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Sequential fractional differential equations with nonlocal integro-multipoint boundary conditions

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Abstract. This paper is concerned with the existence and uniqueness of solutions for sequential Caputo fractional differential equation equipped with integro multipoint boundary conditions. In the proposed problem, the nonlinearity depends on the unknown function as well as its lower order fractional derivatives. We apply standard fixed point theorems to obtain the desired results, which are well-illustrated with the aid of examples.

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1. Introduction

Fractional-order boundary value problems have been extensively studied by many authors during the last few years. In particular, the study of fractional differential equations complemented with nonlocal and integral boundary conditions gained much popularity. It has been mainly due to the importance of the nonlocal conditions in describing some peculiar phenomena taking place at interior points or sub-intervals of the given domain [8]. On the other hand, integral boundary conditions help to model blood flow problems [3] and regularizing ill-posed parabolic backward problems [25].

Fractional calculus is found to be of great value in appropriate modelling of many real-world problems arising in several fields of physical and applied sciences. For examples and details, see [16], [18], [27], [12]. Multi-term fractional differential equations also received considerable attention as these equations appear in the mathematical models related to practical situations, for instance,

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the behavior of real materials [24], an inextensible pendulum with fractional damping terms [26], etc. For further applications, see [10], [17], [15], [9].

A great interest was shown in the study of the boundary value problems involving sequential fractional differential equations (a sub-class of multi-term fractional differential equations). For some recent works on this class of boundary value problems, we refer the reader to the articles [5], [2], [13], [6], [1], [20], [7], [14], [4], [11], [23], [19], [22].

In this paper, motivated by aforementioned work, we investigate the existence of solutions for a nonlinear Liouville-Caputo type fractional differential equation of the form:

$$\mu \, {}^cD^q x(t) + \xi \, {}^cD^{q-1}x(t) = f(t, x(t), {}^cD^p x(t), {}^cD^{p+1}x(t)), \; t \in [a, b], \tag{1.1}$$

$3 < q \leq 4, \; 0 < p \leq 1$, supplemented with nonlocal integro-multipoint boundary conditions

$$x(a) = \sigma(x), \; x'(a) = 0, \; x(b) = 0, \; x'(b) = \sum_{i=1}^{m} \alpha_i x(\eta_i) + \int_a^b x(s)ds, \tag{1.2}$$

where ${}^cD^q$ denotes the Caputo fractional differential operator of order $q \in (3, 4]$, $a < \eta_1 < \eta_2 < \cdots < \eta_m < b$, $f : [a, b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a given continuous function, σ is a nonlinear function defined on $C([a, b], \mathbb{R})$ and $\mu, \xi \; (\mu, \xi \neq 0), \; \alpha_i \in \mathbb{R}, \; i = 1, 2, \cdots, m$.

The rest of the paper is arranged as follows. In Section 2, we prove a basic result related to the linear variant of the problem (1.1)-(1.2), which plays a key role in the forthcoming analysis. We also recall some basic concepts of fractional calculus. The main results are presented in Section 3.

2. Preliminaries and auxiliary result

Before presenting an auxiliary lemma, we recall some basic definitions of fractional calculus [16].

Definition 2.1. The Riemann-Liouville fractional integral of order p with lower limit for a function $g : [a, \infty) \longrightarrow \mathbb{R}$ is defined as

$$I^p g(t) = \frac{1}{\Gamma(p)} \int_a^t \frac{g(s)}{(t-s)^{1-p}} ds, \; p > 0,$$

provided the integral exists.

Definition 2.2. For (n-1)-times absolutely continuous function $g : [a, \infty) \longrightarrow \mathbb{R}$, the Caputo derivative of fractional order p is defined as

$${}^cD^p g(t) = \frac{1}{\Gamma(n-p)} \int_a^t (t-s)^{n-p-1} g^{(n)}(s) ds, \; n-1 < p \leq n, \; n = [p]+1,$$

where $[p]$ denotes the integer part of the real number p.

Lemma 2.3. *[16] For $n-1 < q \leq n$, the general solution of the fractional differential equation ${}^cD^q x(t) = 0,\ t \in [a,b]$, is*

$$x(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \ldots + c_{n-1}(t-a)^{n-1},$$

where $c_i \in \mathbb{R},\ i = 0,1,\ldots,n-1$. Furthermore,

$$I^q\ {}^cD^q x(t) = x(t) + \sum_{i=0}^{n-1} c_i (t-a)^i.$$

Let us consider the linear sequential fractional differential equation

$$\mu\ {}^cD^q x(t) + \xi\ {}^cD^{q-1}x(t) = \psi(t),\ 3 < q \leq 4,\ t \in [a,b], \tag{2.1}$$

equipped with nonlocal integro-multipoint boundary conditions

$$x(a) = \sigma^*,\ x'(a) = 0,\ x(b) = 0,\ x'(b) = \sum_{i=1}^{m} \alpha_i x(\eta_i) + \int_a^b x(s)ds, \tag{2.2}$$

where $\psi \in C([a,b],\mathbb{R}),\ \ a < \eta_1 < \eta_2 < \cdots < \eta_{n-2} < b,\ \ \mu, \xi\ (\mu, \xi \neq 0),\ \alpha_i \in \mathbb{R}$, $i = 1,2,\cdots,m$ and $\sigma^* \in \mathbb{R}$

Lemma 2.4. *Let $\psi \in C([a,b],\mathbb{R})$. Then the unique solution $x \in C^4([a,b],\mathbb{R})$ of the problem (2.1)-(2.2) is given by*

$$\begin{aligned} x(t) &= \frac{\sigma^*}{\Lambda}\omega_1(t) - \frac{\omega_2(t)}{\mu}\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\psi(s)ds \\ &\quad +\omega_3(t)\Big[\frac{\xi}{\mu^2}\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\psi(s)ds - \frac{1}{\mu} I_a^{q-1}\psi(b) \\ &\quad +\frac{1}{\mu}\sum_{i=0}^{m}\alpha_i \int_a^{\eta_i} e^{\frac{-\xi}{\mu}(\eta_i-s)} I_a^{q-1}\psi(s)ds \\ &\quad +\frac{1}{\mu}\int_a^b \Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)} I_a^{q-1}\psi(u)du\Big)ds\Big] + \frac{1}{\mu}\int_a^t e^{\frac{-\xi}{\mu}(t-s)} I_a^{q-1}\psi(s)ds, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \omega_1(t) &= \Big(\frac{A_5}{(b-a)^2} - A_3\Big)e^{\frac{-\xi}{\mu}(t-a)} + A_3 - \frac{A_5}{(b-a)^2} + \Lambda \\ &\quad +\frac{\xi}{\mu}\Big(\frac{A_5}{(b-a)^2} - A_3\Big)(t-a) \\ &\quad +\Big(\frac{A_1A_5}{(b-a)^4} - \frac{A_1A_3}{(b-a)^2} - \frac{\Lambda}{(b-a)^2}\Big)(t-a)^2, \\ \omega_2(t) &= \frac{A_5(1 - e^{\frac{-\xi}{\mu}(t-a)})}{\Lambda(b-a)^2} - \frac{\xi A_5}{\mu\Lambda(b-a)^2}(t-a) \\ &\quad +\frac{(\Lambda(b-a)^2 - A_1A_5)}{\Lambda(b-a)^4}(t-a)^2, \end{aligned}$$

$$\omega_3(t) = \frac{(e^{\frac{-\xi}{\mu}(t-a)}-1)}{\Lambda} + \frac{\xi}{\mu\Lambda}(t-a) + \frac{A_1}{\Lambda(b-a)^2}(t-a)^2, \tag{2.4}$$

$$\begin{aligned} A_1 &= 1 - e^{\frac{-\xi}{\mu}(b-a)} - \frac{\xi(b-a)}{\mu}, \\ A_2 &= \frac{-\xi}{\mu} e^{\frac{-\xi}{\mu}(b-a)} - \sum_{i=1}^{m} \alpha_i e^{\frac{-\xi}{\mu}(\eta_i - a)} - \int_a^b e^{\frac{-\xi}{\mu}(s-a)} ds, \\ A_3 &= -\sum_{i=1}^{m} \alpha_i - \int_a^b ds, \quad A_4 = 1 - \sum_{i=1}^{m} \alpha_i(\eta_i - a) - \int_a^b (s-a) ds, \qquad (2.5) \\ A_5 &= 2(b-a) - \sum_{i=1}^{m} \alpha_i (\eta_i - a)^2 - \int_a^b (s-a)^2 ds, \end{aligned}$$

and it is assumed that

$$\Lambda = A_2 - A_3 + \frac{\xi}{\mu} A_4 + \frac{A_1 A_5}{(b-a)^2} \neq 0. \tag{2.6}$$

Proof. Rewrite the equation $\mu\ {}^cD^q x(t) + \xi\ {}^cD^{q-1} x(t) = \psi(t)$ as

$${}^cD^{q-1}(\mu\ D + \xi) x(t) = \psi(t). \tag{2.7}$$

Applying the integral operator I^{q-1} on both sides of (2.7) and solving the resulting equation, we get

$$\begin{aligned} x(t) &= b_0 e^{\frac{-\xi}{\mu}(t-a)} + b_1 + b_2(t-a) + b_3(t-a)^2 \\ &\quad + \frac{1}{\mu}\int_a^t e^{\frac{-\xi}{\mu}(t-s)} I_a^{q-1}\psi(s) ds, \end{aligned} \tag{2.8}$$

where $b_i \in \mathbb{R},\ i = 0, 1, 2, 3$ are unknown arbitrary constants. From (2.8) we have

$$\begin{aligned} x'(t) &= \frac{-\xi}{\mu} b_0 e^{\frac{-\xi}{\mu}(t-a)} + b_2 + 2b_3(t-a) \\ &\quad - \frac{\xi}{\mu^2}\int_a^t e^{\frac{-\xi}{\mu}(t-s)} I_a^{q-1}\psi(s) ds + \frac{1}{\mu} I_a^{q-1}\psi(t). \end{aligned} \tag{2.9}$$

Using the boundary conditions (2.2) in (2.8) and (2.9), we obtain

$$b_2 = \frac{\xi}{\mu} b_0, \tag{2.10}$$

$$b_1 = -b_0 + \sigma^*, \tag{2.11}$$

$$e^{\frac{-\xi}{\mu}(b-a)} b_0 + b_1 + (b-a) b_2 + (b-a)^2 b_3 = I_1, \tag{2.12}$$

$$A_2 b_0 + A_3 b_1 + A_4 b_2 + A_5 b_3 = I_2, \tag{2.13}$$

where A_i $(i = 1, \dots, 5)$ are given by (2.5), and

$$\begin{aligned} I_1 &= -\frac{1}{\mu}\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\psi(s)ds, \\ I_2 &= \frac{\xi}{\mu^2}\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\psi(s)ds - \frac{1}{\mu} I_a^{q-1}\psi(b) \\ &+\frac{1}{\mu}\sum_{i=1}^{m}\alpha_i \int_a^{\eta_i} e^{\frac{-\xi}{\mu}(\eta_i-s)} I_a^{q-1}\psi(s)ds \\ &+\frac{1}{\mu}\int_a^b \Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)} I_a^{q-1}\psi(u)du\Big)ds. \end{aligned} \tag{2.14}$$

Using (2.10) and (2.11) in (2.12) and (2.13), we get

$$b_3 = \frac{1}{(b-a)^2} I_1 + \frac{A_1}{(b-a)^2} b_0 - \frac{\sigma^*}{(b-a)^2}, \tag{2.15}$$

$$\begin{aligned} b_0 &= \frac{1}{A_2 - A_3 + \frac{\xi}{\mu}A_4} I_2 - \frac{A_5}{A_2 - A_3 + \frac{\xi}{\mu}A_4} b_3 \\ &- \frac{A_3}{A_2 - A_3 + \frac{\xi}{\mu}A_4}\sigma^*. \end{aligned} \tag{2.16}$$

Solving (2.15) and (2.16) together we find that

$$\begin{aligned} b_0 &= \frac{-A_5}{\Lambda(b-a)^2} I_1 + \frac{1}{\Lambda} I_2 + \frac{\sigma^*}{\Lambda}\Big(\frac{A_5}{(b-a)^2} - A_3\Big), \\ b_3 &= \frac{(\Lambda(b-a)^2 - A_1 A_5)}{\Lambda(b-a)^4} I_1 + \frac{A_1}{\Lambda(b-a)^2} I_2 \\ &+ \frac{\sigma^*}{\Lambda}\Big(\frac{A_1 A_5}{(b-a)^4} - \frac{A_1 A_3}{(b-a)^2} - \frac{\Lambda}{(b-a)^2}\Big). \end{aligned} \tag{2.17}$$

Putting (2.17) in (2.10) and (2.11), we find that

$$\begin{aligned} b_1 &= \frac{A_5}{\Lambda(b-a)^2} I_1 - \frac{1}{\Lambda} I_2 + \frac{\sigma^*}{\Lambda}\Big(A_3 - \frac{A_5}{(b-a)^2} + \Lambda\Big), \\ b_2 &= \frac{-\xi A_5}{\mu\Lambda(b-a)^2} I_1 + \frac{\xi}{\mu\Lambda} I_2 + \frac{\xi\sigma^*}{\mu\Lambda}\Big(\frac{A_5}{(b-a)^2} - A_3\Big). \end{aligned}$$

Inserting the values of b_0, b_1, b_2 and b_3 in (2.8) together with notations (2.4), we obtain the solution (2.3). The converse of the lemma follows by direct computation. The proof is completed. □

To simplify the proofs in the forthcoming theorems, we establish the bounds for the integrals arising in the sequel in the following lemma.

Lemma 2.5. *For $\psi \in C([a,b],\mathbb{R})$ we have*

(i) $$\left|\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\psi(s)ds\right| = \left|\int_a^b e^{\frac{-\xi}{\mu}(b-s)}\Big(\int_a^s \frac{(s-u)^{q-2}}{\Gamma(q-1)}\psi(u)du\Big)ds\right| \le \frac{|\mu|(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big(1-e^{\frac{-\xi}{\mu}(b-a)}\Big)\|\psi\|.$$

(ii) $$\left|\frac{1}{\mu}\sum_{i=0}^{m}\alpha_i\int_a^{\eta_i} e^{\frac{-\xi}{\mu}(\eta_i-s)} I_a^{q-1}\psi(s)ds\right| \le \frac{1}{|\xi|\Gamma(q)}\sum_{i=0}^{m}|\alpha_i|(\eta_i-a)^{q-1}\Big(1-e^{\frac{-\xi}{\mu}(\eta_i-a)}\Big)\|\psi\|.$$

(iii) $$\left|\frac{1}{\mu}\int_a^b\Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)} I_a^{q-1}\psi(u)du\Big)ds\right| \le \Big\{\frac{(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big[(b-a) + \frac{|\mu|}{|\xi|}\Big(e^{\frac{-\xi}{\mu}(b-a)}-1\Big)\Big]\Big\}\|\psi\|.$$

(iv) $$\left|\frac{1}{\mu}\int_a^t e^{\frac{-\xi}{\mu}(t-s)} I_a^{q-1}\psi(s)ds\right| \le \frac{(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big(1-e^{\frac{-\xi}{\mu}(b-a)}\Big)\|\psi\|.$$

3. Existence of solutions

For $0 < p \le 1$, let us consider the space $\mathcal{G} = \{x : x,\ {}^cD^px,\ {}^cD^{p+1}x \in C([a,b],\mathbb{R})\}$ endowed with the norm defined by

$$\|x\|^* = \sup_{t\in[a,b]}\{|x(t)| + |{}^cD^px(t)| + |{}^cD^{p+1}x(t)|\}. \tag{3.1}$$

In view of Lemma 2.4, we transform the problem (1.1)-(1.2) into an equivalent fixed point problem as

$$x = \mathcal{H}x, \tag{3.2}$$

where $\mathcal{H} : \mathcal{G} \longrightarrow \mathcal{G}$ is defined by

$$\begin{aligned}(\mathcal{H}x)(t) \;=\;& \frac{\sigma(x)}{\Lambda}\omega_1(t) - \frac{\omega_2(t)}{\mu}\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\widehat{f}(x(s))ds \\ &+\omega_3(t)\Big[\frac{\xi}{\mu^2}\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\widehat{f}(x(s))ds - \frac{1}{\mu}I_a^{q-1}\widehat{f}(x(b)) \\ &+\frac{1}{\mu}\sum_{i=0}^{m}\alpha_i\int_a^{\eta_i} e^{\frac{-\xi}{\mu}(\eta_i-s)} I_a^{q-1}\widehat{f}(x(s))ds \\ &+\frac{1}{\mu}\int_a^b\Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)} I_a^{q-1}\widehat{f}(x(u))du\Big)ds\Big] \\ &+\frac{1}{\mu}\int_a^t e^{\frac{-\xi}{\mu}(t-s)} I_a^{q-1}\widehat{f}(x(s))ds,\ t\in[a,b],\end{aligned} \tag{3.3}$$

where $\omega_i(t),\ i=1,2,3$ are defined by (2.4) and $\widehat{f}(x(t)) = f(t, x(t), {}^cD^p x(t), {}^cD^{p+1}x(t))$. Notice that the fixed points of the operator $\mathcal{H}$ are the solution of (1.1)-(1.2).

From (3.3), we have

$$\begin{aligned}(\mathcal{H}x)'(t) &= \frac{\sigma(x)}{\Lambda}\omega_1'(t) - \frac{\omega_2'(t)}{\mu}\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\widehat{f}(x(s))ds \\ &+\omega_3'(t)\Big[\frac{\xi}{\mu^2}\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\widehat{f}(x(s))ds \\ &-\frac{1}{\mu}I_a^{q-1}\widehat{f}(x(b)) + \frac{1}{\mu}\sum_{i=0}^m \alpha_i \int_a^{\eta_i} e^{\frac{-\xi}{\mu}(\eta_i-s)} I_a^{q-1}\widehat{f}(x(s))ds \\ &+\frac{1}{\mu}\int_a^b\Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)} I_a^{q-1}\widehat{f}(x(u))du\Big)ds\Big] + \frac{1}{\mu}I_a^{q-1}\widehat{f}(x(t)) \\ &-\frac{\xi}{\mu^2}\int_a^t e^{\frac{-\xi}{\mu}(t-s)} I_a^{q-1}\widehat{f}(x(s))ds,\end{aligned}$$

$$\begin{aligned}(\mathcal{H}x)''(t) &= \frac{\sigma(x)}{\Lambda}\omega_1''(t) - \frac{\omega_2''(t)}{\mu}\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\widehat{f}(x(s))ds \\ &+\omega_3''(t)\Big[\frac{\xi}{\mu^2}\int_a^b e^{\frac{-\xi}{\mu}(b-s)} I_a^{q-1}\widehat{f}(x(s))ds \\ &-\frac{1}{\mu}I_a^{q-1}\widehat{f}(x(b)) + \frac{1}{\mu}\sum_{i=0}^m \alpha_i \int_a^{\eta_i} e^{\frac{-\xi}{\mu}(\eta_i-s)} I_a^{q-1}\widehat{f}(x(s))ds \\ &+\frac{1}{\mu}\int_a^b\Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)} I_a^{q-1}\widehat{f}(x(u))du\Big)ds\Big] + \frac{1}{\mu}I_a^{q-2}\widehat{f}(x(t)) \\ &-\frac{\xi}{\mu^2}I_a^{q-1}\widehat{f}(x(t)) + \frac{\xi^2}{\mu^3}\int_a^t e^{\frac{-\xi}{\mu}(t-s)} I_a^{q-1}\widehat{f}(x(s))ds.\end{aligned}$$

By the definition of Caputo fractional derivative with $p \in (0,1)$ we have

$$\begin{aligned}{}^cD^p(\mathcal{H}x)(t) &= \int_a^t \frac{(t-s)^{-p}}{\Gamma(1-p)}(\mathcal{H}x)'(s)ds, \\ {}^cD^{p+1}(\mathcal{H}x)(t) &= \int_a^t \frac{(t-s)^{-p}}{\Gamma(1-p)}(\mathcal{H}x)''(s)ds,\ t\in[a,b]. \qquad (3.4)\end{aligned}$$

We need the following hypotheses in the sequel.

($\mathcal{B}1$) The function $f : [a,b]\times\mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous and there exists $l_1 > 0$ such that

$$\begin{aligned}&|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)| \\ &\quad\le l_1\Big(|x_1-y_1| + |x_2-y_2| + |x_3-y_3|\Big), \\ &\qquad \forall\, t\in[a,b],\ x_i,y_i\in\mathbb{R},\ i=1,2,3;\end{aligned}$$

($\mathcal{B}2$) The function $f : [a, b] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous and there exists a function $\phi \in C([a, b], \mathbb{R}^+)$ such that

$$|f(t, x_1, x_2, x_3)| \leq \phi(t), \ \|\phi\| = \sup_{t \in [a,b]} |\phi(t)|,$$

for $t \in [a, b]$, and each $x_i \in \mathbb{R}, \ i = 1, 2, 3$;

($\mathcal{B}3$) The function $\sigma \in C([a, b], \mathbb{R})$ satisfies the Lipschitz condition:

$$|\sigma(x) - \sigma(y)| \leq l_2 \|x - y\|, \ l_2 > 0, \ \forall \ x, y \in C([a, b], \mathbb{R});$$

($\mathcal{B}4$) The function $\sigma \in C([a, b], \mathbb{R})$ and there exists $k > 0$ such that

$$|\sigma(x)| \leq k \|x\|, \ \forall \ x \in C([a, b], \mathbb{R}).$$

For the sake of computational convenience, we set

$$\lambda_i = \sup_{t \in [a,b]} |\omega_i(t)| > 0, \ \tilde{\lambda}_i = \sup_{t \in [a,b]} |\omega_i'(t)| > 0, \ \lambda_i^* = \sup_{t \in [a,b]} |\omega_i''(t)| > 0, \ i = 1, 2, 3,$$

$$\begin{aligned}
\mathcal{E}_1 &= \frac{\lambda_2 (b-a)^{q-1}}{|\xi|\Gamma(q)} \Big(1 - e^{\frac{-\xi}{\mu}(b-a)}\Big), \\
\mathcal{E}_2 &= \lambda_3 \Big\{ \frac{(b-a)^{q-1}}{|\mu|\Gamma(q)} \Big(1 - e^{\frac{-\xi}{\mu}(b-a)}\Big) + \frac{(b-a)^{q-1}}{|\mu|\Gamma(q)} \\
&\quad + \frac{1}{|\xi|\Gamma(q)} \sum_{i=0}^{m} |\alpha_i| (\eta_i - a)^{q-1} \Big(1 - e^{\frac{-\xi}{\mu}(\eta_i - a)}\Big) \\
&\quad + \frac{(b-a)^{q-1}}{|\xi|\Gamma(q)} \Big[(b-a) + \frac{|\mu|}{|\xi|} \Big(e^{\frac{-\xi}{\mu}(b-a)} - 1\Big)\Big]\Big\}, \\
\mathcal{E}_3 &= \frac{(b-a)^{q-1}}{|\xi|\Gamma(q)} \Big(1 - e^{\frac{-\xi}{\mu}(b-a)}\Big),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\tilde{\mathcal{E}}_1 &= \frac{\tilde{\lambda}_2 (b-a)^{q-1}}{|\xi|\Gamma(q)} \Big(1 - e^{\frac{-\xi}{\mu}(b-a)}\Big), \\
\tilde{\mathcal{E}}_2 &= \tilde{\lambda}_3 \Big\{ \frac{(b-a)^{q-1}}{|\mu|\Gamma(q)} \Big(1 - e^{\frac{-\xi}{\mu}(b-a)}\Big) + \frac{(b-a)^{q-1}}{|\mu|\Gamma(q)} \\
&\quad + \frac{1}{|\xi|\Gamma(q)} \sum_{i=0}^{m} |\alpha_i| (\eta_i - a)^{q-1} \Big(1 - e^{\frac{-\xi}{\mu}(\eta_i - a)}\Big) \\
&\quad + \frac{(b-a)^{q-1}}{|\xi|\Gamma(q)} \Big[(b-a) + \frac{|\mu|}{|\xi|} \Big(e^{\frac{-\xi}{\mu}(b-a)} - 1\Big)\Big]\Big\},
\end{aligned} \tag{3.6}$$

$$\mathcal{E}_1^* = \frac{\lambda_2^* (b-a)^{q-1}}{|\xi|\Gamma(q)} \Big(1 - e^{\frac{-\xi}{\mu}(b-a)}\Big),$$

$$
\begin{aligned}
\mathcal{E}_2^* &= \lambda_3^*\Big\{\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}\Big(1-e^{\frac{-\xi}{\mu}(b-a)}\Big)+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)} \\
&\quad+\frac{1}{|\xi|\Gamma(q)}\sum_{i=0}^{m}|\alpha_i|(\eta_i-a)^{q-1}\Big(1-e^{\frac{-\xi}{\mu}(\eta_i-a)}\Big) \\
&\quad+\frac{(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big[(b-a)+\frac{|\mu|}{|\xi|}\Big(e^{\frac{-\xi}{\mu}(b-a)}-1\Big)\Big]\Big\}, \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_1 &= \lambda_1+\frac{(b-a)^{1-p}}{\Gamma(2-p)}(\tilde{\lambda}_1+\lambda_1^*), \\
\mathcal{Q}_2 &= \mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3+\frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big(\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3 \qquad (3.8) \\
&\quad+\mathcal{E}_1^*+\mathcal{E}_2^*+\frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)}+\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)}+\Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big).
\end{aligned}
$$

Now we prove the existence of solutions for the problem (1.1)- (1.2) by applying Krasnosel'skiǐ fixed point theorem [21].

Theorem 3.1. *(Krasnosel'skiǐ fixed point theorem [21]): Let $\mathcal{M}$ be a closed, convex, bounded and nonempty subset of a Banach space X and let $\mathcal{F}_1, \mathcal{F}_2$ be the operators defined from $\mathcal{M}$ to X such that: (i) $\mathcal{F}_1x+\mathcal{F}_2y\in\mathcal{M}$ wherever $x, y\in\mathcal{M}$; (ii) $\mathcal{F}_1$ is compact and continuous; (iii) $\mathcal{F}_2$ is a contraction. Then there exists $z\in\mathcal{M}$ such that $z=\mathcal{F}_1z+\mathcal{F}_2z$.*

Theorem 3.2. *Assume that $(\mathcal{B}1)-(\mathcal{B}4)$ hold: Then the problems (1.1)-(1.2) has at least one solution on $[a,b]$ if*

$$\frac{k}{|\Lambda|}\mathcal{Q}_1<1, \quad \frac{l_2}{|\Lambda|}\mathcal{Q}_1<1,$$

where $\mathcal{Q}_1$ is defined by (3.8).

Proof. Consider a closed bounded and convex ball $S_r=\{x\in\mathcal{G}:\|x\|^*\leq r\}\subseteq\mathcal{G}$, with

$$r \;\geq\; \frac{\mathcal{Q}_2\|\phi\|}{1-\dfrac{k}{|\Lambda|}\mathcal{Q}_1},$$

where we have used $(\mathcal{B}_2)$. Define operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on S_r as

$$
\begin{aligned}
(\mathcal{H}_1x)(t) &= -\frac{\omega_2(t)}{\mu}\int_a^b e^{\frac{-\xi}{\mu}(b-s)}I_a^{q-1}\widehat{f}(x(s))ds \\
&\quad+\omega_3(t)\Big[\frac{\xi}{\mu^2}\int_a^b e^{\frac{-\xi}{\mu}(b-s)}I_a^{q-1}\widehat{f}(x(s))ds \\
&\quad-\frac{1}{\mu}I_a^{q-1}\widehat{f}(x(b))+\frac{1}{\mu}\sum_{i=0}^{m}\alpha_i\int_a^{\eta_i}e^{\frac{-\xi}{\mu}(\eta_i-s)}I_a^{q-1}\widehat{f}(x(s))ds
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{\mu}\int_a^b\Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)}I_a^{q-1}\widehat{f}(x(u))du\Big)ds\Big] \\
& +\frac{1}{\mu}\int_a^t e^{\frac{-\xi}{\mu}(t-s)}I_a^{q-1}\widehat{f}(x(s))ds,\ t\in[a,b], \\
(\mathcal{H}_2x)(t) &= \frac{\sigma(x)}{\Lambda}\omega_1(t),\ t\in[a,b].
\end{aligned}
$$

Observe that

$$(\mathcal{H}x)(t)=(\mathcal{H}_1x)(t)+(\mathcal{H}_2x)(t),\ t\in[a,b].$$

Now we show that $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy all conditions of Theorem 3.1.

(i) For $x,y\in S_r$, by using Lemma 2.5, we have

$$
\begin{aligned}
&\|\mathcal{H}_1x+\mathcal{H}_2y\|\le\|\mathcal{H}_1x\|+\|\mathcal{H}_2y\| \\
&\le \frac{kr\lambda_1}{|\Lambda|}+\frac{\lambda_2(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big(1-e^{\frac{-\xi}{\mu}(b-a)}\Big)+\lambda_3\Big\{\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}\Big(1-e^{\frac{-\xi}{\mu}(b-a)}\Big) \\
&\quad+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{1}{|\xi|\Gamma(q)}\sum_{i=0}^{m}|\alpha_i|(\eta_i-a)^{q-1}\Big(1-e^{\frac{-\xi}{\mu}(\eta_i-a)}\Big) \\
&\quad+\frac{(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big[(b-a)+\frac{|\mu|}{|\xi|}\Big(e^{\frac{-\xi}{\mu}(b-a)}-1\Big)\Big]\Big\}+\frac{(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big(1-e^{\frac{-\xi}{\mu}(b-a)}\Big) \\
&\le \frac{kr\lambda_1}{|\Lambda|}+(\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3)\|\phi\|.
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\|(\mathcal{H}_1x)'+(\mathcal{H}_2y)'\| &\le \frac{kr\tilde{\lambda}_1}{|\Lambda|}+\Big(\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3\Big)\|\phi\|, \\
\|(\mathcal{H}_1x)''+(\mathcal{H}_2y)''\| &\le \frac{kr\lambda_1^*}{|\Lambda|}+\Big(\mathcal{E}_1^*+\mathcal{E}_2^*+\frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)} \\
&\quad+\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)}+\Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big)\|\phi\|, \\
\|{}^cD^p(\mathcal{H}_1x)+{}^cD^p(\mathcal{H}_2y)\| &\le \frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big[\frac{kr\tilde{\lambda}_1}{|\Lambda|}+\Big(\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2 \\
&\quad+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3\Big)\|\phi\|\Big], \\
\|{}^cD^{p+1}(\mathcal{H}_1x)+{}^cD^{p+1}(\mathcal{H}_2y)\| &\le \frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big[\frac{kr\lambda_1^*}{|\Lambda|}+\Big(\mathcal{E}_1^*+\mathcal{E}_2^*+\frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)} \\
&\quad+\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)}+\Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big)\|\phi\|\Big].
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|\mathcal{H}_1x+\mathcal{H}_2y\|^* &= \|\mathcal{H}_1x+\mathcal{H}_2y\|+\|{}^cD^p(\mathcal{H}_1x)+{}^cD^p(\mathcal{H}_2y)\| \\
&\quad+\|{}^cD^{p+1}(\mathcal{H}_1x)+{}^cD^{p+1}(\mathcal{H}_2y)\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{kr}{|\Lambda|}\Big[\lambda_1+\frac{(b-a)^{1-p}}{\Gamma(2-p)}(\tilde{\lambda}_1+\lambda_1^*)\Big]+\|\phi\|\Big[\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3\\
&+\frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big(\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3+\mathcal{E}_1^*+\mathcal{E}_2^*\\
&+\frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)}+\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)}+\Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big)\Big]\\
&= \frac{kr}{|\Lambda|}\mathcal{Q}_1+\|\phi\|\mathcal{Q}_2\leq r,
\end{aligned}$$

which shows that $\mathcal{H}_1x+\mathcal{H}_2y\in S_r$ for all $x,y\in S_r$.

(*ii*) We prove that the operator $\mathcal{H}_2$ is a contraction. For $x,y\in S_r$, we have

$$\|\mathcal{H}_2x-\mathcal{H}_2y\| = \sup_{t\in[a,b]}|\mathcal{H}_2x(t)-\mathcal{H}_2y(t)|\leq \frac{l_2\lambda_1}{|\Lambda|}\|x-y\|^*. \tag{3.9}$$

Also, for all $t\in[a,b]$, we have

$$\begin{aligned}
\|(\mathcal{H}_2x)'-(\mathcal{H}_2y)'\| &= \sup_{t\in[a,b]}|(\mathcal{H}_2x)'(t)-(\mathcal{H}_2y)'(t)|\\
&\leq \frac{l_2\tilde{\lambda}_1}{|\Lambda|}\|x-y\|^*. \qquad (3.10)
\end{aligned}$$

In view of (3.10), we obtain

$$\begin{aligned}
\|{}^cD^p(\mathcal{H}_2x)-{}^cD^p(\mathcal{H}_2y)\| &= \sup_{t\in[a,b]}|{}^cD^p(\mathcal{H}_2x)(t)-{}^cD^p(\mathcal{H}_2y)(t)|\\
&\leq \sup_{t\in[a,b]}\Big\{\int_a^t\frac{(t-s)^{-p}}{\Gamma(1-p)}|(\mathcal{H}x)'(s)-(\mathcal{H}y)'(s)|ds\Big\}\\
&\leq \frac{l_2\tilde{\lambda}_1}{|\Lambda|}\frac{(b-a)^{1-p}}{\Gamma(2-p)}\|x-y\|^*,\ \forall\ t\in[a,b]. \qquad (3.11)
\end{aligned}$$

Similarly, one can find that

$$\|{}^cD^{p+1}(\mathcal{H}_2x)-{}^cD^{p+1}(\mathcal{H}_2y)\| \leq \frac{l_2\lambda_1^*}{|\Lambda|}\frac{(b-a)^{1-p}}{\Gamma(2-p)}\|x-y\|^*. \tag{3.12}$$

From the inequalities (3.9), (3.11) and (3.12), it follows that

$$\begin{aligned}
\|\mathcal{H}_2x-\mathcal{H}_2y\|^* &= \|\mathcal{H}_2x-\mathcal{H}_2y\|+\|{}^cD^p(\mathcal{H}_2x)-{}^cD^p(\mathcal{H}_2y)\|\\
&\quad+\|{}^cD^{p+1}(\mathcal{H}_2x)-{}^cD^{p+1}(\mathcal{H}_2y)\|\\
&\leq \frac{l_2}{|\Lambda|}\Big[\lambda_1+\frac{(b-a)^{1-p}}{\Gamma(2-p)}(\tilde{\lambda}_1+\lambda_1^*)\Big]\|x-y\|^*\\
&= \frac{l_2}{|\Lambda|}\mathcal{Q}_1\|x-y\|^*,
\end{aligned}$$

for all $x,y\in S_r$, with $\frac{l_2}{|\Lambda|}\mathcal{Q}_1<1$. This shows that $\mathcal{H}_2$ is a contraction.
(*iii*) $\mathcal{H}_1$ is compact.

Continuity of f implies that the operator $\mathcal{H}_1$ is continuous on S_r. Also, the operator $\mathcal{H}_1$ is uniformly bounded on S_r as

$$\begin{aligned}
\|\mathcal{H}_1 x\| &\leq (\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3)\|\phi\|,\\
\|{}^cD^p(\mathcal{H}_1 x)\| &\leq \frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big(\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3\Big)\|\phi\|,\\
\|{}^cD^{p+1}(\mathcal{H}_1 x)\| &\leq \frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big(\mathcal{E}_1^*+\mathcal{E}_2^*+\frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)}+\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)}\\
&\quad +\Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big)\|\phi\|.
\end{aligned}$$

In consequence, we get

$$\begin{aligned}
\|\mathcal{H}_1 x\|^* &\leq \Big[\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3+\frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big(\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3+\mathcal{E}_1^*\\
&\quad +\mathcal{E}_2^*+\frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)}+\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)}+\Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big)\Big]\|\phi\|\\
&= \mathcal{Q}_2\|\phi\|,\ \forall x\in S_r.
\end{aligned}$$

Now we prove that $\mathcal{H}_1$ is equicontinuous on S_r. Let $t_1,\ t_2\in[a,b]$ with $t_1<t_2$ and $x\in S_r$. Then we have

$$\begin{aligned}
&|\mathcal{H}_1 x(t_2)-\mathcal{H}_1 x(t_1)|\\
\leq\ & \frac{|\omega_2(t_2)-\omega_2(t_1)|}{|\mu|}\Big|\int_a^b e^{\frac{-\xi}{\mu}(b-s)}I_a^{q-1}\widehat{f}(x(s))ds\Big|\\
&+|\omega_3(t_2)-\omega_3(t_1)|\Big[\frac{|\xi|}{\mu^2}\Big|\int_a^b e^{\frac{-\xi}{\mu}(b-s)}I_a^{q-1}\widehat{f}(x(s))ds\Big|\\
&+\frac{1}{|\mu|}|I_a^{q-1}\widehat{f}(x(b))|+\frac{1}{|\mu|}\sum_{i=0}^m|\alpha_i|\Big|\int_a^{\eta_i}e^{\frac{-\xi}{\mu}(\eta_i-s)}I_a^{q-1}\widehat{f}(x(s))ds\Big|\\
&+\frac{1}{|\mu|}\Big|\int_a^b\Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)}I_a^{q-1}\widehat{f}(x(u))du\Big)ds\Big|\Big]\\
&+\frac{1}{|\mu|}\Big\{\Big|\int_a^{t_1}\Big(e^{\frac{-\xi}{\mu}(t_2-s)}-e^{\frac{-\xi}{\mu}(t_1-s)}\Big)I_a^{q-1}\widehat{f}(x(s))ds\Big|\\
&+\Big|\int_{t_1}^{t_2}e^{\frac{-\xi}{\mu}(t_2-s)}I_a^{q-1}\widehat{f}(x(s))ds\Big|\Big\}\\
\leq\ & \|\phi\|\Big\{\frac{|\omega_2(t_2)-\omega_2(t_1)|(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big(1-e^{\frac{-\xi}{\mu}(b-a)}\Big)\\
&+|\omega_3(t_2)-\omega_3(t_1)|\Big[\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}\Big(1-e^{\frac{-\xi}{\mu}(b-a)}\Big)\\
&+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{1}{|\xi|\Gamma(q)}\sum_{i=0}^m|\alpha_i|(\eta_i-a)^{q-1}\Big(1-e^{\frac{-\xi}{\mu}(\eta_i-a)}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big((b-a)+\frac{|\mu|}{|\xi|}\Big(e^{\frac{-\xi}{\mu}(b-a)}-1\Big)\Big)\Big]\Big\}\\
&+\frac{\|\phi\|}{|\mu|}\Big\{\Big|\int_a^{t_1}\Big(e^{\frac{-\xi}{\mu}(t_2-s)}-e^{\frac{-\xi}{\mu}(t_1-s)}\Big)\frac{(s-a)^{q-1}}{\Gamma(q)}ds\Big|\\
&+\Big|\int_{t_1}^{t_2}e^{\frac{-\xi}{\mu}(t_2-s)}\frac{(s-a)^{q-1}}{\Gamma(q)}ds\Big|\Big\}\\
\leq\;&\|\phi\|\Big\{\frac{|\omega_2(t_2)-\omega_2(t_1)|(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big(1-e^{\frac{-\xi}{\mu}(b-a)}\Big)\\
&+|\omega_3(t_2)-\omega_3(t_1)|\Big[\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}\Big(1-e^{\frac{-\xi}{\mu}(b-a)}\Big)\\
&+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{1}{|\xi|\Gamma(q)}\sum_{i=0}^{m}|\alpha_i|(\eta_i-a)^{q-1}\Big(1-e^{\frac{-\xi}{\mu}(\eta_i-a)}\Big)\\
&+\frac{(b-a)^{q-1}}{|\xi|\Gamma(q)}\Big((b-a)+\frac{|\mu|}{|\xi|}\Big(e^{\frac{-\xi}{\mu}(b-a)}-1\Big)\Big)\Big]\Big\}\\
&+\frac{\|\phi\|}{|\xi|\Gamma(q)}\Big\{(t_1-a)^{q-1}\Big(e^{\frac{-\xi}{\mu}(t_2-t_1)}-1-e^{\frac{-\xi}{\mu}(t_2-a)}\\
&+e^{\frac{-\xi}{\mu}(t_1-a)})+(t_2-a)^{q-1}(1-e^{\frac{-\xi}{\mu}(t_2-t_1)}\Big)\Big\}. \qquad (3.13)
\end{aligned}
$$

In addition, we have

$$
|(\mathcal{H}_1x)'(t)| \;\leq\; \|\phi\|\Big(\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3\Big).
$$

Thus,

$$
\begin{aligned}
&|^cD^p\mathcal{H}_1x(t_2)-^cD^p\mathcal{H}_1x(t_1)|\\
\leq\;&\Big|\int_a^{t_1}\frac{(t_2-s)^{-p}-(t_1-s)^{-p}}{\Gamma(1-p)}(\mathcal{H}_1x)'(s)ds\Big|\\
&+\Big|\int_{t_1}^{t_2}\frac{(t_2-s)^{-p}}{\Gamma(1-p)}(\mathcal{H}_1x)'(s)ds\Big|\Big]\\
\leq\;&\frac{\|\phi\|}{\Gamma(2-p)}\Big(\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3\Big)\Big[|(t_2-a)^{1-p}\\
&-(t_1-a)^{1-p}|+2(t_2-t_1)^{1-p}\Big]. \qquad (3.14)
\end{aligned}
$$

Similarly, we can find that

$$
|(\mathcal{H}_1x)''(t)| \;\leq\; \|\phi\|\Big(\mathcal{E}_1^*+\mathcal{E}_2^*+\frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)}+\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)}+\Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big),
$$

and thus

$$
|^cD^{p+1}\mathcal{H}_1x(t_2)-^cD^{p+1}\mathcal{H}_1x(t_1)|
$$

$$
\begin{aligned}
&\le \Big|\int_a^{t_1} \frac{(t_2-s)^{-p}-(t_1-s)^{-p}}{\Gamma(1-p)}(\mathcal{H}_1x)''(s)ds\Big| \\
&\quad +\Big|\int_{t_1}^{t_2} \frac{(t_2-s)^{-p}}{\Gamma(1-p)}(\mathcal{H}_1x)''(s)ds\Big| \\
&\le \frac{\|\phi\|}{\Gamma(2-p)}\Big(\mathcal{E}_1^*+\mathcal{E}_2^*+\frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)}+\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)} \\
&\quad +\Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big)\Big[|(t_2-a)^{1-p}-(t_1-a)^{1-p}|+2(t_2-t_1)^{1-p}\Big]. \qquad (3.15)
\end{aligned}
$$

The right hand sides of the inequalities (3.13)-(3.15) tend to zero as $t_2 - t_1 \longrightarrow 0$ independent of x. Thus, the operator $\mathcal{H}_1$ is equicontinuous on S_r. Therefore, by the Arzelá-Ascoli theorem, $\mathcal{H}_1$ is a relatively compact on S_r. So, all conditions of Theorem 3.1 are satisfied, which implies that there exists a fixed point of operator $\mathcal{H}$. Therefore, the problem (1.1)-(1.2) has at least one solution on $[a, b]$. The proof is completed. □

Example 3.3. Consider the fractional boundary value problem.

$$
\left\{
\begin{array}{l}
4\ {}^cD^{\frac{19}{6}}x(t) + 7\ {}^cD^{\frac{13}{6}}x(t) = f(t, x(t), {}^cD^{\frac{1}{3}}x(t), {}^cD^{\frac{4}{3}}x(t)),\ t \in (0,1), \\
x(0) = \sigma(x),\ x'(0) = 0,\ x(1) = 0,\ x'(1) = \displaystyle\sum_{i=1}^{4}\alpha_i x(\eta_i) + \int_0^1 x(s)ds,
\end{array}
\right. \qquad (3.16)
$$

where $q = 19/6,\ p = 1/3,\ a = 0,\ b = 1,\ \mu = 4,\ \xi = 7,\ \alpha_1 = -2,\ \alpha_2 = -5/4,\ \alpha_3 = -1/12,\ \alpha_4 = 2/39,\ \eta_1 = 1/8,\ \eta_2 = 1/4,\ \eta_3 = 1/2,\ \eta_4 = 3/4$. Using the given data, we find that $\Lambda \simeq -0.39151 \neq 0,\ \mathcal{Q}_1 \simeq 3.10393,\ \mathcal{Q}_2 \simeq 2.51497$. Consider $\sigma(x) = \dfrac{|x(\frac{4}{5})|}{23\big(1+|x(\frac{4}{5})|\big)}$ and

$$
\begin{aligned}
f(t, x(t), {}^cD^{\frac{1}{3}}x(t), {}^cD^{\frac{4}{3}}x(t)) &= \frac{\cos t}{t^2+8}\Big(\frac{(x(t)+1)^2}{3+(x(t)+1)^2}\Big) + \sin^2({}^cD^{\frac{1}{3}}x(t)) \\
&\quad + \frac{|{}^cD^{\frac{4}{3}}x(t)|}{2(1+|{}^cD^{\frac{4}{3}}x(t)|)}.
\end{aligned}
$$

For the above functions we have $k = l_2 = 1/23$ and $|f(t, x_1, x_2, x_3)| \le \phi(t)$, for all $t \in [0, 1]$ and $x_i \in \mathbb{R}, i = 1, 2, 3$, with $\phi(t) = \dfrac{|\cos t|}{(t^2+8)} + \dfrac{3}{2},\ t \in [0, 1]$. Furthermore, we obtain $\dfrac{k\mathcal{Q}_1}{|\Lambda|} = \dfrac{l_2\mathcal{Q}_1}{|\Lambda|} \simeq 0.344700 < 1$. Thus, all conditions of Theorem 3.2 are satisfied, so we conclude that the problem (3.16) has at least one solution on $[0, 1]$.

4. Uniqueness of solutions

Next, we prove the uniqueness of solutions for the problem (1.1)-(1.2) via Banach's fixed point theorem.

Theorem 4.1. *Assume that* $(\mathcal{B}1)$ *and* $(\mathcal{B}3)$ *hold. Then the boundary value problem (1.1)-(1.2) has a unique solution on* $[a,b]$ *if*

$$\frac{l_2}{|\Lambda|}\mathcal{Q}_1 + l_1\mathcal{Q}_2 < 1, \tag{4.1}$$

where $\mathcal{Q}_1,\ \mathcal{Q}_2$ *are given by (3.8).*

Proof. Setting $\sup_{t\in[a,b]}|f(t,0,0,0)| = \mathcal{N} < \infty$, $|\sigma(0)| = \sigma_0$, and selecting

$$r^* \quad \geq \quad \frac{\frac{\sigma_0}{|\Lambda|}\mathcal{Q}_1 + \mathcal{N}\mathcal{Q}_2}{1 - \Big(\frac{l_2}{|\Lambda|}\mathcal{Q}_1 + l_1\mathcal{Q}_2\Big)},$$

we define $S_{r^*} = \{x \in \mathcal{A} : \|x\|^* \leq r^*\}$, and show that $\mathcal{H}S_{r^*} \subset S_{r^*}$, where the operator $\mathcal{H}$ is defined by (3.3). For $x \in S_{r^*}$, and using the norm given by (3.1), we find that

$$\begin{aligned}
|\widehat{f}(x(t))| &= |f(t,x(t),{}^cD^px(t),{}^cD^{p+1}x(t))| \\
&\leq |f(t,x(t),{}^cD^px(t),{}^cD^{p+1}x(t)) - f(t,0,0,0)| + |f(t,0,0,0)| \\
&\leq l_1(|x(t)| + |{}^cD^px(t)| + |{}^cD^{p+1}x(t)|) + \mathcal{N} \\
&\leq l_1\|x\|^* + \mathcal{N} \leq l_1 r^* + \mathcal{N},
\end{aligned}$$

and

$$|\sigma(x)| = |\sigma(x)-\sigma(0)+\sigma(0)| \leq |\sigma(x)-\sigma(0)|+|\sigma(0)| \leq l_2\|x\|+\sigma_0 \leq l_2\|x\|^*+\sigma_0.$$

Then we have

$$\begin{aligned}
|\mathcal{H}x(t)| &\leq \frac{|\sigma(x)|}{|\Lambda|}|\omega_1(t)| + \frac{|\omega_2(t)|}{|\mu|}\Big|\int_a^b e^{\frac{-\xi}{\mu}(b-s)}I_a^{q-1}\widehat{f}(x(s))ds\Big| \\
&\quad +|\omega_3(t)|\Big[\frac{|\xi|}{\mu^2}\Big|\int_a^b e^{\frac{-\xi}{\mu}(b-s)}I_a^{q-1}\widehat{f}(x(s))ds\Big| + \frac{1}{|\mu|}|I_a^{q-1}\widehat{f}(x(b))| \\
&\quad +\frac{1}{|\mu|}\sum_{i=0}^{m}|\alpha_i|\Big|\int_a^{\eta_i} e^{\frac{-\xi}{\mu}(\eta_i-s)}I_a^{q-1}\widehat{f}(x(s))ds\Big| \\
&\quad +\frac{1}{|\mu|}\Big|\int_a^b\Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)}I_a^{q-1}\widehat{f}(x(u))du\Big)ds\Big|\Big] \\
&\quad +\frac{1}{|\mu|}\Big|\int_a^t e^{\frac{-\xi}{\mu}(t-s)}I_a^{q-1}\widehat{f}(x(s))ds\Big| \\
&\leq \frac{(l_2r^*+\sigma_0)\lambda_1}{|\Lambda|} + (\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3)(l_1r^*+\mathcal{N}),
\end{aligned}$$

which, on taking the supremum for $t \in [a,b]$, yields

$$\|\mathcal{H}x\| \leq \frac{(l_2r^*+\sigma_0)\lambda_1}{|\Lambda|} + (\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3)(l_1r^*+\mathcal{N}). \tag{4.2}$$

Furthermore, we can find that

$$\|(\mathcal{H}x)'\| \leq \frac{(l_2 r^* + \sigma_0)\tilde{\lambda}_1}{|\Lambda|} + \Big(\tilde{\mathcal{E}}_1 + \tilde{\mathcal{E}}_2 + \frac{(b-a)^{q-1}}{|\mu|\Gamma(q)} + \frac{|\xi|}{|\mu|}\mathcal{E}_3\Big)(l_1 r^* + \mathcal{N}),$$

which implies that

$$\begin{aligned} \|^cD^p\mathcal{H}x\| &= \sup_{t\in[a,b]} |^cD^p\mathcal{H}x(t)| \leq \int_a^t \frac{(t-s)^{-p}}{\Gamma(1-p)}|(\mathcal{H}x)'(s)|ds \\ &\leq \frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big[\frac{(l_2 r^* + \sigma_0)\tilde{\lambda}_1}{|\Lambda|} \\ &\quad + \Big(\tilde{\mathcal{E}}_1 + \tilde{\mathcal{E}}_2 + \frac{(b-a)^{q-1}}{|\mu|\Gamma(q)} + \frac{|\xi|}{|\mu|}\mathcal{E}_3\Big)(l_1 r^* + \mathcal{N})\Big]. \end{aligned} \tag{4.3}$$

Similarly, we have

$$\begin{aligned} \|^cD^{p+1}\mathcal{H}x\| &= \sup_{t\in[a,b]} |^cD^{p+1}\mathcal{F}x(t)| \leq \int_a^t \frac{(t-s)^{-p}}{\Gamma(1-p)}|(\mathcal{H}x)''(s)|ds \\ &\leq \frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big[\frac{(l_2 r^* + \sigma_0)\lambda_1^*}{|\Lambda|} + \Big(\mathcal{E}_1^* + \mathcal{E}_2^* + \frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)} \\ &\quad + \frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)} + \Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big)(l_1 r^* + \mathcal{N})\Big]. \end{aligned} \tag{4.4}$$

From the inequalities (4.2)-(4.4), it follows that

$$\begin{aligned} \|\mathcal{H}x\|^* &= \|\mathcal{H}x\| + \|^cD^p\mathcal{H}x\| + \|^cD^{p+1}\mathcal{H}x\| \\ &\leq \frac{(l_2 r^* + \sigma_0)}{|\Lambda|}\Big(\lambda_1 + \frac{(b-a)^{1-p}}{\Gamma(2-p)}(\tilde{\lambda}_1 + \lambda_1^*)\Big) + \Big[\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 \\ &\quad + \frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big(\tilde{\mathcal{E}}_1 + \tilde{\mathcal{E}}_2 + \frac{(b-a)^{q-1}}{|\mu|\Gamma(q)} + \frac{|\xi|}{|\mu|}\mathcal{E}_3 + \mathcal{E}_1^* + \mathcal{E}_2^* \\ &\quad + \frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)} + \frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)} + \Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big)\Big](l_1 r^* + \mathcal{N}) \\ &= \frac{(l_2 r^* + \sigma_0)}{|\Lambda|}\mathcal{Q}_1 + \mathcal{Q}_2(l_1 r^* + \mathcal{N}) < r^*. \end{aligned}$$

Thus, we conclude that $\mathcal{H}$ maps S_{r^*} into itself for any $x \in S_{r^*}$. Therefore, $\mathcal{H}S_{r^*} \subset S_{r^*}$.

Now we show that $\mathcal{H}$ is a contraction. For $x, y \in \mathcal{G}$ and $t \in [a,b]$, we obtain

$$\begin{aligned} &\Big|(\mathcal{H}x)(t) - (\mathcal{H}y)(t)\Big| \\ \leq\; &\frac{|\sigma(x) - \sigma(y)|}{|\Lambda|}|\omega_1(t)| + \frac{|\omega_2(t)|}{|\mu|}\Big|\int_a^b e^{\frac{-\xi}{\mu}(b-s)}|I_a^{q-1}\widehat{f}(x(s)) - I_a^{q-1}\widehat{f}(y(s))|ds\Big| \\ &+|\omega_3(t)|\Big[\frac{|\xi|}{\mu^2}\Big|\int_a^b e^{\frac{-\xi}{\mu}(b-s)}|I_a^{q-1}\widehat{f}(x(s)) - I_a^{q-1}\widehat{f}(y(s))|ds\Big| \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\mu|}|I_a^{q-1}\widehat{f}(x(b))-I_a^{q-1}\widehat{f}(y(b))|\\
&+\frac{1}{|\mu|}\sum_{i=0}^{m}|\alpha_i|\Big|\int_a^{\eta_i}e^{\frac{-\xi}{\mu}(\eta_i-s)}|I_a^{q-1}\widehat{f}(x(s))\\
&-I_a^{q-1}\widehat{f}(y(s))|ds\Big|+\frac{1}{|\mu|}\Big|\int_a^b\Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)}|I_a^{q-1}\widehat{f}(x(u))\\
&-I_a^{q-1}\widehat{f}(y(u))|du\Big)ds\Big|\Big]+\frac{1}{|\mu|}\Big|\int_a^t e^{\frac{-\xi}{\mu}(t-s)}|I_a^{q-1}\widehat{f}(x(s))-I_a^{q-1}\widehat{f}(y(s))|ds\Big|\\
\leq\;&\frac{l_2\|x-y\|}{|\Lambda|}|\omega_1(t)|+\frac{|\omega_2(t)|(b-a)^{q-1}}{|\mu|\Gamma(q)}\|\widehat{f}(x)-\widehat{f}(y)\|\Big|\int_a^b e^{\frac{-\xi}{\mu}(b-s)}ds\Big|\\
&+|\omega_3(t)|\Big[\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)}\|\widehat{f}(x)-\widehat{f}(y)\|\Big|\int_a^b e^{\frac{-\xi}{\mu}(b-s)}ds\Big|\\
&+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}\|\widehat{f}(x)-\widehat{f}(y)\|\\
&+\frac{1}{|\mu|\Gamma(q)}\|\widehat{f}(x)-\widehat{f}(y)\|\sum_{i=0}^{m}|\alpha_i|(\eta_i-a)^{q-1}\Big|\int_a^{\eta_i}e^{\frac{-\xi}{\mu}(\eta_i-s)}ds\Big|\\
&+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}\|\widehat{f}(x)-\widehat{f}(y)\|\Big|\int_a^b\Big(\int_a^s e^{\frac{-\xi}{\mu}(s-u)}du\Big)ds\Big|\Big]\\
&+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}\|\widehat{f}(x)-\widehat{f}(y)\|\Big|\int_a^t e^{\frac{-\xi}{\mu}(t-s)}ds\Big|.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\|\widehat{f}(x)-\widehat{f}(y)\| &= \sup_{t\in[a,b]}|\widehat{f}(x(s))-\widehat{f}(y(s))|\\
&\leq l_1(|x(s)-y(s)|+|{}^cD^p(x(s))-{}^cD^p(y(s))|\\
&\quad+|{}^cD^{p+1}(x(s))-{}^cD^{p+1}(y(s))|)\\
&\leq l_1(\|x-y\|+\|{}^cD^px-{}^cD^py\|+\|{}^cD^{p+1}x-{}^cD^{p+1}y\|)\\
&\leq l_1\|x-y\|^*,\ \forall\ s\in[a,b],
\end{aligned}
$$

which implies that

$$
\|\mathcal{H}x-\mathcal{H}y\| \;\leq\; \Big[\frac{l_2\lambda_1}{|\Lambda|}+l_1(\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3)\Big]\|x-y\|^*.
$$

Also, for all $t\in[a,b]$, we have

$$
\|(\mathcal{H}x)'-(\mathcal{H}y)'\|\leq\Big[\frac{l_2\tilde{\lambda}_1}{|\Lambda|}+l_1\Big(\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3\Big)\Big]\|x-y\|^*,
$$

which implies that

$$
\|{}^cD^p\mathcal{H}x-{}^cD^p\mathcal{H}y\|
$$

$$
\begin{aligned}
&= \sup_{t\in[a,b]} |{}^cD^p\mathcal{H}x(t) -{}^c D^p\mathcal{H}y(t)| \\
&\le \int_a^t \frac{(t-s)^{-p}}{\Gamma(1-p)}|(\mathcal{H}x)'(s)-(\mathcal{H}y)'(s)|ds \\
&\le \frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big[\frac{l_2\tilde{\lambda}_1}{|\Lambda|}+l_1\Big(\tilde{\mathcal{E}}_1+\tilde{\mathcal{E}}_2+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3\Big)\Big]\|x-y\|^*.
\end{aligned}
$$

In a similar manner, we have

$$
\begin{aligned}
& \|{}^cD^{p+1}\mathcal{H}x -{}^c D^{p+1}\mathcal{H}y\| \\
=\;& \sup_{t\in[a,b]} |{}^cD^{p+1}\mathcal{H}x(t) -{}^c D^{p+1}\mathcal{H}y(t)| \\
\le\;& \int_a^t \frac{(t-s)^{-p}}{\Gamma(1-p)}|(\mathcal{H}x)''(s)-(\mathcal{H}y)''(s)|ds \\
\le\;& \frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big[\frac{l_2\lambda_1^*}{|\Lambda|}+l_1\Big(\mathcal{E}_1^*+\mathcal{E}_2^*+\frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)}+\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)} \\
& +\Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big)\Big]\|x-y\|^*.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
& \|(\mathcal{H}x)-(\mathcal{H}y)\|^* \\
\le\;& \Big[\frac{l_2}{|\Lambda|}\Big(\lambda_1+\frac{(b-a)^{1-p}}{\Gamma(2-p)}(\tilde{\lambda}_1+\lambda_1^*)\Big)+l_1\Big(\mathcal{E}_1+\mathcal{E}_2+\mathcal{E}_3+\frac{(b-a)^{1-p}}{\Gamma(2-p)}\Big(\tilde{\mathcal{E}}_1 \\
& +\tilde{\mathcal{E}}_2+\frac{(b-a)^{q-1}}{|\mu|\Gamma(q)}+\frac{|\xi|}{|\mu|}\mathcal{E}_3+\mathcal{E}_1^*+\mathcal{E}_2^*+\frac{(b-a)^{q-2}}{|\mu|\Gamma(q-1)}+\frac{|\xi|(b-a)^{q-1}}{\mu^2\Gamma(q)} \\
& +\Big(\frac{\xi}{\mu}\Big)^2\mathcal{E}_3\Big)\Big)\Big]\|x-y\|^* \\
=\;& \Big(\frac{l_2}{|\Lambda|}\mathcal{Q}_1+l_1\mathcal{Q}_2\Big)\|x-y\|^*.
\end{aligned}
$$

Thus, by using (4.1), we deduce that the operator $\mathcal{H}$ is a contraction. Therefore, by applying Banach's fixed point theorem we conclude that the boundary value problem (1.1)-(1.2) has a unique solution on $[a,b]$, which completes the proof. □

Example 4.2. Consider the following fractional differential equation

$$
4\;{}^cD^{\frac{19}{6}}x(t)+7\;{}^cD^{\frac{13}{6}}x(t)=f(t,x(t),{}^cD^{\frac{1}{3}}x(t),{}^cD^{\frac{4}{3}}x(t)),\;\; t\in(0,1), \tag{4.5}
$$

supplemented with the boundary conditions of Example (3.3), and let $\sigma(x)=\frac{1}{19}\sin x\Big(\frac{2}{3}\Big)$,

$$
f(t,x(t),{}^cD^{\frac{1}{3}}x(t),{}^cD^{\frac{4}{3}}x(t)) = \frac{1}{11\sqrt{t^4+9}}\big(\arctan x(t)+{}^cD^{\frac{1}{3}}x(t)\big)
$$

$$+\frac{\cos({}^cD^{\frac{4}{3}}x(t))}{33}.$$

Obviously

$$|f(t,x(t),{}^cD^{\frac{1}{3}}x(t),{}^cD^{\frac{4}{3}}x(t)) - f(t,y(t),{}^cD^{\frac{1}{3}}y(t),{}^cD^{\frac{4}{3}}y(t))|$$
$$\leq \frac{1}{33}\Big(|x-y| + |{}^cD^{\frac{1}{3}}x - {}^cD^{\frac{1}{3}}y| + |{}^cD^{\frac{4}{3}}x - {}^cD^{\frac{4}{3}}y|\Big) \leq \frac{1}{33}\|x-y\|,$$

with $l_1 = 1/33$ and from the inequality

$$|\sigma(x(t)) - \sigma(y(t))| \leq \frac{1}{19}\|x-y\|,$$

we have $l_2 = 1/19$ for all $t \in [0,1]$ and $x, y \in \mathbb{R}$. In addition, we obtain $\dfrac{l_2\mathcal{Q}_1}{|\Lambda|} + l_1\mathcal{Q}_2 \simeq 0.493479 < 1$. Therefore, all conditions of Theorem 4.1 are satisfied, and we conclude that there exists a unique solution on $[0,1]$ for the problem (4.5).

5. Conclusions

We have presented the existence and uniqueness criteria for solutions of a sequential Caputo fractional differential equation complemented with nonlocal integro multipoint boundary conditions. In the first step, we convert the given nonlinear problem into a fixed point problem. Once the fixed point operator is available, we make use of Krasnosel'skiĭ's fixed point theorem to obtain an existence result for the problem at hand, while the uniqueness result is established by applying the the contraction mapping principle. Our results are new in the given configuration and enrich the literature on boundary value problems involving sequential fractional differential equations.

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