being forward $\mathcal{I}$-exhaustive.

Example 4.6. Let $X = \mathbb{R}$ with the asymmetric metric

$$d(x,y) = \begin{cases} y - x & \text{if } \ y \geq x \\ 1 & \text{if } \ y < x. \end{cases}$$

Also let $Y = [-1, 1]$ with the asymmetric metric

$$\rho(x,y) = \begin{cases} y - x & \text{if } \ y \geq x \\ 2(y - x) & \text{if } \ y < x. \end{cases}$$

Let $\mathcal{I}$ be any admissible ideal of $\mathbb{N}$. In Y, forward $\mathcal{I}$-convergence implies backward $\mathcal{I}$-convergence (see [11, Theorem 3.2]).

Let $A \in \mathcal{I}$. Now consider a sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ in Y^X defined as follows:

If $k \in A$, then $f_k(x) = 0$ for all $x \in X$.
If $k \notin A$, then

$$f_k(x) = \begin{cases} 0 & \text{if } \ x \in (-\infty, -\frac{1}{k}) \cup \{0\} \cup (\frac{1}{k}, \infty) \\ kx + 1 & \text{if } \ x \in [-\frac{1}{k}, 0) \\ -kx + 1 & \text{if } \ x \in (0, \frac{1}{k}]. \end{cases}$$

Clearly, the sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ forward pointwise $\mathcal{I}$-convergent to the zero function in Y^X. Then by Corollary 4.3, the sequence of functions $\{f_k\}_{k\in\mathbb{N}}$ is forward weakly $\mathcal{I}$-exhaustive.

Now we show that $\{f_k\}_{k\in\mathbb{N}}$ is not forward $\mathcal{I}$-exhaustive at $x = 0$. Let $M \in \mathcal{F}(\mathcal{I})$ and $\delta > 0$. Choose $j \in M \cap (\mathbb{N} \setminus A)$. Let $y \in B^+(0, \delta)$ in X be such that $y < \frac{1}{2j}$. Then $-jy + 1 > \frac{1}{2}$, that is, $\rho(f_j(0), f_j(y)) > \frac{1}{2}$. Therefore, $\{f_k\}_{k\in\mathbb{N}}$ is not forward $\mathcal{I}$-exhaustive at 0.

We now give a generalization of [27, Theorem 2.9]. To do this, we need to introduce the following definition.

Definition 4.7. A sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is said to be backward (resp. forward) $\mathcal{I}$-convergent uniformly to $f \in Y^X$ on X if, for each $\varepsilon > 0$ there exists $A \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(x), f(x)) < \varepsilon$ (resp. $\rho(f(x), f_k(x)) < \varepsilon$) for each $k \in A$ and $x \in X$.

Lemma 4.8. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Assume that backward $\mathcal{I}$-convergence implies forward $\mathcal{I}$-convergence in Y. Let $x \in X$. If for every $\varepsilon > 0$ there exists $\delta > 0$ be such that for all $y \in B^-(x, \delta)$ we have $\rho(f(y), f(x)) < \varepsilon$, then $\rho(f(x), f(y)) < \varepsilon$ for all $y \in B^-(x, \delta)$.*

Proof. Suppose not, then there exists $\varepsilon' > 0$ such that for all $\delta > 0$ there exists $y_0 \in B^-(x, \delta)$ such that $\rho(f(y_0), f(x)) < \varepsilon'$ but $\rho(f(x), f(y_0)) \geq \varepsilon'$. Therefore, for each $n \in \mathbb{N}$ there exists $x_n \in B^-(x, 1/n)$ such that $\rho(f(x_n), f(x)) < \varepsilon'$ but $\rho(f(x), f(x_n)) \geq \varepsilon'$. Clearly, $\{x_n\}_{n\in\mathbb{N}}$ backward converges to x. By the given hypothesis f is backward continuous function at x. Therefore, $\{f(x_n)\}_{n\in\mathbb{N}}$ is backward convergent to $f(x)$ in Y. Since $\mathcal{I}$ is an admissible ideal, $\{f(x_n)\}_{n\in\mathbb{N}}$ is backward $\mathcal{I}$-convergent to $f(x)$ in Y. Thus by the assumption $\{f(x_n)\}_{n\in\mathbb{N}}$ is forward $\mathcal{I}$-convergent to $f(x)$ in Y. But this is impossible because for each $n \in \mathbb{N}$, we have $\rho(f(x), f(x_n)) \geq \varepsilon'$. Hence $\rho(f(x), f(y)) < \varepsilon$ for all $y \in B^-(x, \delta)$. $\square$

From the next example, we can conclude that if backward $\mathcal{I}$-convergence does not imply forward $\mathcal{I}$-convergence in Y, then the above lemma may not hold.

Example 4.9. Let $X = \mathbb{R}$ with the asymmetric metric

$$d(x,y) = \begin{cases} y - x & \text{if } y \geq x \\ 1 & \text{if } y < x. \end{cases}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$. Let $0 < \varepsilon < 1$ be given. Now at $x = 1$, choose $0 < \delta < \varepsilon$. Then for each $y \in B^-(x, \delta)$, we have $d(f(y), f(x)) = d(y, x) < \delta < \varepsilon$. But $d(f(x), f(y)) = d(x, y) = 1 > \varepsilon$.

Theorem 4.10. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Assume that backward $\mathcal{I}$-convergence implies forward $\mathcal{I}$-convergence in Y. If $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is backward pointwise $\mathcal{I}$-convergent to $f \in Y^X$ and $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is backward $\mathcal{I}$-exhaustive on X, then f is f^--continuous on X and $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is backward $\mathcal{I}$-convergent uniformly to f on every backward compact subset of X.*

Proof. First, we show that f is f^- continuous on X. Let $\varepsilon > 0$ be given and $x \in X$. Then $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is backward $\mathcal{I}$-exhaustive at x, which implies the existence of $\delta > 0$ and $A \in \mathcal{F}(\mathcal{I})$ such that for every $y \in B^-(x, \delta)$ we have $\rho(f_k(y), f_k(x)) < \frac{\varepsilon}{3}$ for each $k \in A$. Let $y \in B^-(x, \delta)$. Since $\{f_k\}_{k\in\mathbb{N}}$ is backward pointwise $\mathcal{I}$-convergent to f, thus $f_k(x) \overset{\mathcal{I}^-}{\to} f(x)$ and $f_k(y) \overset{\mathcal{I}^-}{\to} f(y)$. Now $f_k(x) \overset{\mathcal{I}^-}{\to} f(x)$ implies the existence of a set $A_x \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(x), f(x)) < \frac{\varepsilon}{3}$ for all $k \in A_x$. Again by assumption $f_k(y) \overset{\mathcal{I}^-}{\to} f(y)$ implies $f_k(y) \overset{\mathcal{I}^+}{\to} f(y)$. Thus there exists $A_y \in \mathcal{F}(\mathcal{I})$ such that $\rho(f(y), f_k(y)) < \frac{\varepsilon}{3}$ for all $k \in A_y$. Now since $A \cap A_x \cap A_y \in \mathcal{F}(\mathcal{I})$, therefore $A \cap A_x \cap A_y \neq \emptyset$. Choose $j \in A \cap A_x \cap A_y$. Then, for $y \in B^-(x, \delta)$, we have

$$\rho(f(y), f(x)) \leq \rho(f(y), f_j(y)) + \rho(f_j(y), f_j(x)) + \rho(f_j(x), f(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence f is f^--continuous at x.

Let K be a backward compact subset of X. Let $\varepsilon > 0$ be given and $x \in K$. Then f is f^--continuous at x. Therefore, there exists $\delta > 0$ such that for $y \in B^-(x, \delta)$, we have $\rho(f(y), f(x)) < \frac{\varepsilon}{3}$. Since backward $\mathcal{I}$-convergence implies forward $\mathcal{I}$-convergence in Y, by Lemma 4.8, we have $\rho(f(x), f(y)) < \frac{\varepsilon}{3}$.

Now $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-exhaustive at x. Therefore, there are $\delta_x(<\delta)>0$ and $A_x\in\mathcal{F}(\mathcal{I})$ such that for every $y\in B^-(x,\delta_x)$ we have $\rho(f_k(y),f_k(x))<\frac{\varepsilon}{3}$ for each $k\in A_x$.

Since $K\subset\bigcup_{x\in K}B^-(x,\delta_x)$ and K is backward compact, there exist some elements $x_1,x_2,..,x_t$ in K such that $K\subset\bigcup_{i=1}^{t}B^-(x_i,\delta_{x_i})$. Now by the backward pointwise $\mathcal{I}$-convergence of $\{f_k\}_{k\in\mathbb{N}}$ to f, for each i there are $A_i\in\mathcal{F}(\mathcal{I})$ such that $\rho(f_k(x_i),f(x_i))<\frac{\varepsilon}{3}$ for each $k\in A_i$.

Consider B=$\bigcap_{i=1}^{t}(A_i\cap A_{x_i})$. Then $B\in\mathcal{F}(\mathcal{I})$. Let $z\in K$. Then there exist some $i\in\{1,2,...,t\}$ such that $z\in B^-(x_i,\delta_{x_i})$. Therefore $d(z,x_i)<\delta_{x_i}<\delta$. Thus $\rho(f(x_i),f(z))<\frac{\varepsilon}{3}$ and $\rho(f_k(z),f_k(x_i))<\frac{\varepsilon}{3}$ for all $k\in B$. Therefore for all $k\in B$, we have

$$\rho(f_k(z),f(z))\leq\rho(f_k(z),f_k(x_i))+\rho(f_k(x_i),f(x_i))+\rho(f(x_i),f(z))<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

Hence $\{f_k\}_{k\in\mathbb{N}}\subset Y^X$ is backward $\mathcal{I}$-convergent uniformly to f on K. □

All the above results hold similarly for the other types.

5. $\mathcal{I}$-strong uniform convergence in asymmetric metric spaces

In [3], the notion of $\mathcal{I}$-strong uniform convergence of a sequence of functions was introduced, and it turns out to be equivalent to the continuity of the pointwise $\mathcal{I}$-limit function of such a sequence of continuous functions [3, Proposition 2.10]. In this section, we introduce the notions of backward and forward $\mathcal{I}$-strong uniform convergence of sequences of functions, and we show that under some considerations the notion of backward (resp. forward) $\mathcal{I}$-strong uniform convergence is equivalent to the notion of backward (resp. forward) continuity of the backward (resp. forward) pointwise $\mathcal{I}$-limit function of such a sequence.

Let (X,d) be an asymmetric metric space. For $\delta>0$, the backward (resp. forward) δ-enlargement of $A\subset X$ is $A^{\delta^-}:=\bigcup_{x\in A}B^-(x,\delta)$ (resp. $A^{\delta^+}:=\bigcup_{x\in A}B^+(x,\delta)$).

Definition 5.1. Let $\mathcal{N}$ be the family of all finite subsets of an asymmetric metric space (X,d). Let f_k,f be functions from X to an asymmetric metric space (Y,ρ). Then $\{f_k\}_{k\in\mathbb{N}}$ is said to be backward (resp. forward) $\mathcal{I}$-strongly uniformly convergent to f on X if for every $\varepsilon>0$ and $B\in\mathcal{N}$ there are $\delta>0$ and $A\in\mathcal{F}(\mathcal{I})$ such that for each $z\in B^{\delta^-}$ (resp. $z\in B^{\delta^+}$) we have $\rho(f_k(z),f(z))<\varepsilon$ (resp. $\rho(f(z),f_k(z))<\varepsilon$) for all $k\in A$.

Theorem 5.2. *Let (X,d) and (Y,ρ) be asymmetric metric spaces. Assume that backward $\mathcal{I}$-convergence implies forward $\mathcal{I}$-convergence in Y. If $\{f_k\}_{k\in\mathbb{N}}\subset Y^X$ is backward pointwise $\mathcal{I}$-convergent to $f\in Y^X$ and $\{f_k\}_{k\in\mathbb{N}}$ is backward weakly $\mathcal{I}$-exhaustive on X, then $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-strongly uniformly convergent to f on X.*

Proof. Let $\varepsilon > 0$ be given and $B = \{x_1, x_2, ..., x_t\} \in \mathcal{N}$. By the given hypothesis $\{f_k\}_{k\in\mathbb{N}}$ is backward weakly $\mathcal{I}$-exhaustive at every x_j, $1 \le j \le t$. Therefore for x_j there is δ_j such that for every $z \in B^-(x_j, \delta_j)$ there is a set $M_z \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(z), f_k(x_j)) < \frac{\varepsilon}{3}$ for all $k \in M_z$. Again for every x_j, the sequence $\{f_k(x_j)\}_{k\in\mathbb{N}}$ in Y is backward $\mathcal{I}$-convergent to $f(x_j)$. Therefore there exists a set $M_j \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(x_j), f(x_j)) < \frac{\varepsilon}{3}$ for all $k \in M_j$. Also, by Theorem 4.2 f is f^- continuous at every x_j, so for every x_j there is $\eta_j > 0$ such that for each $y \in B^-(x_j, \eta_j)$, we have $\rho(f(y), f(x_j)) < \frac{\varepsilon}{3}$ and by Lemma 4.8 $\rho(f(x_j), f(y)) < \frac{\varepsilon}{3}$. Take

$$\delta = min\{\delta_1, \delta_2, ..., \delta_t, \eta_1, \eta_2, ..., \eta_t\}$$

and let $x \in B^{\delta^-}$. Then $x \in B^-(x_i, \delta)$ for some $1 \le i \le t$. Put $M = M_x \cap (\bigcap_{j=1}^{t} M_j)$. Then $M \in \mathcal{F}(\mathcal{I})$. Thus for all $k \in M$, we have

$$\rho(f_k(x), f(x)) \le \rho(f_k(x), f_k(x_i)) + \rho(f_k(x_i), f(x_i)) + \rho(f(x_i), f(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Hence $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-strongly uniformly convergent to f on X. □

Corollary 5.3. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Assume that backward $\mathcal{I}$-convergence implies forward $\mathcal{I}$-convergence in Y. If $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is backward pointwise $\mathcal{I}$-convergent to $f \in Y^X$ and f is backward continuous function, then $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-strongly uniformly convergent to f on X.*

Proof. The corollary holds because of Theorem 4.2 and Theorem 5.2. □

Similarly, the following result holds for the other types.

Corollary 5.4. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Assume that forward $\mathcal{I}$-convergence implies backward $\mathcal{I}$-convergence in Y. If $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is forward pointwise $\mathcal{I}$-convergent to $f \in Y^X$ and $\{f_k\}_{k\in\mathbb{N}}$ is forward weakly $\mathcal{I}$-exhaustive on X, then $\{f_k\}_{k\in\mathbb{N}}$ is forward $\mathcal{I}$-strongly uniformly convergent to f on X.*

Theorem 5.5. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Assume that (Y, ρ) satisfies the property (AMA) and the corresponding function C is bounded. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of backward continuous functions from X to Y. If $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-strongly uniformly convergent to f on X, then f is backward continuous on X.*

Proof. Let $x \in X$ and $\varepsilon > 0$ be given. We show that f is f^--continuous at x. Since C is bounded, we can choose a positive real number r such that $C(z) < r$, $\forall z \in X$. Now since $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-strongly uniformly convergent to f, there are $\delta_x > 0$ and $A \in \mathcal{F}(\mathcal{I})$ such that for every $z \in \{x\}^{\delta_x^-} = B^-(x, \delta_x)$, we have $\rho(f_k(z), f(z)) < \frac{\varepsilon}{3(r+1)}$ for all $k \in A$. Again every f_k, $k \in \mathbb{N}$, is backward continuous. Thus for every $k \in A$ there exists $\delta_k > 0$ such that for every

$z \in B^-(x, \delta_k)$, we have $\rho(f_k(z), f_k(x)) < \frac{\varepsilon}{3}$. Let $n \in A$. Set $\delta = min\{\delta_x, \delta_n\}$. Since (Y, ρ) satisfies the property (AMA), therefore for $y \in B^-(x, \delta)$, we have

$$\begin{aligned}
\rho(f(y), f(x)) &\leq \rho(f(y), f_n(y)) + \rho(f_n(y), f_n(x)) + \rho(f_n(x), f(x)) \\
&< c(f_n(y), f(y))\rho(f_n(y), f(y)) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3(r+1)} \\
&< C(f_n(y))\rho(f_n(y), f(y)) + \frac{2\varepsilon}{3} \\
&< r\rho(f_n(y), f(y)) + \frac{2\varepsilon}{3} \\
&< r\frac{\varepsilon}{3(r+1)} + \frac{2\varepsilon}{3} \\
&< \varepsilon.
\end{aligned}$$

Hence f is backward continuous at x.

□

Corollary 5.6. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Assume that (Y, ρ) satisfies the property (AMA) and the corresponding function C is bounded. Let $\{f_k\}_{k \in \mathbb{N}}$ be a sequence of backward continuous functions from X to Y. If $\{f_k\}_{k \in \mathbb{N}}$ is backward $\mathcal{I}$-strongly uniformly convergent to f on X, then $\{f_k\}_{k \in \mathbb{N}}$ is backward pointwise $\mathcal{I}$-convergent to f and $\{f_k\}_{k \in \mathbb{N}}$ is backward weakly $\mathcal{I}$-exhaustive.*

Proof. The Corollary holds because of Theorem 4.2 and Theorem 5.5.

□

One can get the above results analogously for the other types.

6. $\mathcal{I}$-Alexandroff convergence in asymmetric metric spaces

In [3], Athanassiadou et al. introduced another notion of convergence, the notion of $\mathcal{I}$-Alexandroff convergence, and studied it with other types of convergence. In this section, we introduce the notions of backward and forward $\mathcal{I}$-Alexandroff convergence. And we show backward $\mathcal{I}$-Alexandroff preservers backward continuity of the limit function. In this continuation backward $\mathcal{I}$-Alexandroff turns out to be equivalent to backward $\mathcal{I}$-strong uniform convergence under some considerations.

Definition 6.1. Let (X, d) and (Y, ρ) be asymmetric metric spaces. A sequence $\{f_k\}_{k \in \mathbb{N}} \subset Y^X$ of backward (resp. forward) continuous functions is said to be backward (resp. forward) $\mathcal{I}$-Alexandroff convergent to $f \in Y^X$ if $\{f_k\}_{k \in \mathbb{N}}$ is backward (resp. forward) pointwise $\mathcal{I}$-convergent to f and for every $\varepsilon > 0$ and any $A \in \mathcal{F}(\mathcal{I})$ there exist an infinite set $M_A = \{m_1 < m_2 < ...\} \subset A$ and an open cover $\mathcal{U} = \{U_k : k \in A\}$ in the backward (resp. forward) topology of X such that for every $x \in U_k$ we have $\rho(f_{m_k}(x), f(x)) < \varepsilon$ (resp. $\rho(f(x), f_{m_k}(x)) < \varepsilon$).

Theorem 6.2. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Assume that (Y, ρ) satisfies the property (AMA) and the corresponding function C is bounded. If $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ is backward $\mathcal{I}$-Alexandroff convergent to $f \in Y^X$, then f is backward continuous.*

Proof. Let $\{f_k\}_{k\in\mathbb{N}} \subset Y^X$ be backward $\mathcal{I}$-Alexandroff convergent to $f \in Y^X$. Let $x \in X$ and $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in X backward converging to x. It is sufficient to prove that $\{f(x_j)\}_{j\in\mathbb{N}}$ backward converges to $f(x)$. Let $\varepsilon > 0$ be given. Since $f_k(x) \xrightarrow{\mathcal{I}^-} f(x)$, there exists $A_x \in \mathcal{F}(\mathcal{I})$ such that $\rho(f_k(x), f(x)) < \frac{\varepsilon}{3}$ for all $k \in A_x$. Again since $C(z)$ is bounded, we can choose a positive real number r such that $C(z) < r$, $\forall z \in X$. Now $A_x \in \mathcal{F}(\mathcal{I})$. Therefore there are an infinite set $M_{A_x} = \{m_1 < m_2 < ...\} \subset A_x$ and an open cover $\mathcal{U} = \{U_k : k \in A_x\}$ in the backward topology of X such that $y \in U_k$, $\rho(f_{m_k}(y), f(y)) < \frac{\varepsilon}{3(r+1)}$. Since $\mathcal{U} = \{U_k : k \in A_x\}$ is an open cover, we can choose $k \in \mathbb{N}$ such that $x \in U_k$. Now since f_{m_k} is backward continuous at x and $\{x_j\}_{j\in\mathbb{N}}$ is backward convergent x, thus there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$, $x_j \in U_k$ we have $\rho(f_{m_k}(x_j), f_{m_k}(x)) < \frac{\varepsilon}{3}$. Since (Y, ρ) satisfies the property (AMA), thus for all $j \geq j_0$, we have

$$\begin{aligned}
\rho(f(x_j), f(x)) &\leq \rho(f(x_j), f_{m_k}(x_j)) + \rho(f_{m_k}(x_j), f_{m_k}(x)) + \rho(f_{m_k}(x), f(x)) \\
&< c(f_{m_k}(x_j), f(x_j))\rho(f_{m_k}(x_j), f(x_j)) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\leq C(f_{m_k}(x_j))\rho(f_{m_k}(x_j), f(x_j)) + \frac{2\varepsilon}{3} \\
&< r\rho(f_{m_k}(x_j), f(x_j)) + \frac{2\varepsilon}{3} \\
&< r\frac{\varepsilon}{3(r+1)} + \frac{2\varepsilon}{3} \\
&< \varepsilon.
\end{aligned}$$

Hence $\{f(x_j)\}_{j\in\mathbb{N}}$ is backward convergent to $f(x)$. □

Theorem 6.3. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. If $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-strongly uniformly convergent to f on X, then $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-Alexandroff convergent to f on X.*

Proof. Let $\varepsilon > 0$ and $B \in \mathcal{F}(\mathcal{I})$ be given. Since $\{f_k\}_{k\in\mathbb{N}}$ is backward $\mathcal{I}$-strongly uniformly convergent to f, thus there are $\delta_x > 0$ and $A_x \in \mathcal{F}(\mathcal{I})$ such that for each $y \in B^-(x, \delta_x)$ and each $k \in A_x$ we have $\rho(f_k(y), f(y)) < \varepsilon$. Set $A = \bigcup_{x\in X} A_x$. Now for each $k \in B \cap A$, we define

$$M_k = \{x \in X : \rho(f_m(y), f(y)) < \varepsilon \forall m \in A_x \cap B, m \geq k, \forall y \in B^-(x, \delta_x)\}.$$

Clearly, $X = \bigcup_{k\in B\cap A} M_k$. For each $k \in B$, we define the open set U_k in the backward topology of X as follows:

$$U_k = \begin{cases} \emptyset & \text{if } k \in B \setminus A \\ \bigcup_{x\in M_k} B^-(x, \delta_x) & \text{if } k \in B \cap A. \end{cases}$$

Then $\{U_k : k \in B\}$ is an open cover of X in the backward topology of X. Let $n \in B$ and $y \in U_n$. Then $n \in A$ and $y \in B^-(x, \delta_x)$ for some $x \in M_n$. Thus for each $y \in U_n$, we have $\rho(f_m(y), f(y)) < \varepsilon$ for some $m \in A \cap B$ and $m \geq n$. Hence for given $\varepsilon > 0$ and $B \in \mathcal{F}(\mathcal{I})$, there are an infinite set $A \cap B = \{m_1 < m_2 < ...\} \subset B$ and an open cover $\{U_n : n \in B\}$ in the backward topology of X such that for every $y \in U_n$ we have $\rho(f_{m_n}(y), f(y)) < \varepsilon$. This completes the proof.

□

Corollary 6.4. *Let (X, d) and (Y, ρ) be asymmetric metric spaces. Assume that backward $\mathcal{I}$-convergence implies forward $\mathcal{I}$-convergence in Y. If the sequence of backward continuous functions $\{f_k\}_{k \in \mathbb{N}} \subset Y^X$ is backward pointwise $\mathcal{I}$-convergent to $f \in Y^X$ and f is backward continuous function, then $\{f_k\}_{k \in \mathbb{N}}$ is backward $\mathcal{I}$-Alexandroff convergent to f on X.*

Proof. The corollary holds because of Corollary 5.3 and Theorem 6.3. □

One can get the above results analogously for the other types.

Open problem: It is not clear whether Theorem 5.5 and Theorem 6.2 can be proved without the condition (AMA) and boundedness of the corresponding function in the target space. Thus it is reasonable to ask whether these theorems can be proved under weaker condition or without any condition in the target space.

Declarations

No conflict of interest and no funding source.

Acknowledgment

The author is grateful to the anonymous referee for his/her observations, which aid improvement of some notations and phrases of this study.

References

[1] Alexandroff, P. S. *Einführung in die Mengenlehre und die Theorie der reellen Funktionen.* VEB deutscher Verlag der Wissenschaften, Berlin, 1956.

[2] Arzelà, C. Intorno alla continuitá della somma di infinitá di funzioni continue. *Rend. R. Accad. Sci. Istit. Bologna* (1883/1884), 79–84.

[3] Athanassiadou, E., Boccuto, A., Dimitriou, X., and Papanastassiou, N. Ascoli-type theorems and ideal (α)-convergence. *Filomat 26*, 2 (2012), 397–405.

[4] Balcerzak, M., Dems, K., and Komisarski, A. Statistical convergence and ideal convergence for sequences of functions. *J. Math. Anal. Appl. 328*, 1 (2007), 715–729.

[5] Beer, G., and Levi, S. Strong uniform continuity. *J. Math. Anal. Appl. 350*, 2 (2009), 568–589.

[6] Carathéodory, C. Stetige Konvergenz und normale Familien von Funktionen. *Math. Ann. 101*, 1 (1929), 515–533.

[7] Caserta, A., Di Maio, G., and Holá, L. Arzelà's theorem and strong uniform convergence on bornologies. *J. Math. Anal. Appl. 371*, 1 (2010), 384–392.

[8] Caserta, A., Di Maio, G., and Kočinac, L. D. R. Statistical convergence in function spaces. *Abstr. Appl. Anal.* (2011), Art. ID 420419, 11.

[9] Caserta, A., and Kočinac, L. D. R. On statistical exhaustiveness. *Appl. Math. Lett. 25*, 10 (2012), 1447–1451.

[10] Collins, J., and Zimmer, J. An asymmetric Arzelà-Ascoli theorem. *Topology Appl. 154*, 11 (2007), 2312–2322.

[11] Das, P., Ghosal, S., and Pal, S. K. Extending asymmetric convergence and Cauchy condition using ideals. *Math. Slovaca 63*, 3 (2013), 545–562.

[12] Das, P., and Ghosal, S. K. On $\mathcal{I}$-Cauchy nets and completeness. *Topology Appl. 157*, 7 (2010), 1152–1156.

[13] Das, P., and Ghosal, S. K. Some further results on $\mathcal{I}$-Cauchy sequences and condition (AP). *Comput. Math. Appl. 59*, 8 (2010), 2597–2600.

[14] Di Maio, G., and Kočinac, L. D. R. Statistical convergence in topology. *Topology Appl. 156*, 1 (2008), 28–45.

[15] Dutta, H., and Rhoades, B. E., Eds. *Current topics in summability theory and applications.* Springer, [Singapore], 2016.

[16] Fast, H. Sur la convergence statistique. *Colloq. Math. 2* (1951), 241–244 (1952).

[17] Fridy, J. A. On statistical convergence. *Analysis 5*, 4 (1985), 301–313.

[18] Gregoriades, V., and Papanastassiou, N. The notion of exhaustiveness and Ascoli-type theorems. *Topology Appl. 155*, 10 (2008), 1111–1128.

[19] Hahn, H. Theorie der reellen funktionen. *J. Springer, Berlin* (1921).

[20] Kostyrko, P., Šalát, T., and Wilczyński, W. $\mathcal{I}$-convergence. *Real Anal. Exchange 26*, 2 (2000/01), 669–685.

[21] Lahiri, B. K., and Das, P. $\mathcal{I}$ and $\mathcal{I}^*$-convergence of nets. *Real Anal. Exchange 33*, 2 (2008), 431–442.

[22] Mainik, A., and Mielke, A. Existence results for energetic models for rate-independent systems. *Calc. Var. Partial Differential Equations 22*, 1 (2005), 73–99.

[23] Mielke, A., and Roubíček, T. A rate-independent model for inelastic behavior of shape-memory alloys. *Multiscale Model. Simul. 1*, 4 (2003), 571–597.

[24] Papachristodoulos, C., Papanastassiou, N., and Wilczynski, W. $\mathcal{I}$-exhaustive sequences of functions. *Selected papers of the ICTA* (2010).

[25] Steinhaus, H. Sur la convergence ordinaire et la convergence asymptotique. *Colloq. Math. 2*, 1 (1951), 73–74.

[26] Stoilov, S. Continuous convergence. *Rev. Math. Pures Appl. 4* (1959), 341–344.

[27] Toyganözü, Z. H., and Pehlivan, S. Some results on exhaustiveness in asymmetric metric spaces. *Filomat 29*, 1 (2015), 183–192.

[28] Šalát, T. On statistically convergent sequences of real numbers. *Math. Slovaca 30*, 2 (1980), 139–150.

[29] Wilson, W. A. On Quasi-Metric Spaces. *Amer. J. Math. 53*, 3 (1931), 675–684.

Received by the editors February 14, 2021
First published online June 17, 2021