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Young measure theory for steady problems in Orlicz-Sobolev spaces

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Abstract. In this paper, we study the existence of weak solutions for Dirichlet boundary-value problems given in the following quasilinear elliptic system

$$\begin{cases} -\operatorname{div}\sigma(x,u,Du)+b(x,u,Du) &= f(x,u,Du) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

We prove the needed result, relying on the theory of Young measures, Galerkin's approximation and weak monotonicity assumptions on σ, in reflexive Orlicz-Sobolev spaces.

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1. Introduction

Let Ω be a bounded open set of $\mathbb{R}^n$ with $n \geq 2$. In this paper we are interested in establishing an existence result for the following elliptic problem:

$$\begin{cases} -\operatorname{div}\sigma(x,u,Du)+b(x,u,Du) &= f(x,u,Du) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $u : \Omega \to \mathbb{R}^m$ ($m \in \mathbb{N}^*$) is a vector-valued function and Du its gradient and belongs to $\mathbb{M}^{m\times n}$ which stands for the real vector space of $m \times n$ matrices equipped with the inner product $A : B = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij}$. The functions $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$, $b : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ and $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ will be assumed to satisfy some conditions.

Consider first b independent of its third variable and $b(x, s) = 0$ ($s \in \mathbb{R}^m$) and the framework of Sobolev spaces. In [36], Zhang Ke-Wei proved the existence of solutions by introducing the notions of "quasimonotone" mappings and "semiconvex" functions. Pucci and Servadei [33] established several regularity results for weak solutions by using the Moser iteration scheme and the translation method due to Nirenberg. See also [32] for related topic. The

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existence of positive solutions was studied in [25] relying on the method of sub-supersolution, nonlinear regularity theory and strong maximum principle.

In the setting of a Sobolev space with weight, Azroul et al. [2] studied the corresponding quasilinear elliptic problem and proved the existence of weak solutions. When the exponent p which defines the growth and coercivity conditions is dependent on x, i.e. $p = p(x)$, the existence of solutions has been proved in [16] in Sobolev spaces with variable exponents (always $b \equiv 0$).

In the same case and in Orlicz spaces, Youngqiang et al. [34] proved the existence of weak solutions for the concerned elliptic partial differential systems. An existence theorem for weak solutions in general Orlicz-Sobolev spaces has been proved by Dong in [20]. When the function f is independent of u and Du, we have proved in [3] the existence of weak solutions to the system $-\text{div}\sigma(x, u, Du) = f$, by using the theory of Young measures and weak monotonicity assumptions on σ. By the same theory and where f depends on u and Du, the result of existence was established in [5]. For more results where the theory of Young measures has been applied, we refer the reader to [11, 4, 12, 14, 27] for an elliptic case and [7, 8, 13, 9] for evolutionary problems.

Now, consider the case where $b(x, s) \neq 0$. Dong and Fang [21] studied the existence of weak solutions for (1.1) in the case of differential equations, $\sigma(x, s, \xi) = a_1(x, \xi)$ and in Musielak-Orlicz-Sobolev spaces, with b independent of its third variable. When f is independent of s and ξ, Benkirane and Elmahi [17] established the existence result under the condition that the N-function M, which defines the functional space, satisfies the Δ_2-condition near infinity. Without this condition, Aharouch et al. [1] proved existence result for the associated unilateral problem. See also [22, 23, 26, 6] for related topics.

Our purpose, in this study, is to prove the existence result for (1.1) in the setting of the Orlicz-Sobolev spaces $W_0^1 L_M(\Omega; \mathbb{R}^m)$, where M is an N-function that satisfies the Δ_2-condition near infinity (see the next section). Asuming the lower order term $b(x, s, \xi)$ to satisfy the sign condition $b(x, s, \xi) \cdot s \geq 0$, we extended our previous results [5, 3, 10] by using again the theory of Young measures to achieve the needed result.

Finally, this work is organized as follows: In Section 2, we recall some well-known preliminaries, properties of Orlicz-Sobolev spaces and Young measures. Section 3 is devoted to specify the assumptions on $\sigma(.)$, $b(.)$ and $f(.)$. In Section 4, we state the existence theorem and its proof.

2. Preliminaries

In this section, we start by recalling some definitions and properties about Orlicz-Sobolev spaces (see e.g. [19, 29] and references therein).

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, i.e. M is continuous, convex, with $M(t) > 0$ for $t > 0$, $M(t)/t \to 0$ as $t \to 0$ and $M(t)/t \to \infty$ as $t \to \infty$. Equivalently, M admits the representation

$$M(t) = \int_0^t a(s)ds,$$

where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to \infty$ as $t \to \infty$. The conjugate to M is defined by

$$\overline{M}(t) = \int_0^t \overline{a}(s)ds$$

and is an N-function, where $\overline{a}(t) = \sup_{a(s)\leq t} s$. The N-function M is said to satisfy the Δ_2-condition near infinity if for some $\epsilon > 0$ and $t_0 > 0$,

$$M(2t) \leq \epsilon M(t), \qquad \forall t \geq t_0. \tag{2.1}$$

For two N-functions P and M, we say that P grows essentially less rapidly than M if $\lim_{t\to\infty} P(t)/M(kt) = 0$ for all $k > 0$, and we write $P \ll M$. Moreover, if $P \ll M$ then there exists $t_0 > 0$ such that

$$P(t) \leq M(\gamma^* t) \qquad \forall t \geq t_0, \tag{2.2}$$

where γ^* is the constant of Poincaré's inequality (see Eq. (2.3)).

Let Ω be a domain of $\mathbb{R}^n$. The module of a vector-valued function $u : \Omega \to \mathbb{R}^m$ is given by $\rho_M(u) = \int_\Omega M(|u|)dx$. The classes $W^1L_M(\Omega;\mathbb{R}^m)$ and $W^1E_M(\Omega;\mathbb{R}^m)$ consist of all functions in the Orlicz spaces

$$L_M(\Omega;\mathbb{R}^m) = \{u : \Omega \to \mathbb{R}^m \text{ measurable}/ \int_\Omega M\big(\frac{u(x)}{\beta}\big)dx < \infty \text{ for some } \beta > 0\}$$

or $E_M(\Omega;\mathbb{R}^m)$, such that $Du \in L_M(\Omega;\mathbb{M}^{m\times n})$ or $Du \in E_M(\Omega;\mathbb{M}^{m\times n})$ (resp.). The Orlicz spaces $L_M(\Omega;\mathbb{R}^m)$ are endowed with the Luxemburg norm

$$\|u\|_M = \inf\{\beta > 0/ \int_\Omega M\big(\frac{|u(x)|}{\beta}\big)dx \leq 1\}.$$

Moreover, the classes $W^1L_M(\Omega;\mathbb{R}^m)$ and $W^1E_M(\Omega;\mathbb{R}^m)$ are endowed with the norm

$$\|u\|_{1,M} = \|u\|_M + \|Du\|_M.$$

They are Banach spaces under this norm. The space $E_M(\Omega;\mathbb{R}^m)$ is the closure of all measurable, simple functions in $L_M(\Omega;\mathbb{R}^m)$. Let $W_0^1E_M(\Omega;\mathbb{R}^m)$ be the (norm) closure of $C_0^\infty(\Omega;\mathbb{R}^m)$ in $W^1E_M(\Omega;\mathbb{R}^m)$. The equality $W_0^1L_M(\Omega;\mathbb{R}^m) = W_0^1E_M(\Omega;\mathbb{R}^m)$ holds if M satisfies Eq. (2.1). Moreover, if $M \in \Delta_2$-condition near infinity, then there exists $\gamma^* > 0$ such that for all $u \in W_0^1L_M(\Omega;\mathbb{R}^m)$

$$\int_\Omega M(\gamma^*|u|)dx \leq \int_\Omega M(|Du|)dx, \tag{2.3}$$

where $\gamma^* = 1/\text{diam}(\Omega)$ and $\text{diam}(\Omega)$ is the diameter of Ω (see [32]).

For convenience of the readers not familiar with the concept of Young measures, we give here an overview which will be needed in the sequel (see e.g. [15, 24, 28]). By $C_0(\mathbb{R}^m)$ we denote the closure of the space of continuous functions on $\mathbb{R}^m$ with compact support with respect to the $\|.\|_\infty$-norm. Its dual can be identified with $\mathcal{M}(\mathbb{R}^m)$, the space of signed Radon measures with finite mass. The related duality pairing is given for $\nu : \Omega \to \mathcal{M}(\mathbb{R}^m)$, by

$$\langle \nu, g\rangle = \int_{\mathbb{R}^m} g(\lambda) d\nu(\lambda).$$

Lemma 2.1. *[24] Let $\{z_j\}_{j\geq 1}$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exists a subsequence $\{z_k\}_k \subset \{z_j\}_j$ and a Borel probability measure ν_x on $\mathbb{R}^m$ for almost every $x\in\Omega$, such that for almost each $g \in C(\mathbb{R}^m)$ we have*

$$g(z_k) \rightharpoonup^* \overline{g} \quad \textit{weakly in } L^\infty(\Omega;\mathbb{R}^m),$$

where $\overline{g}(x) = \langle \nu_x, g\rangle = \int_{\mathbb{R}^m} g(\lambda)d\nu_x(\lambda)$ for a.e. $x \in \Omega$, and $\nu = \{\nu_x\}_{x\in\Omega}$ is any family of Young measures associated with the subsequence $\{z_k\}_k$.

Remark 2.2. (1) In [15], it is shown that for any Carathéodory function $g : \Omega\times\mathbb{R}^m \to \mathbb{R}$ and $\{z_k\}_k$ generates a Young measure ν_x, we have

$$g(x, z_k) \rightharpoonup \langle \nu_x, g(x, .)\rangle = \int_{\mathbb{R}^m} g(x,\lambda)d\nu_x(\lambda)$$

weakly in $L^1(\Omega')$ for all measurable $\Omega' \subset \Omega$, provided that the negative part $g^-(x, z_k)$ is equiintegrable.
(2) The above properties remain true if we replace z_k by Dv_k for $v_k : \Omega \to \mathbb{R}^m$.

Lemma 2.3 ([28])**.** *(i) If $|\Omega| < \infty$ then*

$$z_k \to z \textit{ in measure} \Leftrightarrow \nu_x = \delta_{z(x)} \quad \textit{for a.e. } x\in\Omega.$$

(ii) Moreover, if v_k generates the Young measure $\delta_{v(x)}$, then (z_k, v_k) generates the Young measure $\nu_x \otimes \delta_{v(x)}$.

Lemma 2.4 ([18])**.** *Let $g : \Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and $z_k : \Omega \to \mathbb{R}^m$ a sequence of measurable functions such that $z_k \to z$ in measure and such that Dz_k generates the Young measure ν_x, with $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for almost every $x \in \Omega$. Then*

$$\liminf_{k\to\infty} \int_\Omega g(x, z_k, Dz_k)dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} g(x, z, \lambda)d\nu_x(\lambda)dx$$

provided that the negative part $g^-(x, z_k, Dz_k)$ is equiintegrable.

We conclude this section by recalling the following lemma:

Lemma 2.5 ([5])**.** *If the sequence (Dz_k) is bounded in $L_M(\Omega;\mathbb{M}^{m\times n})$, then the Young measure ν_x generated by Dz_k satisfies:*

(i) ν_x is a probability measure, i.e. $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ for almost every $x \in \Omega$.

(ii) The weak L^1-limit of Dz_k is given by $\langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)$.

(iii) ν_x satisfies $\langle \nu_x, id\rangle = Dz(x)$ for almost every $x \in \Omega$.

3. Main assumptions

Let Ω be a bounded open set of $\mathbb{R}^n$ $(n \geq 2)$ and let M and P be two N-functions such that $P \ll M$, and M, $\overline{M} \in \Delta_2$. Our assumptions are the following:
(H0)(Continuity) $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$, $b : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ and $f : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}^m$ are Carathéodory functions, i.e. measurable w.r.t first variable and continuous w.r.t other variables.
(H1)(Growth, coercivity and sign condition) There exist $d_1, d_2, d_3 \in E_{\overline{M}}(\Omega)$, $d_4(x) \in L^1(\Omega)$, $\gamma_i \geq 0$ $(i = 1, .., 6)$ and $\gamma_0 > 0$ (γ_5 and γ_6 are small) such that for all $(s, A) \in \mathbb{R}^m \times \mathbb{M}^{m\times n}$ and a.e. $x \in \Omega$

$$
\begin{aligned}
|\sigma(x, s, A)| &\leq d_1(x) + \gamma_1 \overline{M}^{-1} P(|s|) + \gamma_2 \overline{M}^{-1} M(|A|),\\
|b(x, s, A)| &\leq d_2(x) + \gamma_3 \overline{M}^{-1} P(|s|) + \gamma_4 \overline{M}^{-1} M(|A|),\\
|f(x, s, A)| &\leq d_3(x) + \gamma_5 \overline{M}^{-1} P(|s|) + \gamma_6 \overline{M}^{-1} M(|A|),\\
\sigma(x, s, A) : A &\geq \gamma_0 M(|A|) - d_4(x),\\
b(x, s, A) \cdot s &\geq 0.
\end{aligned}
$$

(H2)(Monotonicity) σ satisfies one of the following conditions:

(a) For a.e. $x \in \Omega$ and all $u \in \mathbb{R}^m$, $A \mapsto \sigma(x, u, A)$ is a C^1-function and is monotone, i.e.

$$\big(\sigma(x, u, A) - \sigma(x, u, B)\big) : (A - B) \geq 0$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}^m$ and $A, B \in \mathbb{M}^{m\times n}$.

(b) There exists a function (potential) $W : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x, u, A) = \frac{\partial W}{\partial A}(x, u, A) =: D_A W(x, u, A)$, and $A \mapsto W(x, u, A)$ is convex and C^1.

(c) σ is strictly monotone, i.e. $\sigma(x, u, .)$ is monotone and

$$\big(\sigma(x, u, A) - \sigma(x, u, B)\big) : (A - B) = 0 \Longrightarrow A = B.$$

(d) σ is strictly M-quasimonotone, i.e.

$$\int_{\mathbb{M}^{m\times n}} \big(\sigma(x, u, \lambda) - \sigma(x, u, \overline{\lambda})\big) : (\lambda - \overline{\lambda}) d\nu_x(\lambda) > 0$$

for $\overline{\lambda} = \langle \nu_x, id \rangle$ and $\nu = \{\nu_x\}_{x\in\Omega}$ is any family of Young measures generated by a sequence in $L_M(\Omega)$ and not a Dirac measure for almost every $x \in \Omega$.

Remark 3.1. 1) As in [30], P is introduced instead of M in (H1) only to guarantee the boundedness in $L_{\overline{M}}(\Omega)$ of $\overline{M}^{-1} P(|u_k|)$ and whenever u_k is

bounded in $L_M(\Omega)$, one usually takes $P = M$ in the term $\overline{M}^{-1}P(|u_k|)$.
2) γ_5 and γ_6 (in (H1)) are small means that their values ensures that

$$\gamma_0 - \frac{2\gamma_5}{\gamma^*} - \frac{2\gamma_6}{\gamma^*} - \frac{1}{\theta\gamma^*} > 0,$$

where $\theta = \sup\{\theta_1 > 0;\ \rho_{\overline{M}}(\theta_1 d_3) < \infty\}$ and γ^* is the smallest constant defined in the equation (2.3).

4. Existence result

Let Ω be a bounded open set of $\mathbb{R}^n$ and let M and P be two N-functions such that $P \ll M$ and satisfies the Δ_2-condition (2.1). Let us define first the weak solution for problem (1.1). A function $u \in W_0^1L_M(\Omega;\mathbb{R}^m)$ is said to be a weak solution for (1.1) if

$$\int_\Omega \big(\sigma(x,u,Du) : D\varphi + b(x,u,Du)\cdot\varphi\big)dx = \int_\Omega f(x,u,Du)\cdot\varphi dx$$

holds for all $\varphi \in W_0^1L_M(\Omega;\mathbb{R}^m)$.

The main theorem of existence result reads as follows:

Theorem 4.1. *If σ, b and f satisfy the conditions (H0)-(H2), then problem (1.1) has a weak solution $u \in W_0^1L_M(\Omega;\mathbb{R}^m)$.*

Proof. The proof is devided into 3 steps. In Step 1, we introduce the approximating solution by the Galerkin method and some a priori estimates . Step 2 is devoted to prove an inequality of div-curl type which permits to pass to the limit in the approximating equations in Step 3.

Step 1:

Let us define the operator

$$T : W_0^1L_M(\Omega;\mathbb{R}^m) \longrightarrow W^{-1}L_{\overline{M}}(\Omega;\mathbb{R}^m)$$
$$u \mapsto \Big(\varphi \mapsto \int_\Omega \big(\sigma(x,u,Du) : D\varphi + b(x,u,Du)\cdot\varphi\big)dx - \int_\Omega f(x,u,Du)\cdot\varphi dx\Big).$$

For arbitrary $u \in W_0^1L_M(\Omega;\mathbb{R}^m)$, $T(u)$ is trivially linear. Let us take $\alpha = \max\{\gamma_1, \gamma_2, \frac{1}{\alpha_1}\}$, where $\alpha_1 > 0$ such that $\rho_{\overline{M}}(\alpha_1 d_1) < \infty$. By the virtue of (2.2), we deduce the existence of $t_0 > 0$ such that $P(|u|) \leq M(\gamma^*|u|)$ when $|u| > t_0$. The condition (H1) and the equation (2.3) implies

$$\begin{aligned}
&\rho_{\overline{M}}\Big(\frac{1}{3\alpha}\sigma(x,u,Du)\Big)\\
&\qquad\leq \int_\Omega \overline{M}\Big(\frac{\alpha_1}{3\alpha\alpha_1}d_1(x) + \frac{\gamma_1}{3\alpha}\overline{M}^{-1}P(|u|) + \frac{\gamma_2}{3\alpha}\overline{M}^{-1}M(|Du|)\Big)dx\\
&\qquad\leq \frac{1}{3}\int_\Omega \Big(\overline{M}\big(\alpha_1 d_1(x)\big) + P(|u|) + M(|Du|)\Big)dx\\
&\qquad\leq \frac{1}{3}\int_\Omega \Big(\overline{M}\big(\alpha_1 d_1(x)\big) + 2M(|Du|)\Big)dx < \infty.
\end{aligned} \tag{4.1}$$

Similarly, we take $\beta = \max\{\gamma_3, \gamma_4, \frac{1}{\beta_1}\}$ and $\theta = \max\{\gamma_5, \gamma_6, \frac{1}{\theta_1}\}$ (resp.) such that $\rho_{\overline{M}}(\beta_1 d_2) < \infty$ and $\rho_{\overline{M}}(\theta_1 d_3) < \infty$ (resp.), then

$$\rho_{\overline{M}}\big(\frac{1}{3\beta} b(x,u,Du)\big) \leq \frac{1}{3}\int_\Omega \Big(\overline{M}\big(\beta_1 d_2(x)\big) + 2M(|Du|)\Big)dx < \infty \tag{4.2}$$

and

$$\rho_{\overline{M}}\big(\frac{1}{3\theta} f(x,u,Du)\big) \leq \frac{1}{3}\int_\Omega \Big(\overline{M}\big(\theta_1 d_3(x)\big) + 2M(|Du|)\Big)dx < \infty. \tag{4.3}$$

Consequently, $\sigma(.,u,Du)$, $b(.,u,Du)$, $f(.,u,Du) \in L_{\overline{M}}(\Omega)$. By using the Hölder inequality and the above inequalities, it follows that

$$\big|\langle T(u), \varphi\rangle\big| \leq c\|D\varphi\|_M,$$

for a positive constant c. Hence T is well defined and bounded.

Now, let $V = \text{span}\{w_1, .., w_r\}$ be a finite subspace of $W_0^1 L_M(\Omega;\mathbb{R}^m)$, where $(w_i)_{i=1,..,r}$ is a basis of V. For simplicity, we denote the restriction $T_{|_V}$ as T. We claim that T is continuous. Let $(u_k = a_k^i w_i)$ be a sequence in V such that $u_k \to u$ in V (with conventional summation). Then $u_k \to u$ and $Du_k \to Du$ almost everywhere. The continuity property in (H0) implies for $\varphi \in V$ that $\sigma(x,u_k,Du_k) : D\varphi \to \sigma(x,u,Du) : D\varphi$, $b(x,u_k,Du_k)\cdot\varphi \to b(x,u,Du)\cdot\varphi$ and $f(x,u_k,Du_k)\cdot\varphi \to f(x,u,Du)\cdot\varphi$ almost everywhere for $k \to \infty$. Since $u_k \to u$ strongly in V, then

$$\int_\Omega M(2|u_k - u|)dx \to 0 \quad \text{and} \quad \int_\Omega M(2|Du_k - Du|)dx \to 0.$$

Therefore, there is a subsequence (still denoted $(u_k)_k$) and $l_1, l_2 \in L^1(\Omega)$ such that $M(2|u_k - u|) \leq l_1$ and $M(2|Du_k - Du|) \leq l_2$. By the virtue of the convexity of M, we then get

$$\begin{aligned} M(|u_k|) = M(|u_k - u + u|) &\leq \frac{1}{2}M(2|u_k - u|) + \frac{1}{2}M(2|u|) \\ &\leq \frac{l_1}{2} + \frac{1}{2}M(2|u|). \end{aligned}$$

In the same way, we have $M(|Du_k|) \leq \frac{l_2}{2} + \frac{1}{2}M(2|Du|)$. Hence $\|u_k\|_M$ and $\|Du_k\|_M$ are bounded. By the equations (4.1)-(4.3) and the boundedness of $\|u_k\|_M$ and $\|Du_k\|_M$, we get that $\big(\sigma(x,u_k,Du_k) : D\varphi\big)$, $\big(b(x,u_k,Du_k)\cdot\varphi\big)$ and $\big(f(x,u_k,Du_k)\cdot\varphi\big)$ are equiintegrable over a measurable subset Ω' of Ω. The Vitali theorem yields that T is continuous.

Now, let us take $\varphi = u$ in the definition of T, this implies by the coercivity and sign condition that

$$
\begin{aligned}
\langle T(u),u\rangle &= \int_\Omega \Big(\sigma(x,u,Du):Du+b(x,u,Du)\cdot u\Big)dx-\int_\Omega f(x,u,Du)\cdot u dx\\
&\geq \gamma_0\int_\Omega M(|Du|)dx-\int_\Omega d_4(x)dx\\
&-\int_\Omega \Big(d_3(x)|u|+\gamma_5\overline{M}^{-1}P(|u|)|u|+\gamma_6\overline{M}^{-1}M(|Du|)|u|\Big)dx\\
&\geq \gamma_0\int_\Omega M(|Du|)dx-\int_\Omega d_4(x)dx-\frac{1}{\theta\gamma^*}\int_\Omega M\big(\theta d_3(x)\big)dx\\
&-\frac{1}{\theta\gamma^*}\int_\Omega M\big(\gamma^*|u|\big)dx-\frac{\gamma_5}{\gamma^*}\int_\Omega P(|u|)dx-\frac{\gamma_5}{\gamma^*}M(\gamma^*|u|)dx\\
&-\frac{\gamma_6}{\gamma^*}\int_\Omega M(|Du|)dx-\frac{\gamma_6}{\gamma^*}\int_\Omega M(\gamma^*|u|)dx\\
&\geq \underbrace{\big(\gamma_0-\frac{2\gamma_5}{\gamma^*}-\frac{2\gamma_6}{\gamma^*}-\frac{1}{\theta\gamma^*}\big)}_{>0}\int_\Omega M(|Du|)dx\\
&-\int_\Omega d_4(x)dx-\frac{1}{\theta_1\gamma^*}\int_\Omega M(\theta d_3(x))dx.
\end{aligned}
$$

Hence T is coercive in the following sense: $\langle T(u),u\rangle \longrightarrow +\infty$ as $\|u\|_{1,M}\to+\infty$. Therefore T is surjective. Thanks to [31], there exists a Galerkin solution u_k of (1.1) in $V=\text{span}\{w_1,..,w_r\}$, that is

$$
\langle T(u_k),\varphi\rangle = 0 \quad \text{for all } \varphi\in V. \tag{4.4}
$$

Step 2:

As $\langle T(u),u\rangle\to+\infty$ when $\|u\|_{1,M}\to+\infty$, we can deduce the existence of $R>0$ for which $\langle T(u),u\rangle>1$ whenever $\|u\|_{1,M}>R$. Hence, for the sequence of Galerkin approximations $u_k\in V$ which satisfy Eq. (4.4), we get

$$
\|u_k\|_{1,M}\leq R \quad \text{for all } k\in\mathbb{N}. \tag{4.5}
$$

Since Du_k is bounded in $L_M(\Omega;\mathbb{M}^{m\times n})$, it follows by Lemma 2.1 the existence of a Young measure ν_x associated to Du_k in $L_M(\Omega;\mathbb{M}^{m\times n})$ such that ν_x satisfies the properties of Lemma 2.5.

Let us fix k and consider u_k, the sequence defined above such that $V_k=\text{span}\{w_1,..,w_r\}$. We shall prove the following lemma, namely div-curl inequality, which will be the key ingredient to pass to the limit in the approximating equations.

Lemma 4.2. *The Young measure ν_x satisfies the following inequality:*

$$
\int_\Omega\int_{\mathbb{M}^{m\times n}}\big(\sigma(x,u,\lambda)-\sigma(x,u,Du)\big):(\lambda-Du)d\nu_x(\lambda)dx\leq 0.
$$

Proof. Consider the sequence

$$\begin{aligned}\sigma_k &:= \big(\sigma(x,u_k,Du_k)-\sigma(x,u,Du)\big):(Du_k-Du)\\ &= \sigma(x,u_k,Du_k):(Du_k-Du)-\sigma(x,u,Du):(Du_k-Du)\\ &= \sigma_{k,1}+\sigma_{k,2}.\end{aligned}$$

Since by equation (4.1), $\sigma(.,u,Du)\in L_{\overline{M}}(\Omega)$, it follows then by the weak convergence defined in Lemma 2.5 that

$$\begin{aligned}\liminf_{k\to\infty}\int_\Omega \sigma_{k,2}dx &= \liminf_{k\to\infty}\int_\Omega \sigma(x,u,Du):(Du_k-Du)dx\\ &= \int_\Omega\int_{\mathbb{M}^{m\times n}}\sigma(x,u,Du):(\lambda-Du)d\nu_x(\lambda)dx \qquad (4.6)\\ &= \int_\Omega \sigma(x,u,Du):\Big(\underbrace{\int_{\mathbb{M}^{m\times n}}\lambda d\nu_x(\lambda)}_{=:Du(x)}-Du\Big)dx=0.\end{aligned}$$

On the one hand, since $(u_k)_k$ is bounded in $W_0^1L_M(\Omega;\mathbb{R}^m)$ then $u_k\to u$ in $L_M(\Omega;\mathbb{R}^m)$ (for a proper subsequence). Consequently,

$$\begin{aligned}\int_\Omega M(|u_k-u|)dx &\geq \int_{\{x\in\Omega:\,|u_k-u|\geq\epsilon\}} M(|u_k-u|)dx\\ &\geq c\int_{\{x\in\Omega:\,|u_k-u|\geq\epsilon\}}|u_k-u|dx\\ &\geq c\epsilon\big|\{x\in\Omega:\,|u_k-u|\geq\epsilon\}\big|,\end{aligned}$$

where c is the constant of the embedding $L_M\subset L^1$ and ϵ is some positive constant. Therefore $u_k\to u$ in measure in Ω for $k\to\infty$. Now, from Step 1, since $\big(\sigma(x,u_k,Du_k):D\varphi\big)$ is equiintegrable, then $\big(\sigma(x,u_k,Du_k):Du\big)$ is equiintegrable. To get the equiintegrability of $\big(\sigma(x,u_k,Du_k):Du_k\big)$, we choose $\Omega'\subset\Omega$ to be measurable and by the coercivity condition in (H1) and the boundedness of $(u_k)_k$, we get

$$\int_{\Omega'}\big|\min\big(\sigma(x,u_k,Du_k):Du_k,0\big)\big|dx\leq\gamma_0\int_{\Omega'}M(|Du|)dx+\int_{\Omega'}\big|d_4(x)\big|dx<\infty.$$

Therefore $\big(\sigma(x,u_k,Du_k):Du_k\big)$ is equiintegrable. Thanks to Lemma 2.4,

$$\begin{aligned}I:=\liminf_{k\to\infty}\int_\Omega\sigma_k dx &= \liminf_{k\to\infty}\int_\Omega\sigma_{k,1}dx\\ &\geq \int_\Omega\int_{\mathbb{M}^{m\times n}}\sigma(x,u,\lambda):(\lambda-Du)d\nu_x(\lambda)dx.\end{aligned}$$

To get the needed inequality, it is sufficient to show that $I\leq 0$. To do this, we use Mazur's theorem (see e.g. [35, Theorem 2, page 120]) to deduce the existence of $v_k\in W_0^1L_M(\Omega;\mathbb{R}^m)$ such that $v_k\to u$ in $W_0^1L_M(\Omega;\mathbb{R}^m)$, where v_k

is a convex linear combination of $\{u_1, .., u_k\}$, thus $v_k \in V_k$. Take $\varphi = u_k - v_k$ in Eq. (4.4). By the boundedness of $(u_k)_k$ in $W_0^1 L_M(\Omega; \mathbb{R}^m)$ and Eq. (4.3), it follows that

$$\Big| \int_\Omega f(x, u_k, Du_k) \cdot (u_k - v_k) dx \Big| \leq c \int_\Omega M(|u_k - v_k|) dx, \tag{4.7}$$

where c is a constant depend on θ. Since

$$\|u_k - v_k\|_M \leq \|u_k - u\|_M + \|v_k - u\|_M \to 0 \quad \text{as } k \to \infty,$$

then the right hand side of (4.7) tends to zero for $k \to \infty$. By a similar argument, we deduce

$$\Big| \int_\Omega b(x, u_k, Du_k) \cdot (u_k - v_k) dx \Big| \leq c \int_\Omega M(|u_k - v_k|) dx \longrightarrow 0 \quad \text{as } k \to \infty.$$

Consequently, the term

$$\begin{aligned} &\int_\Omega \sigma(x, u_k, Du_k) : (Du_k - Dv_k) dx \\ &\qquad = \int_\Omega f(x, u_k, Du_k) \cdot (u_k - v_k) dx - \int_\Omega b(x, u_k, Du_k) \cdot (u_k - v_k) dx \end{aligned}$$

tends to zero as $k \to \infty$. This implies that

$$\begin{aligned} I &= \liminf_{k\to\infty} \int_\Omega \sigma(x, u_k, Du_k) : (Du_k - Du) dx \\ &= \liminf_{k\to\infty} \Big(\int_\Omega \sigma(x, u_k, Du_k) : (Du_k - Dv_k) dx \\ &\qquad + \int_\Omega \sigma(x, u_k, Du_k) : (Dv_k - Du) dx \Big) \\ &= \liminf_{k\to\infty} \int_\Omega \sigma(x, u_k, Du_k) : (Dv_k - Du) dx \\ &\leq \liminf_{k\to\infty} c \big\| |\sigma(x, u_k, Du_k)| \big\|_{\overline{M}} \|v_k - u\|_{1,M} = 0 \end{aligned}$$

and the desired inequality follows. □

Step 3:

As a consequence of Lemma 4.2 and monotonicity of σ (see [5, Lemma 9]), we have

$$\big(\sigma(x, u, \lambda) - \sigma(x, u, Du)\big) : (\lambda - Du) = 0 \quad \text{on } \operatorname{supp} \nu_x. \tag{4.8}$$

Now, we have all ingredients to pass to the limit in the Galerkin equations and prove Theorem 4.1 by considering the cases (a)-(d) listed in (H2).

Case (a): In this case, we claim that

$$\sigma(x,u,\lambda):A=\sigma(x,u,Du):A+\big(\nabla\sigma(x,u,Du)A\big):(Du-\lambda)$$

holds on $\operatorname{supp}\nu_x$, for $A\in\mathbb{M}^{m\times n}$ and where ∇ is the derivative of σ with respect to its third variable. By the monotonicity of σ, it follows for all $\tau\in\mathbb{R}^m$ and $A\in\mathbb{M}^{m\times n}$ that

$$\big(\sigma(x,u,\lambda)-\sigma(x,u,Du+\tau A)\big):(\lambda-Du-\tau A)\geq 0,$$

which implies by Eq. (4.8)

$$\begin{aligned}&-\sigma(x,u,\lambda):\tau A\\&\quad\geq-\sigma(x,u,\lambda):(\lambda-Du)+\sigma(x,u,Du+\tau A):(\lambda-Du-\tau A)\\&\quad=-\sigma(x,u,Du):(\lambda-Du)+\sigma(x,u,Du+\tau A):(\lambda-Du-\tau A).\end{aligned}$$

Using the fact that $\sigma(x,u,Du+\tau A)=\sigma(x,u,Du)+\nabla\sigma(x,u,Du)\tau A+o(\tau)$ and deduce that

$$-\sigma(x,u,\lambda):\tau A\geq\tau\Big(\big(\nabla\sigma(x,u,Du)A\big):(\lambda-Du)-\sigma(x,u,Du):A\Big)+o(\tau).$$

Since τ is arbitrary in $\mathbb{R}$, then our claim follows. By the equiintegrability of $\sigma(x,u_k,Du_k)$, it follows by Remark 2.2 that its weak L^1-limit is given by

$$\begin{aligned}\overline{\sigma}&:=\int_{\mathbb{M}^{m\times n}}\sigma(x,u,\lambda)d\nu_x(\lambda)\\&=\int_{\operatorname{supp}\nu_x}\sigma(x,u,\lambda)d\nu_x(\lambda)\\&=\int_{\operatorname{supp}\nu_x}\Big(\sigma(x,u,Du)+\big(\nabla\sigma(x,u,Du)\big):(Du-\lambda)\Big)d\nu_x(\lambda)\\&=\sigma(x,u,Du)\underbrace{\int_{\operatorname{supp}\nu_x}d\nu_x(\lambda)}_{=:1}+\big(\nabla\sigma(x,u,Du)\big)^t\underbrace{\Big(\int_{\operatorname{supp}\nu_x}(Du-\lambda)d\nu_x(\lambda)\Big)}_{=0}\\&=\sigma(x,u,Du).\end{aligned}$$

Since $\sigma(x,u_k,Du_k)$ is bounded in $L_{\overline{M}}(\Omega;\mathbb{M}^{m\times n})$ reflexive, then $\sigma(x,u_k,Du_k)$ is weakly convergent in $L_{\overline{M}}(\Omega;\mathbb{M}^{m\times n})$ and its weak $L_{\overline{M}}$-limit is also $\sigma(x,u,Du)$. Therefore, for arbitrary $\varphi\in W_0^1L_M(\Omega;\mathbb{R}^m)$, we have

$$\int_\Omega\big(\sigma(x,u_k,Du_k)-\sigma(x,u,Du)\big):D\varphi dx\longrightarrow 0\quad\text{as } k\to\infty.$$

Case (b): We show that $\operatorname{supp}\nu_x\subset K_x$, where

$$K_x=\big\{\lambda\in\mathbb{M}^{m\times n}:\ W(x,u,\lambda)=W(x,u,Du)+\sigma(x,u,Du):(\lambda-Du)\big\}.$$

Let $\lambda\in\operatorname{supp}\nu_x$, then by Eq. (4.8)

$$(1-\tau)\big(\sigma(x,u,\lambda)-\sigma(x,u,Du)\big):(\lambda-Du)=0\quad\forall\tau\in[0,1].$$

This equation together with the monotonicity of σ implies

$$0 \leq (1-\tau)\big(\sigma(x,u,Du+\tau(\lambda-Du))-\sigma(x,u,\lambda)\big):(Du-\lambda)$$
$$= (1-\tau)\big(\sigma(x,u,Du+\tau(\lambda-Du))-\sigma(x,u,Du)\big):(Du-\lambda). \tag{4.9}$$

Using again the monotonicity of σ yields

$$\big(\sigma(x,u,Du+\tau(\lambda-Du))-\sigma(x,u,Du)\big):\tau(\lambda-Du)\geq 0,$$

which implies since $\tau\in[0,1]$ that

$$\big(\sigma(x,u,Du+\tau(\lambda-Du))-\sigma(x,u,Du)\big):(1-\tau)(\lambda-Du)\geq 0. \tag{4.10}$$

From (4.9) and (4.10) it follows that

$$(1-\tau)\big(\sigma(x,u,Du+\tau(\lambda-Du))-\sigma(x,u,Du)\big):(\lambda-Du)=0 \quad \forall\tau\in[0,1],$$

i.e.

$$\sigma(x,u,Du+\tau(\lambda-Du)):(\lambda-Du)=\sigma(x,u,Du):(\lambda-Du),$$

whenever $\lambda\in\operatorname{supp}\nu_x$. Integrate the above equality over $[0,1]$ and use the fact that $\sigma:=D_AW$, it results that

$$W(x,u,\lambda)=W(x,u,Du)+\int_0^1\sigma(x,u,Du+\tau(\lambda-Du)):(\lambda-Du)d\tau$$
$$=W(x,u,Du)+\sigma(x,u,Du):(\lambda-Du).$$

Therefore $\lambda\in K_x$. The convexity of W implies for all $\lambda\in\mathbb{M}^{m\times n}$ that

$$\underbrace{W(x,u,\lambda)}_{=:F(\lambda)}\geq\underbrace{W(x,u,Du)+\sigma(x,u,Du):(\lambda-Du)}_{=:G(\lambda)}.$$

Since $\lambda\mapsto F(\lambda)$ is a C^1-function, then for $A\in\mathbb{M}^{m\times n}$ and $\tau\in\mathbb{R}$ we have

$$\frac{F(\lambda+\tau A)-F(\lambda)}{\tau}\geq\frac{G(\lambda+\tau A)-G(\lambda)}{\tau}\quad\text{for }\tau>0,$$
$$\frac{F(\lambda+\tau A)-F(\lambda)}{\tau}\leq\frac{G(\lambda+\tau A)-G(\lambda)}{\tau}\quad\text{for }\tau<0.$$

Therefore $D_\lambda F(\lambda)=D_\lambda G(\lambda)$, i.e.

$$\sigma(x,u,\lambda)=\sigma(x,u,Du)\quad\forall\lambda\in K_x\supset\operatorname{supp}\nu_x. \tag{4.11}$$

Hence

$$\overline{\sigma}=\int_{\mathbb{M}^{m\times n}}\sigma(x,u,\lambda)d\nu_x(\lambda)=\int_{\operatorname{supp}\nu_x}\sigma(x,u,\lambda)d\nu_x(\lambda)$$
$$\overset{(4.11)}{=}\int_{\operatorname{supp}\nu_x}\sigma(x,u,Du)d\nu_x(\lambda)$$
$$=\sigma(x,u,Du). \tag{4.12}$$

Consider the Carathéodory function $g(x,s,\lambda) = |\sigma(x,s,\lambda) - \overline{\sigma}(x)|$. The equiintegrability of $\sigma(x,u_k,Du_k)$ implies that $g_k(x) := g(x,u_k,Du_k)$ is equiintegrable, and its weak L^1-limit is given as

$$\begin{aligned}\overline{g}(x) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} g(x,s,\lambda)d\delta_{u(x)}(s)\otimes d\nu_x(\lambda)\\ &= \int_{\operatorname{supp}\nu_x} |\sigma(x,u,\lambda) - \overline{\sigma}(x)|d\nu_x(\lambda) = 0 \qquad \text{(by (4.11) and (4.12)).}\end{aligned}$$

The weak L^1-limit of g_k is in fact strong since $g_k \geq 0$. Hence

$$g_k \longrightarrow 0 \quad \text{in } L^1(\Omega).$$

Case (c): The strict monotonicity of σ together with Eq. (4.8) implies that $\nu_x = \delta_{Du(x)}$ for almost every $x \in \Omega$. By the virtue of Lemma 2.3, it follows that $Du_k \to Du$ in measure for $k \to \infty$. In Step 2 we have $u_k \to u$ in measure. Hence, after extraction of a suitable subsequence, if necessary,

$$u_k \to u \quad \text{and} \quad Du_k \to Du \quad \text{almost everywhere for } k \to \infty.$$

The continuity of σ yields

$$\sigma(x,u_k,Du_k) \to \sigma(x,u,Du) \quad \text{almost everywhere for } k \to \infty.$$

The Vitali convergence theorem implies

$$\int_\Omega \big(\sigma(x,u_k,Du_k) - \sigma(x,u,Du)\big) : D\varphi dx \longrightarrow 0 \text{ as } k \to \infty,$$

since $\sigma(x,u_k,Du_k)$ is equiintegrable.

Case (d): We suppose by contradiction that ν_x is not a Dirac measure on a set $x \in \Omega' \subset \Omega$ of positive Lebesgue measure. We have by the strict monotone of σ and $\overline{\lambda} = \langle \nu_x, id\rangle = Du(x)$ that

$$\begin{aligned}0 &< \int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,u,\lambda) - \sigma(x,u,\overline{\lambda})\big) : (\lambda - \overline{\lambda})d\nu_x(\lambda)dx\\ &= \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : (\lambda - \overline{\lambda})d\nu_x(\lambda)dx,\end{aligned}$$

where we have used

$$\begin{aligned}&\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\overline{\lambda}) : (\lambda - \overline{\lambda})d\nu_x(\lambda)dx\\ &\qquad\qquad = \int_\Omega \sigma(x,u,\overline{\lambda}) : \Big(\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) - \overline{\lambda}\Big)dx = 0.\end{aligned}$$

Hence

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda)dx > \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\lambda) : \overline{\lambda}d\nu_x(\lambda)dx.$$

By the virtue of Lemma 4.2, we get together with the above inequality that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\lambda) : \overline{\lambda} d\nu_x(\lambda)dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda)dx$$
$$> \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,t,\lambda) : \overline{\lambda} d\nu_x(\lambda)dx$$

which is a contradiction. Hence ν_x is a Dirac measure and we can write $\nu_x = \delta_{h(x)}$. Therefore

$$h(x) = \int_{\mathbb{M}^{m\times n}} \lambda d\delta_{h(x)}(\lambda) = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda) = Du(x).$$

Consequently, $\nu_x = \delta_{Du(x)}$. The remainder of the proof is similar then to that in case (c).

To conclude and complete the proof of Theorem 4.1, it remains to pass to the limit on $b(x,u_k,Du_k)$ and $f(x,u_k,Du_k)$. We have $u_k \to u$ and $Du_k \to du$ in measure (see Step 2) for $k \to \infty$. Then $u_k \to u$ and $Du_k \to Du$ almost everywhere (for a proper subsequence). The continuity of the functions b and f implies for arbitrary $\varphi \in W_0^1L_M(\Omega;\mathbb{R}^m)$ that

$$b(x,u_k,Du_k)\cdot\varphi \to b(x,u,Du)\cdot\varphi \quad \text{and} \quad f(x,u_k,Du_k)\cdot\varphi \to f(x,u,Du)\cdot\varphi$$

almost everywhere. Since, by (4.2) and (4.3), $b(x,u_k,Du_k)$ and $f(x,u_k,Du_k)$ are equiintegrable, it follows that $b(x,u_k,Du_k)\cdot\varphi \to b(x,u,Du)\cdot\varphi$ and $f(x,u_k,Du_k)\cdot\varphi \to f(x,u,Du)\cdot\varphi$ in $L^1(\Omega)$ by the Vitali convergence theorem.

Now, we take a test function $\varphi \in \bigcup_{i\in\mathbb{N}} V_i$ in (4.4) and pass to the limit $k\to\infty$. The resulting equation is

$$\int_\Omega \big(\sigma(x,u,Du) : D\varphi + b(x,u,Du)\cdot\varphi\big)dx = \int_\Omega f(x,u,Du)\cdot\varphi dx$$

for arbitrary $\varphi \in \bigcup_{i\in\mathbb{N}} V_i$. By density of the linear span of these functions in $W_0^1L_M(\Omega;\mathbb{R}^m)$, this proves that u is in fact a weak solution. The proof of Theorem 4.1 is complete. □

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