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Coverings with horo- and hyperballs generated by simply truncated orthoschmes

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Abstract. After having investigated the packings derived by horo- and hyperballs related to simple frustum Coxeter orthoscheme tilings, we consider the corresponding covering problems (briefly hyp-hor coverings) in n-dimensional hyperbolic spaces $\mathbb{H}^n$ $(n = 2, 3)$.

In the $2-$ and $3-$dimensional hyperbolic spaces we construct hyp-hor coverings generated by simply truncated Coxeter orthochemes, and we determine their thinnest covering configurations and their densities.

We prove, that in the hyperbolic plane $(n = 2)$ the density of the above thinnest hyp-hor covering arbitrarily approximates the universal lower bound of the hypercycle or horocycle covering density $\frac{\sqrt{12}}{\pi}$, and in $\mathbb{H}^3$ the optimal configuration belongs to the $\{7, 3, 6\}$ Coxeter tiling with density ≈ 1.27297, that is less than the previously known famous horosphere covering density 1.280 due to L. Fejes Tóth and K. Böröczky.

Moreover, we study the hyp-hor coverings in truncated orthoschemes $\{p, 3, 6\}$ $(6 < p < 7,\ p \in \mathbb{R})$, whose density function attains its minimum at parameter $p \approx 6.45962$, with density ≈ 1.26885. That means, that this locally optimal hyp-hor configuration provide smaller covering density than the former determined ≈ 1.27297, but this hyp-hor packing configuration can not be extended to the entire hyperbolic space $\mathbb{H}^3$.

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1. Introduction

The packing and covering problems with solely horo- or hyperballs (horo- or hypespheres) are intensively investigated in earlier works in n-dimensional $(n \geq 2)$ hyperbolic space $\mathbb{H}^n$.

In n-dimensional hyperbolic space $\mathbb{H}^n$ $(n \geq 2)$ there are 3 kinds of "balls (spheres)": the classical balls (spheres), horoballs (horospheres) and hyperballs (hyperspheres).

In this paper we consider the coverings with horo- and hyperballs and their densities in 2- and 3-dimensional hyperbolic space, where the coverings are derived from simply truncated Coxeter orthoscheme tilings.

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A Coxeter simplex is an n-dimensional simplex in $X \in \{\mathbb{S}^n, \mathbb{H}^n, \mathbb{E}^n\}$ with dihedral angles either submultiples of π or zero. The group generated by reflections on the sides of a Coxeter simplex is called a Coxeter simplex reflection group. Such reflections determine a discrete group of isometries of X with the Coxeter simplex as its fundamental domain; hence such groups generate a tessellation of X.

First we shortly survey the previous results related to this topic.

1. **On horoball packings and coverings**

   In the case of periodic ball or horoball packings and coverings, the local density defined e.g. in [3] can be extended to the entire hyperbolic space. This local density is related to the simplicial density function, that was generalized in [18] and [19]. In this paper we will use such definition of covering density.

   In the n-dimensional space $X \in \{\mathbb{E}^n, \mathbb{S}^n, \mathbb{H}^n\}$ of constant curvature ($n \geq 2$), define the simplicial density function $d_n(r)$ to be the density of $n+1$ spheres of radius r mutually touching one another with respect to the regular simplex spanned by the centers of the spheres. L. Fejes Tóth and H. S. M. Coxeter conjectured, that the packing density of balls of radius r in X cannot exceed $d_n(r)$. Rogers [13] proved this conjecture in Euclidean space $\mathbb{E}^n$. The 2-dimensional spherical case was settled by L. Fejes Tóth [6], and Böröczky [3], who proved the following extension:

   Theorem 1.1 (K. Böröczky)**.** *In an n-dimensional space of constant curvature, consider a packing of spheres of radius r. In the case of spherical space, assume that $r < \frac{\pi}{4}$. Then the density of each sphere in its Dirichlet–Voronoi cell cannot exceed the density of $n+1$ spheres of radius r mutually touching one another with respect to the simplex spanned by their centers.*

   In hyperbolic space $\mathbb{H}^3$, the monotonicity of $d_3(r)$ was proved by Böröczky and Florian in [4].

   This upper bound for packing density in hyperbolic space $\mathbb{H}^3$ is ≈ 0.85327, which is not realized by packing regular balls. However, it is attained by a horoball packing of $\overline{\mathbb{H}}^3$, where the ideal centers of horoballs lie on the absolute figure of $\overline{\mathbb{H}}^3$; for example, they may lie at the vertices of the ideal regular simplex tiling with Coxeter-Schläfli symbol $\{3,3,6\}$. The known least dense ball or horoball covering configuration with density ≈ 1.280 can be derived from this regular ideal tetrahedron tiling (see [6]).

   The authors proved in [9], that the optimal ball packing arrangement in $\overline{\mathbb{H}}^3$, mentioned above, is not unique, and gave several new examples of horoball packing arrangements based on totally asymptotic Coxeter tilings, that yield the Böröczky–Florian upper bound [4].

   Furthermore, in [18], [19] the author found, that by allowing horoballs of different types at each vertex of a totally asymptotic simplex and generalizing the simplicial density function to $\mathbb{H}^n$ $(n \geq 2)$, the Böröczky-type density upper bound is no longer valid for the fully asymptotic simplices for $n \geq 3$. For example, in $\overline{\mathbb{H}}^4$ the locally optimal packing density is ≈ 0.77038, higher than the Böröczky-type density upper bound ≈ 0.73046. However these ball packing configurations are only locally optimal and cannot be extended to the entirety of the hyperbolic spaces $\overline{\mathbb{H}}^n$. Further open problems and conjectures on 4-dimensional hyperbolic packings are discussed in [5] and [1]. Using horoball packings in $\mathbb{H}^4$, allowing horoballs of different types, the authors in [10] found seven counterexamples (realized by allowing up to three horoball types) to one of L. Fejes Tóth's conjectures stated in his foundational book Regular Figures.

Continuing the investigations of ball packings in hyperbolic spaces of dimensions $n = 4, \dots, 9$, using horoball packings, allowing horoballs of different types when applicable, the authors found several interesting and dense packing configuratons with respect to the Coxeter simplex cells in [10], [11] and [12].

The second-named author has several additional results on globally and locally optimal ball packings in the eight Thurston geometries arising from Thurston's geometrization conjecture see e.g. [17], [20].

2. On hyperball packings and coverings

In hyperbolic plane $\mathbb{H}^2$ the universal upper bound of the congruent hypercycle packing density is $\frac{3}{\pi}$, proved by I. Vermes in [30]. He initiated this topic and determined also the universal lower bound of the congruent hypercycle covering density in [31], equal to $\frac{\sqrt{12}}{\pi}$.

The author analyzed in [15], [16] and [22] the regular prism tilings (simple truncated Coxeter orthoscheme tilings) and the corresponding optimal hyperball packings in $\mathbb{H}^n$ $(n = 3, 4, 5)$. Recently (to the best of authors' knowledge) these have been the densest packings with congruent hyperballs.

The n-dimensional hyperbolic regular prism honeycombs and the corresponding coverings by congruent hyperballs were studied in [21], and the author determined their least dense covering. Furthermore, there were formulated conjectures for the candidates of the least dense covering by congruent hyperballs in the 3- and 5-dimensional hyperbolic space.

Congruent and non-congruent hyperball packings to the truncated regular tetrahedron tilings were discussed in [24]. These are derived from the truncated Coxeter simplex tilings $\{3, 3, p\}$ $(7 \leq p \in \mathbb{N})$ and $\{3, 3, 3, 3, 5\}$ in 3- and 5-dimensional hyperbolic space, respectively. The author determined the densest packing arrangement and its density with congruent hyperballs in $\mathbb{H}^5$, and determined the smallest density upper bounds of non-congruent hyperball packings generated by the above tilings.

Packings by horo- and hyperballs (briefly hyp-hor packings) in $\mathbb{H}^n$ $(n = 2, 3)$ were considered in [23].

The author studied in [25] a large class of hyperball packings in $\mathbb{H}^3$, that can be derived from truncated tetrahedron tilings. It is proved, that if the truncated tetrahedron is regular $\{3, 3, p\}$, but allowing also $6 < p \in \mathbb{R}$, then the density of the locally densest packing is ≈ 0.86338. This is larger than the Böröczky-Florian density upper bound, but the locally optimal hyperball packing configuration obtained in [25] cannot be extended to the entirety of $\mathbb{H}^3$. However, a hyperball packing construction was described by the regular truncated tetrahedron tiling under the extended Coxeter group $\{3, 3, 7\}$, with maximal density ≈ 0.82251.

The author developed a decomposition algorithm in [28] that for each saturated hyperball packing provides a decomposition of $\mathbb{H}^3$ into truncated tetrahedra. Therefore, in order to get a density upper bound for hyperball packings, it is sufficient to determine the density upper bound of hyperball packings in truncated simplices.

It is proved, that the density upper bound of the saturated congruent hyperball packings, related to corresponding truncated tetrahedron cells is locally realized in a regular truncated tetrahedon with density ≈ 0.86338 ([26]). Furthermore, the author in [26] proved, that the density of locally optimal congruent hyperball arrangement in regular truncated tetrahedron is not monotonically increasing function of the height of corresponding optimal hyperball, contrary to the ball and horoball packings.

Hyperball packings related to truncated regular cube and octahedron tilings, that are derived from the Coxeter truncated orthoscheme tilings $\{4, 3, p\}$ $(6 < p \in \mathbb{N})$ and $\{3, 4, p\}$ $(4 < p \in \mathbb{N})$ in hyperbolic space $\mathbb{H}^3$, were considered in [29]. If we allow $p \in \mathbb{R}$ as well, then the locally densest (non-congruent half) hyperball configuration belongs to the truncated cube, with density ≈ 0.86145. This is larger than the Böröczky-Florian density upper bound for balls and horoballs. But the locally optimal non-congruent hyperball packing configuration obtained in [29] cannot be extended to the entire $\mathbb{H}^3$. The extendable densest non-congruent hyperball packing arrangement related to the truncated cube tiling $\{4, 3, 7\}$ was determined in [29], and the density is ≈ 0.84931.

In [27] the author studied congruent and non-congruent hyperball packings generated by doubly truncated Coxeter orthoscheme tilings in the 3-dimensional hyperbolic space. It is proved, that the densest congruent hyperball packing belongs to the Coxeter orthoscheme tiling of parameter $\{7, 3, 7\}$ with density ≈ 0.81335. This density is equal – by our conjecture – with the upper bound density of the corresponding non-congruent hyperball arrangements.

We have considered in this paper the coverings with horo- and hyperballs (briefly hyp-hor coverings) in the n-dimensional hyperbolic spaces $\mathbb{H}^n$ $(n = 2, 3)$, which form a new class of the classical covering problems.

We construct in the 2− and 3−dimensional hyperbolic spaces hyp-hor coverings, that are generated by complete Coxeter tilings of degree 1 i.e. the fundamental domains of these tilings are simple frustum orthoschemes with a principal vertex lying on the absolute quadric and the other principal vertex is outer point. We determine their thinnest covering configurations and their densities. These considered Coxeter tilings exist in the 2−, 3− and 5−dimensional hyperbolic spaces (see [7], [14]) and are given by their Coxeter-Schläfli graph in Fig. 1.

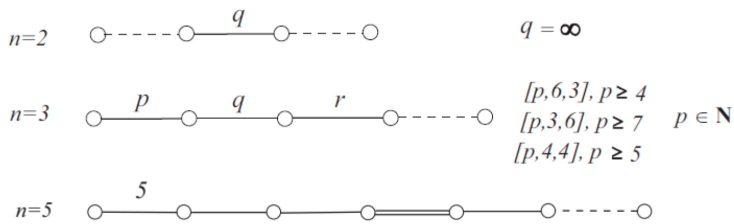


Figure 1: Coxeter-Schläfli graph of Coxeter tilings of degree 1.

We prove that in the hyperbolic plane $n = 2$ the density of the above hyp-hor coverings arbitrarily approximate the universal upper bound of the hypercycle or horocycle packing density $\frac{\sqrt{12}}{\pi}$ and in $\mathbb{H}^3$ the thinnest hyp-hor configuration belongs to the $\{7, 3, 6\}$ Coxeter tiling with density ≈ 1.27297.

Moreover, we consider the hyp-hor coverings in truncated orthoschemes $\{p, 3, 6\}$ $(6 < p < 7,\ p \in \mathbf{R})$. Its density function is attained its minimum for parameter $p \approx 6.45962$, and the corresponding minimal covering density is ≈ 1.26885 less than ≈ 1.280. That means, that this locally optimal hyp-hor configurations provide less densities than the previously known Fejes Tóth-Böröczky-Florian covering density for ball and horoball packings but this hyp-hor covering configurations can not be extended to the entirety of hyperbolic space $\mathbb{H}^3$.

2. Basic notions

For $\mathbb{H}^n$ we use the projective model in the Lorentz space $\mathbb{E}^{1,n}$ of signature $(1, n)$, i.e. $\mathbb{E}^{1,n}$ denotes the real vector space $\mathbf{V}^{n+1}$ equipped with the bilinear form of signature $(1, n)$: $\langle\ \mathbf{x},\ \mathbf{y}\rangle = -x^0y^0 + x^1y^1 + \dots + x^ny^n$, where the non-zero vectors $\mathbf{x} = (x^0, x^1, \dots, x^n) \in \mathbf{V}^{n+1}$ and $\mathbf{y} = (y^0, y^1, \dots, y^n) \in \mathbf{V}^{n+1}$ are determined up to real factors, for representing points of $\mathcal{P}^n(\mathbb{R})$. Then, $\mathbb{H}^n$ can be interpreted as the interior of the quadric $Q = \{[\mathbf{x}] \in \mathcal{P}^n | \langle\ \mathbf{x},\ \mathbf{x}\rangle = 0\} =: \partial\mathbb{H}^n$ in the real projective space $\mathcal{P}^n(\mathbf{V}^{n+1}, \boldsymbol{V}_{n+1})$.

The points of the boundary $\partial\mathbb{H}^n$ in $\mathcal{P}^n$ are called points at infinity of $\mathbb{H}^n$, the points lying outside $\partial\mathbb{H}^n$ are said to be outer points of $\mathbb{H}^n$ relative to Q. Let $P([\mathbf{x}]) \in \mathcal{P}^n$, a point $[\mathbf{y}] \in \mathcal{P}^n$ is said to be conjugate to $[\mathbf{x}]$ relative to Q if $\langle\ \mathbf{x},\ \mathbf{y}\rangle = 0$ holds. The set of all points which are conjugate to $P([\mathbf{x}])$ form a projective (polar) hyperplane $\boldsymbol{x} = pol(\mathbf{x}) := \{[\mathbf{y}] \in \mathcal{P}^n | \langle \mathbf{x}, \mathbf{y} \rangle = 0\}$. Thus the quadric Q induces a bijection (linear polarity $\mathbf{V}^{n+1} \to \boldsymbol{V}_{n+1}$) from the points of $\mathcal{P}^n$ onto its hyperplanes.

The distance s of two proper points $[\mathbf{x}]$ and $[\mathbf{y}]$ is calculated by the formula:

$$\cosh \frac{s}{k} = \frac{-\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}, \tag{2.1}$$

The constant $k = \sqrt{\frac{-1}{K}}$ *is the natural length unit in* $\mathbb{H}^n$*.* K *will be the constant negative sectional curvature. In the following we assume that* $k = 1$*.*

2.1. Complete orthoschemes

A n-dimensional tiling $\mathcal{P}$ (or solid tessellation, honeycomb) is an infinite set of congruent polyhedra (polytopes), that fit together to fill all space ($\mathbb{H}^n$ $(n \geqq 2)$) exactly once, so that every face of each polyhedron (polytope) belongs to another polyhedron as well. At present the cells are congruent orthoschemes (see [8]).

Geometrically, complete orthoschemes of degree d can be described as follows:

1. For $d = 0$, they coincide with the class of classical orthoschemes introduced by Schläfli. The initial and final vertices, A_0 and A_n of the orthogonal edge-path $A_iA_{i+1},\ i = 0, \dots, n-1$, are called principal vertices of the orthoscheme.

2. A complete orthoscheme of degree $d = 1$ can be interpreted as an orthoscheme with one outer principal vertex, say A_n, which is truncated by its polar plane $pol(A_n)$ (see Fig. 2 and 5). In this case the orthoscheme is called simply truncated with outer vertex A_n.

3. For degree $d = 2$ a complete orthoscheme can be interpreted as an orthoscheme with two outer principal vertices, $A_0,\ A_n$, which is truncated by its polar hyperplanes $pol(A_0)$ and $pol(A_n)$. In this case the orthoscheme is called doubly truncated. We distinguish two different types of orthoschemes, but we will not enter into the details (see [8]).

The complete Coxeter orthoschemes were classified by Im Hof in [7] by generalizing the method of Coxeter and Böhm, who showed that they exist only for dimensions ≤ 9. From this classification it follows, that the complete orthoschemes of degree $d = 1$ exist up to 5 dimensions.

In this paper we consider the orthoschemes of degree 1 where the initial vertex A_0 lies on the absolute quadric Q. These orthoschemes and the corresponding Coxeter tilings exist in the 2-, 3$-$ and 5$-$dimensional hyperbolic spaces and are characterized by their Coxeter-Schläfli symbols and graphs (see Fig. 1).

It can be seen that in n-dimensional hyperbolic space $\mathbb{H}^n$ $(n \geq 2)$ $\mathcal{S}$ is a complete orthoscheme of degree $d = 1$, with vertices $A_0A_1A_2 \dots A_{n-1}$

$P_0P_1P_2 \dots P_{n-1}$. A simply frustum orthoscheme where A_n is an outer vertex of $\mathbb{H}^n$ and then the points P_i $(i \in 0, \dots, n-1)$ lie in intersection the polar hyperplane π with A_iA_n.

We consider the images of $\mathcal{S}$ under reflections on its side facets. The union of these n-dimensional orthoschames (having the common π hyperplane) forms an infinite polyhedron denoted by $\mathcal{G}$. $\mathcal{G}$ and its images under reflections on its ,,cover facets" fill hyperbolic space $\mathbb{H}^n$ without overlap and generate n-dimensional tilings $\mathcal{T}$.

2.2. Volumes of the n-dimensional Coxeter orthoschemes

1. 2-dimensional hyperbolic space $\mathbb{H}^2$

   In the hyperbolic plane a simple frustum orthoscheme is a Lambert quadrilateral with exactly three right angles and its fourth angle is acute $\frac{\pi}{q}$ $(q \geq 3)$ (see Fig. 1 and 2). In our case the Lambert quadrilateral has a vertex at the infinity i.e. the angle at this vertex is 0. Its area can be determined by the well-known defect formula of hyperbolic triangles:

$$Vol_2(\mathcal{S}) = \frac{\pi}{2}. \tag{2.2}$$

2. 3-dimensional hyperbolic space $\mathbb{H}^3$:

   Our polyhedron $A_0A_1A_2P_0P_1P_2$ is a simple frustum orthoscheme with outer vertex A_3 (see Fig. 5.a) whose volume can be calculated by the following theorem of R. Kellerhals [8]:

   Theorem 2.1. *The volume of a three-dimensional hyperbolic complete orthoscheme (except Lambert cube cases) $\mathcal{S}$ is expressed with the essential angles $\alpha_{01}, \alpha_{12}, \alpha_{23}$, $(0 \leq \alpha_{ij} \leq \frac{\pi}{2})$ (Fig. 1 and 2) in the following form:*

$$\begin{gathered} Vol_3(\mathcal{S}) = \frac{1}{4}\{\mathcal{L}(\alpha_{01}+\theta) - \mathcal{L}(\alpha_{01}-\theta) + \mathcal{L}(\frac{\pi}{2}+\alpha_{12}-\theta) + \\ + \mathcal{L}(\frac{\pi}{2}-\alpha_{12}-\theta) + \mathcal{L}(\alpha_{23}+\theta) - \mathcal{L}(\alpha_{23}-\theta) + 2\mathcal{L}(\frac{\pi}{2}-\theta)\}, \end{gathered} \tag{2.3}$$

   where $\theta \in [0, \frac{\pi}{2})$ is defined by the following formula:

$$\tan(\theta) = \frac{\sqrt{\cos^2\alpha_{12} - \sin^2\alpha_{01}\sin^2\alpha_{23}}}{\cos\alpha_{01}\cos\alpha_{23}}$$

   and where $\mathcal{L}(x) := -\int\limits_0^x \log|2\sin t| dt$ denotes the Lobachevsky function.

   For our prism tilings $\mathcal{T}_{pqr}$ we have: $\alpha_{01} = \frac{\pi}{p}$, $\alpha_{12} = \frac{\pi}{q}$, $\alpha_{23} = \frac{\pi}{r}$.

2.3. On hyperballs

The equidistant surface (or hypersphere) is a quadratic surface that lies at a constant distance from a plane in both halfspaces. The infinite body of the hypersphere is called a hyperball. The n-dimensional *half-hypersphere* $(n = 2, 3)$ with distance h to a hyperplane π is denoted by $\mathcal{H}_n^h$. The volume of a bounded hyperball piece $\mathcal{H}_n^h(\mathcal{A}_{n-1})$ bounded by an $(n-1)$-polytope $\mathcal{A}_{n-1} \subset \pi$, $\mathcal{H}_n^h$ and by hyperplanes orthogonal to π derived from the facets of $\mathcal{A}_{n-1}$ can be determined by the formulas (2.4) and (2.5) that follow from the suitable extension of the classical method of J. Bolyai ([2]):

$$Vol_2(\mathcal{H}_2^h(\mathcal{A}_1)) = Vol_1(\mathcal{A}_1)\sinh(h), \tag{2.4}$$

$$Vol_3(\mathcal{H}_3^h(\mathcal{A}_2)) = \frac{1}{4}Vol_2(\mathcal{A}_2)\left[\sinh(2h) + 2h\right], \tag{2.5}$$

where the volume of the hyperbolic $(n-1)$-polytope $\mathcal{A}_{n-1}$ lying in the plane π is $Vol_{n-1}(\mathcal{A}_{n-1})$.

2.4. On horoballs

A horosphere in $\mathbb{H}^n$ $(n \geq 2)$ is a hyperbolic n-sphere with infinite radius centered at an ideal point on $\partial\mathbb{H}^n$. Equivalently, a horosphere is an $(n-1)$-surface orthogonal to the set of parallel straight lines passing through a point of the absolute quadratic surface. A horoball is a horosphere together with its interior.

We consider the usual Beltrami-Cayley-Klein ball model of $\mathbb{H}^n$ centered at $O(1, 0, 0, \ldots, 0)$. The equation of a horosphere with center $T_0(1, 0, \ldots, 1)$ passing through the point $S(1, 0, \ldots, s)$ is derived from the equation of the the absolute sphere $-x^0x^0 + x^1x^1 + x^2x^2 + \cdots + x^nx^n = 0$, and the plane $x^0 - x^n = 0$ tangent to the absolute sphere at T_0. The general equation of the horosphere in cartesian coordinates is the following:

$$\frac{2\left(\sum_{i=1}^{n-1} h_i^2\right)}{1-s} + \frac{4\left(h_n - \frac{s+1}{2}\right)^2}{(1-s)^2} = 1. \tag{2.6}$$

In n-dimensional hyperbolic space any two horoballs are congruent in the classical sense. However, it is often useful to distinguish between certain horoballs of a packing. We use the notion of horoball type with respect to the packing as introduced in [18].

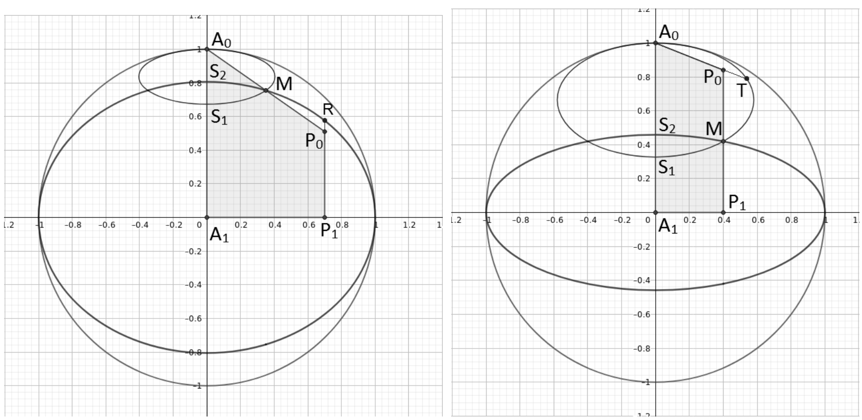
The intrinsic geometry of a horosphere is Euclidean, so the $(n-1)$-dimensional volume $\mathcal{A}$ of a polyhedron A on the surface of the horosphere can be calculated as in $\mathbb{E}^{n-1}$. The volume of the horoball piece $\mathcal{H}(A)$ determined by A and the aggregate of axes drawn from A to the center of the horoball is ([2])

$$Vol(\mathcal{H}(A)) = \frac{1}{n-1}\mathcal{A}. \tag{2.7}$$

3. Hyp-hor coverings in hyperbolic plane

We consider the usual Beltrami-Cayley-Klein ball modell of $\mathbb{H}^2$ centered at $O(1,0,0)$ with a given vector basis $\mathbf{e}_i$ $(i = 0,1,2)$ and set the 2-dimensional Coxeter orthoscheme $A_0A_1A_2$ in this coordinate system with coordinates $A_0(1,0,1)$; $A_1(1,0,0)$; $A_2(1,\frac{1}{a},0)$. Here the initial principal vertex of the orthoscheme A_0 is lying on the absolute quadric Q and the other principal vertex A_2 is an outer point of the model, so $0 < a < 1, a \in \mathbb{R}$.

The polar line of the outer vertex A_2 is $\pi = \boldsymbol{u}_2(1,-\frac{1}{a},0)^T$. By the truncation of the orthoscheme $A_0A_1A_2$ by the polar line π we get the Lambert quadrilateral $A_0A_1P_1P_0$ (see Fig. 2), where the further vertices are: $\pi \cap A_0A_2 = P_0(1,a,1-a^2); \pi \cap A_1A_2 = P_1(1,a,0)$. Its images under reflections on its sides fill hyperbolic plane $\mathbb{H}^2$ without overlap, hence we get the previously described 2-dimensional Coxeter tilings, given by the Coxeter symbol $[\infty]$ (see Fig. 1). The tilings contain the free parameter a, so we denote the tilings by $\mathcal{T}_a$, and the Lambert quadrilaterals $A_0A_1P_0P_1$ by $\mathcal{F}_a$, which serve as the fundamental domain of the above tilings.



a) b)

Figure 2: a) $\mathcal{C}_a^1$-type hyp-hor covering at present $a = 0.7, t = 0.5$ b) $\mathcal{C}_a^2$-type hyp-hor covering at present $a = 0.4, t = 0.5$

We construct hyp-hor coverings to $\mathcal{F}_a$, by the following:

1. Let M be the intersection point of the horo- and hypercycle which lies on the A_0P_0 or P_0P_1 side of $\mathcal{F}_a$ (see Fig. 2).

2. The center of the horocycle can only be the vertex A_0. Let the intersection of the horocycle with A_0A_1 line be $S_1 = (1,0,s_1)$ $(-1 < s_1 < 1)$ and with A_0P_0 line the intersection is $T = (1,ta,1-ta^2)$ $(0 < t < \frac{2}{1+a^2})$. We denote by $\mathfrak{H}_a(t)$ the horocycle-piece determined by points A_0, S_1, T (see Fig. 2a with $M = T$, and Fig. 2b).

3. Let A_1P_1 be the base straight line of a hypercycle, and let the intersection of the hypercycle with the positive segment of A_0A_1 line be $S_2 = (1,0,s_2)$

$(0 < s_2 < 1)$ and with P_0P_1 line is $R = (1, a, r)$ $(0 < r < \sqrt{1-a^2})$. We denote by $\mathcal{H}_a(t)$ the hypercycle-piece settled by points P_1, R, S_2, A_1 (see Fig. 2a, and Fig. 2b with $M = R$).

We can see, that if the horo- and hypercycles satisfy the above requirements, than they cover $\mathcal{F}_a$. Thus the images of $\mathfrak{H}_a(t)$ and $\mathcal{H}_a(t)$ under reflection on the sides of $\mathcal{F}_a$ provide a hyp-hor covering of hyperbolic plane $\mathbb{H}^2$. The fundamental domain $\mathcal{F}_a$ (i.e. parameter a) and point M (i.e. point T, and parameter t, because the position of T determines the position of M) determine the covering. We distinguish two main types of hyp-hor coverings, denoted by $\mathcal{C}_a^1(t)$ if $M \in A_0P_0$ and by $\mathcal{C}_a^2(t)$ if $M \in P_0P_1$ (see Fig. 2).

Definition 3.1. The density of the above hyp-hor coverings $\mathcal{C}_a^i(t)$ $(i = 1, 2)$ are:

$$\delta(\mathcal{C}_a^i(t)) = \frac{Vol(\mathcal{H}_a(t)) + Vol(\mathfrak{H}_a(t))}{Vol(\mathcal{F}_a)} \tag{3.1}$$

It is obvious, that if the point M lies on the perimeter of $\mathcal{F}_a$, the density of the covering is smaller, than if it lies out of $\mathcal{F}_a$. Thus we get the coverings with minimal densities in the above two main cases.

3.1. The densities of coverings $\mathcal{C}_a^1(t)$.

In this case $M \in A_0P_0$ is the intersection point of the cycles, so $M = (1, ta, 1 - ta^2)$ $(0 < t \leq 1)$. The coordinates of S_1 can be expressed using (2.6) and the distance of M and S_1 can be calculated by (2.1), thus we can determine the volume of $\mathfrak{H}_a(t)$ by formula (2.6). The length of A_1P_1 and the distance of M and the $x-$axis can be calculated also by (2.1), thus we can determine the volume of $\mathcal{H}_a(t)$ by formula (2.4). We obtain by 3.1, that the density of $\mathcal{C}_a^1(t)$ can be expressed by the following formula:

$$\delta(\mathcal{C}_a^1(t)) = \frac{\operatorname{arccosh}\left(\frac{1}{\sqrt{1-a^2}}\right)\frac{1-ta^2}{a\sqrt{2t-t^2-a^2t^2}} + 2\sinh\left(\frac{1}{2}\operatorname{arccosh}\left(\frac{2ta^2+t-4}{2t-4+2ta^2}\right)\right)}{\frac{\pi}{2}}$$

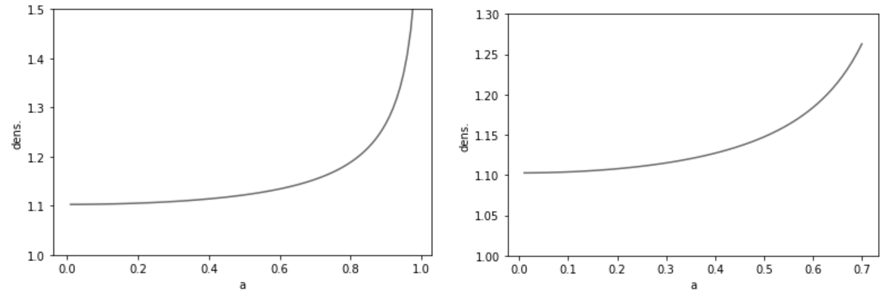
where $0 < a < 1$, $0 < t \leq 1$.

Analyzing the above density formula we obtain the following.

Theorem 3.2. *In the hyperbolic plane $\mathbb{H}^2$ the universal lower bound density of ball and horoball coverings can be arbitrary accurate approximate with the densities $\delta\left(\mathcal{C}_a^1\left(\frac{1}{2}\right)\right)$ of hyp-hor packings of type 1:*

$$\lim_{a\to 0} \delta\left(\mathcal{C}_a^1\left(\frac{1}{2}\right)\right) = \frac{\sqrt{12}}{\pi}$$

and $\delta\left(\mathcal{C}_a^1\left(\frac{1}{2}\right)\right) > \frac{\sqrt{12}}{\pi}$ for parameter $0 < a < 1$ (see Fig. 3a).



a) b)

Figure 3: a) The density function of hyp-hor covering $\mathcal{C}_a^1(t)$ in case $t = 0.5$ b) The density function of hyp-hor covering $\mathcal{C}_a^2(t)$ in case $t \approx 1.142$

3.2. The densities of coverings $\mathcal{C}_a^2(t)$.

In this case $M \in P_0P_1$ the intersection point of the cycles, so the intersection point of the horocycle and line A_0P_0 is $(1, ta, 1-ta^2)$ $(0 < t < \frac{2}{1+2a^2-a^4})$, by the condition, that M lies on the positive segment of P_0P_1. We get the volume of $\mathfrak{H}_a(t)$ just like in the previous section. The coordinates of M and the h_2 length of MP_1 can be calculated by (2.6) and (2.1). We can determine the volume of $\mathcal{H}_a(t)$ by formula (2.4). We obtain by Definition (3.1), that the density of $\mathcal{C}_a^2(t)$ can be expressed by the following formula:

$$\delta(\mathcal{C}_a^2(t)) = \frac{\operatorname{arccosh}\left(\frac{1}{\sqrt{1-a^2}}\right)\sinh h_2 + 2\sinh\left(\frac{1}{2}\operatorname{arccosh}\left(\frac{2ta^2+t-4}{2t-4+2ta^2}\right)\right)}{\frac{\pi}{2}}$$

where $0 < a < 1$, $0 < t < \frac{2}{1+2a^2-a^4}$.

Analyzing the above density formula (using also numerical approximation methods) we obtain that it provides its minimum in case $t \approx 1.142$, $a \to 0$ (see Fig. 3b), and the minimum value is $\frac{\sqrt{12}}{\pi}$. Therefore, we obtain the following

Theorem 3.3. *In the hyperbolic plane $\mathbb{H}^2$ the universal lower bound density of ball coverings can be arbitrary accurate approximated with densities $\delta\left(\mathcal{C}_a^2(t)\right)$ of hyp-hor packings of type 2.*

4. Hyp-hor coverings in hyperbolic space $\mathbb{H}^3$

In the 3-dimensional hyperbolic space $\mathbb{H}^3$ there are 3 infinite series of the simple frustum Coxeter orthoschemes with vertex at the infinity, that are listed in Fig. 1, and characterized in Section 2.1. The Coxeter-Schläfli symbol of these orthoschemes are $\{p, q, r\}$, where $(q, r) = (3, 6), (4, 4), (6, 3)$, and p is an appropriate integer parameter: $p \geq 7$ if $(q, r) = (3, 6)$, $p \geq 5$ if $(q, r) = (4, 4)$, $p \geq 4$ if $(q, r) = (6, 3)$. These conditions came from the geometry of the orthoschemes and can be computed by the inverse Coxeter-Schläfli matrix. We denote the orthoscheme by $\mathcal{F}_p^{(q,r)}$, and its vertices are denoted by A_0, A_1, A_2, P_0, P_1, P_2 (see Fig. 5.a).

We consider the usual Beltrami-Cayley-Klein ball modell of $\mathbb{H}^3$ centred at $O(1,0,0,0)$ with a given vector basis $\mathbf{e}_i$ $(i = 0,1,2,3)$ (see Section 2.1) and with the 3-dimensional complete Coxeter orthoscheme $A_0A_1A_2A_3$ in which initial principal vertex A_0 is lying on the absolute quadric Q and the other principal vertex A_3 is an outer point of the model. By the truncation of the orthoscheme with π (the polar plane of A_3) we get the proper vertices $P_k[\mathbf{p}_k] = \pi \cap A_kA_3, (i = 0,1,2)$, therefore $\mathbf{p}_k \sim c \cdot \mathbf{a}_3 + \mathbf{a}_k$ for some $c \in \mathbb{R}$. $P_k[\mathbf{p}_k]$ lies on $\boldsymbol{a}^3 = pol(\mathbf{a}_3)$ if and only if $\mathbf{p}_k \boldsymbol{a}^3 = 0$, thus:

$$c \cdot \mathbf{a}_3\boldsymbol{a}^3 + \mathbf{a}_k\boldsymbol{a}^3 = 0 \Leftrightarrow c = -\frac{\mathbf{a}_k\boldsymbol{a}^3}{\mathbf{a}_3\boldsymbol{a}^3} \tag{4.1}$$

$$\Leftrightarrow \mathbf{p}_k \sim -\frac{\mathbf{a}_k\boldsymbol{a}^3}{\mathbf{a}_3\boldsymbol{a}^3} \cdot \mathbf{a}_3 + \mathbf{a}_k \sim \mathbf{a}_k(\mathbf{a}_3\boldsymbol{a}^3) - \mathbf{a}_3(\mathbf{a}_k\boldsymbol{a}^3) \tag{4.2}$$

We consider the Coxeter-Schläfli matrix (c^{ij}) of the orthoscheme, and its inverse (h_{ij}), where the elements of the matrices: $c^{ij} = \boldsymbol{a}^i\boldsymbol{a}^j$, $h_{ij} = \mathbf{a}_i\mathbf{a}_j$. The polar hyperplane of A_3 is $\boldsymbol{a}^3$, thus $h_{k3} = \mathbf{a}_k\boldsymbol{a}^3$, hence by (4.1) $\mathbf{p}_k = \mathbf{a}_k h_{33} - \mathbf{a}_3 h_{k3}$.

We set the above simple frustum orthoscheme $\mathcal{F}_p^{(q,r)}$ in the usual coordinate system with vertices: $P_0(1,0,0,0)$, $P_1(1,0,y,0)$, $P_2(1,x,y,0)$, $A_0(1,0,0,1)$, $A_1(1,0,y,z_1)$, $A_2(1,x,y,z_2)$ (see Fig. 5.a). We get the following equations, using the formulas (2.1), (4.1) and $h_{ij} = \mathbf{a}_i\mathbf{a}_j$:

$$\cosh(d(P_0P_1)) = \frac{h_{03}h_{13} - h_{01}h_{33}}{\sqrt{(h_{11}h_{33} - h_{13}^2)(h_{00}h_{33} - h_{03}^2)}} = \frac{1}{\sqrt{(-1)(-1+y^2)}}, \tag{4.3}$$

$$\cosh(d(P_0P_2)) = \frac{h_{03}h_{23} - h_{02}h_{33}}{\sqrt{(h_{22}h_{33} - h_{23}^2)(h_{00}h_{33} - h_{03}^2)}} = \frac{1}{\sqrt{(-1)(-1+y^2+x^2)}}, \tag{4.4}$$

$$\cosh(d(A_1P_1)) = \sqrt{\frac{h_{11}h_{33} - h_{13}^2}{h_{11}h_{33}}} = \frac{1-y^2}{\sqrt{(-1+y^2+z_1^2)(-1+y^2)}}, \tag{4.5}$$

$$\cosh(d(A_2P_2)) = \sqrt{\frac{h_{22}h_{33} - h_{23}^2}{h_{22}h_{33}}} = \frac{1-y^2-x^2}{\sqrt{(-1+y^2+x^2+z_2^2)(-1+y^2+x^2)}}. \tag{4.6}$$

We can determine the coordinates $x, y, z_k, (k = 1,2)$ by solving these equations, and the volume of the orthoschemes $\mathcal{F}_p^{(q,r)}$ by (2.3).

The images of the above orthoscheme $\mathcal{F}_p^{(q,r)}$ under reflections on its facets fill the hyperbolic space $\mathbb{H}^3$ without overlap, so we get the Coxeter tiling $\mathcal{T}_p^{(q,r)}$ of $\mathbb{H}^3$ with fundamental domain $\mathcal{F}_p^{(q,r)}$.

We construct hyp-hor coverings to $\mathcal{F}_p^{(q,r)}$ using the following requirements:

1. The center of the horoball can only be the ideal vertex A_0. Let S_1, T_1, Q_1 be the intersection points of the horoball with lines A_0P_0, A_0A_2, A_0A_1 lines, respectively. We denote by $\mathfrak{H}_p^{(q,r)}$ the horoball-piece determined by points A_0, S_1, T_1, Q_1 (see Fig. 5.b).

2. Plane $P_0P_1P_2$ can be the base hyperplane of a hyperball. Let S_2, V_2, R_2 be the intersection points of the hyperball with the line segments of A_0P_0, A_1P_1, A_2P_2, respectively. We denote by $\mathcal{H}_p^{(q,r)}$ the hyperball-piece bounded by the base hyperplane, the surface of the hyperball and the hyperplanes perpendicular to the base hyperplane derived from edges P_0P_1, P_1P_2, P_2P_0 (see Fig. 5.b).

3. The intersection curve (which is a circle parallel with $[xy]$ plane in Euclidean sense) of the horo- and hyperball passes through one of the edges of the orthoscheme $A_0A_1, A_0A_2, A_1A_2, A_0P_0, A_1P_1, A_2P_2$ (see Fig. 5.a).

We can see that, if the horo- and hyperballs satisfy the above requirements, then they cover $\mathcal{F}_p^{(q,r)}$ if and only if they cover all the edges of $\mathcal{F}_p^{(q,r)}$. Hence, if a covering arrangement covers the edges of the orthoscheme, than the images of $\mathfrak{H}_p^{(q,r)}$ and $\mathcal{H}_p^{(q,r)}$ under reflection on the facets of $\mathcal{F}_p^{(q,r)}$ provide a hyp-hor covering of hyperbolic space $\mathbb{H}^3$, denoted by $\mathcal{C}_p^{(q,r)}$.

Definition 4.1. The density of the above hyp-hor coverings $\mathcal{C}_p^{(q,r)}$ is:

$$\delta(\mathcal{C}_p^{(q,r)}) = \frac{Vol(\mathcal{H}_p^{(q,r)}) + Vol(\mathfrak{H}_p^{(q,r)})}{Vol(\mathcal{F}_p^{(q,r)})} \tag{4.7}$$

It is obvious, that if the intersection curve passes through one of the edge of $\mathcal{F}_p^{(q,r)}$, the density of the covering is smaller, than if it goes out of $\mathcal{F}_p^{(q,r)}$. Thus we get the coverings with minimal densities if the above requiements hold. Based on the above, we have to distinguish and study six cases.

4.1. Non-covering cases

- If the intersection curve of the balls passes through A_0P_0 (see Fig. 5.a), then the balls touch each other, thus the hyp-hor covering is obviously not realized.

- If the intersection curve of the balls intersects the edge A_0A_2 (see Fig. 5.a), then we can parametrize their common point: $T(t) = (1, tx, ty, tz_2 + (1 - t)), t \in [0, 1]$. By substituting this in the equation of the balls, we get the coordinates of $S_1 \in P_0A_0, S_2 \in P_0A_0$ points. If the horoball covers A_1, we can determine the intersection points U_1, U_2 by solving the corresponding equations. By inspecting the z-coordinates of U_i $(i = 1, 2)$ in the model, we can see, that U_1 is always higher than U_2, which means (using the convexity of the ellipsoids) that they together do not cover the edge A_1A_2. If the hyperball covers A_1, we can determine the intersection points Q_1, Q_2 by solving the corresponding equations. By inspecting the

z-coordinates of Q_i $(i = 1, 2)$ in the model, we can see, that Q_1 is always higher than Q_2, which means (using the convexity of the ellipsoids) that they together do not cover the edge A_0A_1. Thus in this case the hyp-hor covering is not realized.

- If the intersection curve of the balls contains a point of A_1P_1 edge (see Fig. 5.a) then we can parametrize the intersection point V: $V(v) = (1, 0, y, vz_1), v \in [0, 1]$. Very similarly to the above case, we can see, that if the horoball covers A_2, than the balls do not cover the edge A_2P_2, and if the hyperball covers A_2, than the balls do not cover the edge A_1A_2. Thus, in this case the hyp-hor covering is not realized.

4.2. Thinnest covering, if the intersection point lies on A_0A_1 edge

In this case, A_0A_1 edge has a common point with the intersection curve of the balls (see Fig. 5.a), so we can parametrize the intersection point Q: $Q(u) = (1, 0, uy, uz_1 + (1 - u)),\ u \in [0, 1]$. By substituting this in the equation of the balls, we get the coordinates of $S_1, S_2 \in P_0A_0$. After that, we can determine the intersection points $T_1, T_2 \in A_0A_2$ by solving the corresponding equations. We prove, that the balls cover the edges of the orthoscheme, so the hyp-hor covering is realized in this case.

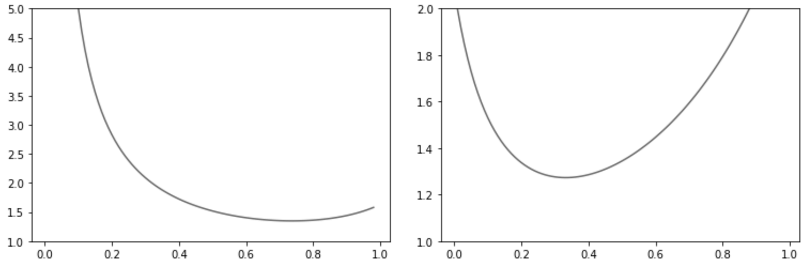
Since $P_0A_0A_1P_1$ is a 2-dimensional Coxeter orthoscheme, thus A_0A_1 is covered as we have seen in Section 3. The hyperball covers A_1. So we can see, that the hyperbolic length of A_1P_1 edge is always bigger than the length of A_2P_2 edge. That means, the hyperball covers A_2, and because of its convexity A_1P_1, A_2P_2, A_1A_2 edges as well. By inspecting the z-coordinates of S_i and T_i $(i = 1, 2)$ in the model, we can see, that S_2 is always "higher" than S_1 and T_2 is always "higher" than T_1, which means (using the convexity of the ellipsoids) that they together cover the edges A_0P_0 and A_0A_2.

We know the coordinates of points Q, T_i, S_i $(i = 1, 2)$, so we can determine the $Vol(\mathcal{H}_p^{(q,r)})$, $Vol(\mathfrak{H}_p^{(q,r)})$ using (2.5), (2.7) and the density of the covering using (4.7), which depends on free parameter u. Analyzing this density function we can compute the optimal densities (see Fig. 4.a). The results for tiling $\mathcal{T}_p^{(6,3)}$ (which provides the lowest density in this case) are summarized in the table below.

Type of tiling	δ_{min}	u
$\mathcal{T}_4^{(6,3)}$	1.3482413	0.7369142
$\mathcal{T}_5^{(6,3)}$	1.4432379	0.7655641
$\mathcal{T}_6^{(6,3)}$	1.5178400	0.7814085

4.3. Thinnest covering, if the intersection point lies on A_1A_2 edge

Now, the intersection curve of the balls passes through A_1A_2 (see Fig. 5.a), so we can parametrize the intersection point $U \in A_1A_2$: $U(u) = (1, ux, y, uz_2 + (1 - u)z_1), u \in [0, 1]$. By substituting this in the equation of the balls, we

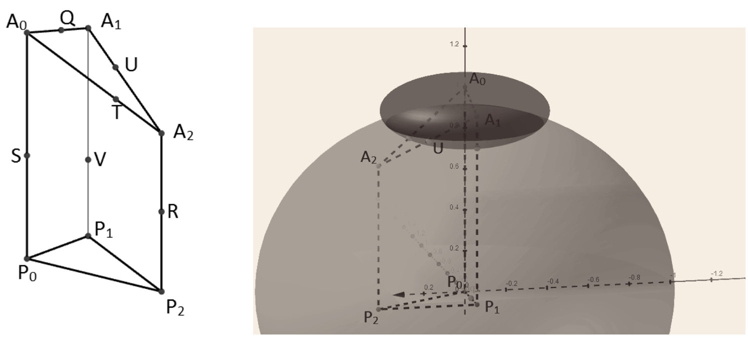


a) b)

Figure 4: a) The density function $\delta(\mathcal{C}_4^{(6,3)}(u))$ b) The density function $\delta(\mathcal{C}_7^{(3,6)}(u))$

get the coordinates of points $S_1, S_2 \in P_0A_0$. After that we can determine the intersection points $V_1, V_2 \in A_1P_1$, $Q_1, Q_2 \in A_0A_1$ and $T_1, T_2 \in A_0A_2$ by solving the corresponding equations. We can prove, similarly as in the above case, that the balls cover the edges of the orthoscheme, so the hyp-hor covering is realized in this case. The horoball covers A_0A_1, the hyperball covers A_2P_2, and together they cover A_1A_2 (see Fig. 5.b). By inspecting the z-coordinates of S_i, V_i and T_i $(i = 1, 2)$ in the model, we can see in this case also, that the balls cover A_0P_0, A_1P_1, A_0A_2 edges (see Fig. 5.b).

We know points Q_i, T_i, S_i $(i = 1, 2)$, so we can determine the $Vol(\mathcal{H}_p^{(q,r)})$, $Vol(\mathfrak{H}_p^{(q,r)})$ using (2.5), (2.7) and the density of the covering using (4.7), which depends on free parameter u. Analysing this density function we can compute the optimal densities (see Fig. 4.b). The results for tiling $\mathcal{T}_p^{(3,6)}$ (which provides the lowest density in this case) are summarized in the next table.



a) b)

Figure 5: a) Simple truncated orthoscheme, and the intersection points of the balls, in the 6 cases b) Hyp-hor covering of $\mathcal{F}_7^{(3,6)}$ with smallest density ≈ 1.27297

Type of tiling	δ_{min}	u
$\mathcal{T}_7^{(3,6)}$	1.27297329	0.3324288
$\mathcal{T}_8^{(3,6)}$	1.288832	0.3337034
$\mathcal{T}_9^{(3,6)}$	1.3065421	0.3358650

Remark 4.2. To any parameter p $(6 < p < 7, p \in \mathbb{R})$ belongs a simple frustum orthoscheme $\mathcal{F}_p^{(3,6)}$ as well, therefore we can determine the densities of the corresponding hyp-hor coverings using the above computation method. The density function depends on free parameters u and p, and analyzing this function we get the minimal density in case $p \approx 6.459617$, $u \approx 0.33248$ with $\delta \approx 1.268853$. This hyp-hor covering is just locally optimal, because the corresponding tiling can not be extended to $\mathbb{H}^3$.

4.4. Thinnest covering, if the intersection point lies on A_2P_2 edge

In this case, A_2P_2 passes through the intersection curve of the balls (see Fig. 5.a), so we can parametrize the intersection point of the curve and the edge: $R(u) = (1, x, y, uz_2), u \in [0,1]$. The further computations of this case is very similar to the above two cases. We can determine the coordinates of Q_i, T_i, S_i $(i = 1, 2)$ points, see that the horo- and hyperball cover the edges, so the hyp-hor covering is realized, and compute the density of the covering by (4.7). The results for tiling $\mathcal{T}_p^{(4,4)}$ (which provides the smallest density in this case) are summarized in the next table.

Type of tiling	δ_{min}	u
$\mathcal{T}_5^{(4,4)}$	1.8383911	0.8114832
$\mathcal{T}_6^{(4,4)}$	2.3821677	0.7332720
$\mathcal{T}_7^{(4,4)}$	3.0569894	0.7025236

Finally, summarizing the results so far, we get the following theorems

Theorem 4.3. *In $\mathbb{H}^3$, among the hyp-hor coverings generated by simple truncated orthoschemes, the $\mathcal{C}_7^{(3,6)}$ covering configuration (see Subsection 4.3) provides the lowest covering density ≈ 1.27297. The above density is smaller than the so far known lowest covering density ≈ 1.280 in the 3-dimensional hyperbolic space, which was described by L. Fejes Tóth and K. Böröczky.*

Theorem 4.4. *In hyperbolic space $\mathbb{H}^3$ the function $\delta(\mathcal{C}_p^{(3,6)})$ $(6 < p < 7, p \in \mathbb{R})$ attains its mimimum in case $p \approx 6.459617$, with density $\delta \approx 1.268853$, but the corresponding hyp-hor covering can not be extended to the entirety of hyperbolic space $\mathbb{H}^3$.*

We note here, that the discussion of the densest horoball packings in the n-dimensional hyperbolic space $n \geq 3$ with horoballs of different types and hyperballs has not been settled yet.

Optimal sphere packings in other homogeneous Thurston geometries represent another huge class of open mathematical problems. For these non-Euclidean geometries only very few results are known (e.g. [17], [20]). Detailed studies are the objective of ongoing research.

References

[1] Bezdek, K. Sphere packings revisited. *Eur. J. Combin. 27*, 6 (2006), 864–883.

[2] Bolyai, J. *The science absolute of space.* 1891.

[3] Böröczky, K. Packing of spheres in spaces of constant curvature. *Acta Math. Acad. Sci. Hungar. 32* (1978), 243–261.

[4] Böröczky, K., and Florian, A. über die dichteste kugelpackung im hyperbolischen raum. *Acta Math. Acad. Sci. Hungar. 15* (1964), 237–245.

[5] Fejes Tóth, G., Kuperberg, G., and Kuperberg, W. Highly saturated packings and reduced coverings. *Monatsh. Math. 125*, 2 (1998), 127–145.

[6] Fejes Tóth, L. *Regular Figures.* Macmillian (New York), 1964.

[7] Im Hof, H.-C. Napier cycles and hyperbolic coxeter groups. *Bull. Soc. Math. Belgique 42* (1990), 523–545.

[8] Kellerhals, R. On the volume of hyperbolic polyhedra. *Math. Ann. 245* (1989), 541–569.

[9] Kozma, R., and Szirmai, J. Optimally dense packings for fully asymptotic coxeter tilings by horoballs of different types. *Monatsh. Math. 168*, 1 (2012), 27–47.

[10] Kozma, R., and Szirmai, J. New lower bound for the optimal ball packing density of hyperbolic 4-space. *Discrete Comput. Geom. 53*, 1 (2015), 182–198.

[11] Kozma, R., and Szirmai, J. New horoball packing density lower bound in hyperbolic 5-space. *Geometriae Dedicata 206* (2019), 1–25.

[12] Kozma, R., and Szirmai, J. Horoball packing density lower bounds in higher dimensional hyperbolic n-space for $6 \leq n \leq 9$. *Geom. Dedicata 206* (2020), 1–25.

[13] Rogers, C. *Packing and Covering, Cambridge Tracts in Mathematics and Mathematical Physics 54.* Cambridge University Press, 1964.

[14] Stojanović, M. Hyperbolic space groups with truncated simplices as fundamental domains. *Filomat. 33*, 4 (2019), 1107–1116.

[15] Szirmai, J. The p-gonal prism tilings and their optimal hypersphere packings in the hyperbolic 3-space. *Acta Math. Hungar. 111*, 1-2 (2006), 65–76.

[16] Szirmai, J. The regular prism tilings and their optimal hyperball packings in the hyperbolic n-space. *Publ. Math. Debrecen 69*, 1-2 (2006), 195–207.

[17] Szirmai, J. The densest geodesic ball packing by a type of nil lattices. *Beitr. Algebra Geom. 48*, 2 (2007), 383–397.

[18] Szirmai, J. Horoball packings and their densities by generalized simplicial density function in the hyperbolic space. *Acta Math. Hung. 136*, 1-2 (2012), 39–55.

[19] Szirmai, J. Horoball packings to the totally asymptotic regular simplex in the hyperbolic n-space. *Aequat. Math. 85* (2013), 471–482.

[20] Szirmai, J. A candidate for the densest packing with equal balls in thurston geometries. *Beitr. Algebra Geom. 55*, 2 (2014), 441– 452.

[21] Szirmai, J. The least dense hyperball covering to the regular prism tilings in the hyperbolic n-space. *Ann. Mat. Pur. Appl. 195* (2016), 235–248.

[22] Szirmai, J. The optimal hyperball packings related to the smallest compact arithmetic 5-orbifold. *Kragujevac Journal of Mathematics 40*, 2 (2016), 260–270.

[23] Szirmai, J. Packings with horo- and hyperballs generated by simple frustum orthoschemes. *Acta Math. Hungar. 152*, 2 (2017), 365–382.

[24] Szirmai, J. Density upper bound of congruent and non-congruent hyperball packings generated by truncated regular simplex tilings. *Rendiconti del Circolo Matematico di Palermo Series 2 67* (2018), 307–322.

[25] Szirmai, J. Hyperball packings in hyperbolic 3-space. *Mat. Vesn. 70*, 3 (2018), 211–221.

[26] Szirmai, J. Upper bound of density for packing of congruent hyperballs in hyperbolic 3−space. *arXiv e-prints* (Dec. 2018), arXiv:1812.06785.

[27] Szirmai, J. Congruent and non-congruent hyperball packings related to doubly truncated coxeter orthoschemes in hyperbolic 3-space. *Acta Univ. Sapientiae Math. 11*, 2 (2019), 437–459.

[28] Szirmai, J. Decomposition method related to saturated hyperball packings. *Ars Math. Contemp. 16* (2019), 349–358.

[29] Szirmai, J. Hyperball packings related to cube and octahedron tilings in hyperbolic space. *Contributions to Discrete Mathematics 15*, 2 (2020), 42–59.

[30] Vermes, I. Ausfüllungen der hyperbolischen ebene durch kongruente hyperzykelbereiche. *Period. Math. Hungar. 10*, 4 (1979), 217–229.

[31] Vermes, I. über reguläre überdeckungen der bolyai-lobatschewskischen ebene durch kongruente hyperzykelbereiche. *Period. Mech. Eng. 25*, 3 (1981), 249–261.

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