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Bornology and duality in locally $\mathbb{K}$-convex sequential spaces[1]

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Abstract. The present paper is concerned with the concept of sequential topologies in non-archimedean analysis. We give characterizations of such topologies in case of bornological spaces and inductive limits spaces. We also study equicontinuity and duality in this class of spaces.

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1. Introduction

In [21], Venkataramen posed the problem of characterization of " the class of topological spaces which can be specified completely by the knowledge of their convergent sequences ". Severals authors then agreed to provide a solution, based on the concept of sequential spaces. In [10] and [11] Franklin gave some properties of sequential spaces, examples, and a relationship with the Fréchet spaces. Later Snipes, in [19], studied a new class of spaces called $T-$sequential spaces and relationships with sequential spaces. In [2], Boone and Siwiec gave a characterization of sequential spaces by sequential quotient mappings. In [4], Caldas Cueva and Maia Vinagre have studied the $K-c-$Sequential spaces and the $K-s-$bornological spaces and adapted the results established by Snipes using linear mappings. Thereafter Katsaras and Benekas, in [15], starting with a topological vector space $(t.v.s.)$ (E, τ) , have built up the finest of topologies on E having the same convergent sequences as τ, and the thinnest of topologies on E having the same precompact sets as τ, using the concept of String (this study is a generalization of the study led by Weeb [23] in case of locally convex spaces). In [9], Ferrer, Morales and Sánchez Ruiz, have reproduced previous

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work by introducing the concept of maximal sequential topology. Goreham, in [12], has conducted a study linking sequentiality and countable subsets in a topological space.

In [6] A. El Amrani has studied, for a locally $\mathbb{K}$-convex space E in the non-archimedean case, the finest sequential locally $\mathbb{K}$-convex topology on E having the same convergent sequences as the original topology. In this work, we continue this study by characterizing bornological sequential spaces and inductive limit of sequential spaces. Furthermore we study the equicontinuity and duality in sequential spaces. Mention that spaces E characterized by being the inductive limit of any sequence of subspaces covering E were introduced in [8] and [17] under local and non-local convexity conditions, respectively. We would like to point out that there is a great difference between a sequential space and a space of sequences, such spaces have been studied in [3] and [13] in the classical case or [5] , [7] and [1] in the non-archimedean case.

2. Preliminaries

Throughout this paper $\mathbb{K}$ is a non-archimedean(n.a) non trivially valued complete field with the valuation $|.|$ and the valuation ring is $B(0,1) = \{\lambda \in K : |\lambda| \leq 1\}$. The field $\mathbb{K}$ is spherically complete if any decreasing sequence of closed balls in $\mathbb{K}$ has a non-empty intersection. For the basic notions and properties concerning topologies and $\mathbb{K}$-vector spaces we refer to [18] and [22] and those concerning locally $\mathbb{K}$-convex spaces we refer to [16] or [20] if $\mathbb{K}$ is spherically complete and to [18] if $\mathbb{K}$ is not spherically complete. However we recall the following:

Let E be a $\mathbb{K}$-vector space, a nonempty subset A of E is called $\mathbb{K}$-convex if $\lambda x+\mu y+\gamma z \in A$ whenever $x, y, z \in A$, $\lambda, \mu, \gamma \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$, $|\gamma| \leq 1$ and $\lambda + \mu + \gamma = 1$. A is said to be absolutely $\mathbb{K}$-convex if $\lambda x + \mu y \in A$ whenever $x, y \in A$, $\lambda, \mu \in K$, $|\lambda| \leq 1$, $|\mu| \leq 1$. For a nonempty set $A \subset E$ its absolutely $\mathbb{K}$-convex hull $\Gamma(A)$ is the smallest absolutely $\mathbb{K}$-convex set that contains A. If A is a finite set $\{x_1, ..., x_n\}$ we sometimes write $\Gamma(x_1, ..., x_n)$ instead of $\Gamma(A)$. A topology on a vector space E over $\mathbb{K}$ is said to be locally $\mathbb{K}$-convex ($lKcs$) if there exists in E a fundamental system of zero-neighbourhoods consisting of absolutely $\mathbb{K}$-convex subsets of E.

Let E be a locally $\mathbb{K}$-convex space with topology τ. We denote by $E^{'}$, E^{*}, $\sigma(E, E^{'})$ and $\sigma(E^{'}, E)$ the topological dual, the algebraic dual, and the weak topology of E and $E^{'}$ respectively. We denote also by $\mathcal{P}_E$ (or by $\mathcal{P}$ if no confusion can arise) a family of semi-norms determining the topology τ. We always assume that (E, τ) is a Hausdorff space. If A is a subset of E we denote by $[A]$ the vector space spanned by A. We remark that if A is absolutely $\mathbb{K}$-convex, then $[A] = \mathbb{K}A$. For an absolutely $\mathbb{K}$-convex subset A of E we denote by p_A the Minkowski functional on $[A]$, that is for $x \in [A]$, $p_A(x) = \inf\{| \lambda | : x \in \lambda A\}$. If A is bounded then p_A is a norm on $[A]$. We then denote by E_A the space $[A]$ normed by p_A. A subset A of a locally $\mathbb{K}$-convex space E is said to be compactoid if for each neighbourhood U of 0 there exist $x_1, x_2, ..., x_n$ in E such that $A \subset U + \Gamma(x_1, x_2, ..., x_n)$.

Let $\langle , \rangle$ be a duality between E and F where E and F are two vector spaces over $\mathbb{K}$ (see [14] for general results), if A is a subset of E, the polar of A is a subset of F defined by $A^\circ = \{y \in F \ / \ (\forall x \in A) \ |\langle x, y\rangle| \leq 1\}$. We define also the polar of a subset B of F in the same way. A subset A of E is said to be a polar set if $A^{\circ\circ} = A$ ($A^{\circ\circ}$ is the bipolar of A). A continuous semi-norm p on E is called a polar seminorm if the corresponding zero-neighbourhood $B_p(0,1) = \{x \in E : p(x) \leqslant 1\}$ is a polar set. The space E is called polar if there exists $\mathcal{P}_E$ such that every $p \in \mathcal{P}_E$ is polar. If E is a polar space then the weak topology $\sigma(E, E')$ is Hausdorff [18, Proposition 5.6]. In that case we have a dual pair (E, E'), the value of the bilinear form on $E \times E'$ (and similarly on $E' \times E$) is denoted by $\langle x, a\rangle$, $x \in E$, $a \in E'$. For more about polar and spaces see [18].

Let (E, τ) be a locally $\mathbb{K}$-convex space. A subset U of E is called a sequential neighborhood of zero if every null sequence in E lies eventually in U. E is called a sequential space if every sequential neighborhood of zero is a neighborhood of zero. Consider $\mathcal{U}$ the set of all sequential $\mathbb{K}$-convex neighbourhood of zero and $\mathcal{V}$ the familly of all $\mathbb{K}$-convex subsets A of E which are polar and sequential neighbourhood of zero in (E, τ). Then, $\mathcal{U}$ is a base of neighbourhoods of zero for a locally $\mathbb{K}$-convex topology on E which is denoted by τ^s and $\mathcal{V}$ is a base of neighbourhoods of zero for a locally $\mathbb{K}$-convex topology on E which we denoted by τ^{ps}. We will recall that τ^s is always finer than τ ($\tau \leqslant \tau^s$), since every neighborhood of zero is a sequential neighborhood of zero [6].

3. Polarly Bornological Space and Sequential Inductive Topology

(E, τ) is said to be bornological if every absolutely $\mathbb{K}$-convex set U, which absorbs all τ-bounded sets of E, is τ-neighbourhood of zero. In the following, we define the polarly bornological space.

Definition 3.1. Let E be a locally $\mathbb{K}$-convex space. E is called polarly bornological (or simply P-bornological) if every polar set that absorbs every bounded set is a neighbourhood of 0.

Proposition 3.2. *E is P-bornological if, and only if, every n.a. polar seminorm that is bounded over every bounded set of E is continuous.*

Proof. Assume that E is P-bornological. Let p be a $n.a.$ polar semi-norm which is bounded over all bounded subset of E. We have to show that $p : E \to \mathbb{R}^+$ is continuous. It is sufficient to show that $A = B_p(0,1)$ is a neighbourhood of zero in E. Since p is polar, then A is polar [18, Proposition 3.4]. Let B a bounded set of E, then there exists $\lambda \in \mathbb{K}$ such that $p(B) \leqslant |\lambda|$, so $B \subset \lambda A$. Therefore A absorbs B and it is a neighborhood of zero.
Conversely, assume that every $n.a.$ polar semi-norm which is bounded over the bounded sets of E is continuous. Let B a polar which absorbs all bounded sets of E, then p_B is a $n.a.$ polar semi-norm on E. Let D be a bounded set of E, then there exists $\lambda \in \mathbb{K}$ such that $D \subset \lambda B$. Then, $p_B(D) \leqslant |\lambda|$ or

p_B is bounded on D. Therefore, p_B is bounded on every bounded set of E. Hence, p_B is continuous and then B is a neighbourhood of zero. Thus, E is P-bornological. □

Proposition 3.3. *If* (E, τ) *is bornological, then* $\tau = \tau^s$.

Proof. The inequality $\tau \leqslant \tau^s$ is always true. Let U be a sequential absolutely $\mathbb{K}$-convex set U, which absorbs all τ^s-bounded sets of E. By [6, Proposition 6] every τ^s-bounded set B of (E, τ) is τ-bounded. Since E is bornologocal, then U is a neighbourhood of zero in (E, τ). Thus, $\tau \geqslant \tau^s$. □

Proposition 3.4. *If* (E, τ) *is* P*-bornological, then* $\tau^{ps} \leqslant \tau$.

Proof. Assume that (E, τ) is P-bornological. Let U a $\mathbb{K}$-convex polar sequential neighbourhood of zero and B be a τ-bounded subset of E. To obtain a contradiction, we suppose that B is not absorbed by U. Let $(\lambda_n)_n$ be a sequence in $\mathbb{K}$ such that $\lim\limits_{n \to +\infty} \mid \lambda_n \mid = +\infty$, then for all $n \in \mathbb{N}$, $B \not\subset \lambda_n U$ from which there exists a sequence (x_n) in B such that $x_n \notin \lambda_n U$. Since $(\frac{1}{\lambda_n})_n$ converges to 0 and $(x_n)_n$ is bounded, then $(\frac{1}{\lambda_n} x_n)_n$ converges to zero in (E, τ) and for all $n \in \mathbb{N}$, $\frac{1}{\lambda_n} x_n \notin U$, which is not the case by the fact that U is a sequential neighborhood of zero. Then B is absorbed by U which is therefore a neighborhood of zero. Thus, $\tau^{ps} \leqslant \tau$. □

Let E, $(E_\alpha)_{\alpha \in A}$ be $\mathbb{K}$-vector spaces, $f_\alpha : E_\alpha \longrightarrow E$ $(\alpha \in A)$ be linear mappings and τ_α be the locally $\mathbb{K}$-convex topology on E_α. The inductive topology on E relative to the family $(E_\alpha, \tau_\alpha, f_\alpha)_{\alpha \in A}$ is the finest locally $\mathbb{K}$-convex topology making f_α continuous for all $\alpha \in A$. We note this topology by τ_{ind}. A neighborhood basis of zero for τ_{ind} is given by all absorbing, absolutely $\mathbb{K}$-convex subsets U on E such that for all $\alpha \in A$, $f_\alpha^{-1}(U)$ is a neighborhood of zero in (E_α, τ_α).

Theorem 3.5. *Let* $(E_\alpha, \tau_\alpha)_{\alpha \in A}$ *be a family of locally* $\mathbb{K}$*-convex spaces,* E *be a* $\mathbb{K}$*-vector space and* $(f_\alpha)_{\alpha \in A}$ *be a family of linear mappings. If for every* $\alpha \in A$, E_α *is sequential, then* E *equipped with the inductive topology* τ_{ind} *is sequential.*

Proof. Let V be a sequential $\mathbb{K}$-convex neighbourhood of zero in (E, τ_{ind}), then for all $\alpha \in A$, $f_\alpha^{-1}(V)$ is $\mathbb{K}$-convex. Let $(x_n)_n$ be a sequence which converges to zero in (E_α, τ_α) where $\alpha \in A$. Then, the continuity of $f_\alpha : (E_\alpha, \tau_\alpha) \to (E, \tau_{ind})$ implies that the sequence $(f_\alpha(x_n))_n$ converges to zero in (E, τ_{ind}) i.e. there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $f_\alpha(x_n) \in V$, so for all $n \geqslant n_0$, $x_n \in f_\alpha^{-1}(V)$. Therefore, $f_\alpha^{-1}(V)$ is a sequential neighborhood of zero in (E_α, τ_α) and then $f_\alpha^{-1}(V)$ is neighborhood of zero in (E_α, τ_α). Hence V is a neighborhood of zero in (E, τ_{ind}). Thus, (E, τ_{ind}) is sequential. □

Corollary 3.6. *The quotient and the direct sum spaces of sequential locally* $K-$*convex spaces are sequential.*

Proof. The quotient topology and direct sum topology are inductive topologies. □

4. Equicontinuity and duality in sequential spaces

Definition 4.1. Let (E, τ) be a locally $\mathbb{K}$-convex space

(i) Let B a subset of E^*, B is said to be sequentially τ-equicontinuous if and only if, for any sequence $(x_n)_n$ which converges to zero in (E, τ) and for all f in B, the sequence $(f(x_n))_n$ converges uniformly to zero on B; that is to say that the sequence $\sup\limits_{f \in B} (f(x_n))_n$ converges to zero.

(ii) E^s is the space of all linear forms sequentially τ-continuous on E.

(iii) $\tau_b(E^s, E)$ is the strong topology on E^s for the separated duality $\langle E^s, E \rangle$.

Remark 4.2. 1. If B is sequentially τ-equicontinuous, then $B \subset E^s$.

2. $E^s = (E, \tau^s)'$ [6, Proposition 7].

3. $\tau_b(E^s, E)$ admits as a basis of neighborhoods of zero the sets A° where A is a subset of E which is $\sigma\left(E, E'\right)$ $-$bounded.

Lemma 4.3. *Let B be a subset of E^*. If B is sequentially τ-equicontinuous, then $B^{\circ\circ}$ is sequentially τ-equicontinuous.*

Proof. We can assume that $\mathbb{K}$ is dense. Let us show that for all $a \in E$

$$\sup_{f \in B^{\circ\circ}} | f(a) | = \sup_{f \in B} | f(a) |$$

Let a be an element of E, so $\sup\limits_{f \in B^{\circ\circ}} | f(a) | \geqslant \sup\limits_{f \in B} | f(a) |$. Suppose, towards a contradiction, that $\sup\limits_{f \in B^{\circ\circ}} | (f(a) | > \sup\limits_{f \in B} | f(a) |$. Then there exists $\lambda \in \mathbb{K}$ such that $\sup\limits_{f \in B^{\circ\circ}} | f(a) | > | \lambda | > \sup\limits_{f \in B} | f(a) |$ ($\mathbb{K}$ *is dense*). Let g be an element of $B^{\circ\circ}$, so $\forall f \in B$ $| f\left(\frac{a}{\lambda}\right) | = \frac{|f(a)|}{|\lambda|}$ from which $| f\left(\frac{a}{\lambda}\right) | < 1$ and consequently $\frac{a}{\lambda} \in B^\circ$ from which $| g\left(\frac{a}{\lambda}\right) | \leqslant 1$ or $| g(a) | \leqslant | \lambda |$, so $\sup\limits_{f \in B^{\circ\circ}} | f(a) | \leqslant | \lambda |$ which is not the case. Therefore $\sup\limits_{f \in B^{\circ\circ}} | f(a) | = \sup\limits_{f \in B} | f(a) |$. □

Proposition 4.4. *If τ is polar, then any sequentially τ-equicontinuous subset H of E^s is $\tau_b(E^s, E)$-bounded.*

Proof. Let H be a sequentially τ-equicontinuous subset of E^s. Let us show that H is absorbed by any neighborhood of zero in $(E^s, \tau_b(E^s, E))$.
Let A be $\sigma(E, E^s)$-bounded in E. Since $E' \subset E^s$ then A is $\sigma\left(E, E'\right)$-bounded, so A is τ-bounded because τ is polar [18, Corollary 7.7].
Suppose, to derive a contradiction, that H is not absorbed by A°. Then, for all $\lambda \in \mathbb{K}$, $H \not\subset \lambda A^\circ$. Let $(\lambda_n)_n \subset \mathbb{K}$ such that $\lim\limits_{n \to +\infty} | \lambda_n | = +\infty$. For all $n \in \mathbb{N}$, $H \not\subset \lambda_n A^\circ$, from which for all $n \in \mathbb{N}$, there exists $f_n \in H$ such that $f_n \notin \lambda_n A^\circ$,

so there exists $x_n \in A$ such that $\mid f_n\left(\frac{x_n}{\lambda_n}\right) \mid > 1$. Since A is τ-bounded and the sequence $\left(\frac{1}{\lambda_n}\right)_n$ converges to zero, then the sequence $\left(\frac{x_n}{\lambda_n}\right)_n$ converges to zero in (E,τ), from which the sequence $\left(\sup\limits_{f\in H} \mid f\left(\dfrac{x_n}{\lambda_n}\right) \mid\right)_n$ converges to zero,which is not the case because $\sup\limits_{f\in H} \mid f\left(\dfrac{x_n}{\lambda_n}\right) \mid \geqslant \sup\limits_{m\in\mathbb{N}} \mid f_m\left(\dfrac{x_n}{\lambda_n}\right) \mid \geqslant 1$.
Therefore H is $\tau_b\left(E^s, E\right)$ $-$bounded. □

Proposition 4.5. $E^s = \left(E,\tau^{ps}\right)'$.

Proof. We know that $E^s = \left(E, E^s\right)'$ and $\tau^{ps} \leqslant \tau^s$ from which $\left(E,\tau^{ps}\right)' \subset E^s$. Let $f \in E^s$, so $f : (E,\tau^s) \longrightarrow K$ is continuous. Let $U \in \mathcal{U}$ such that $\mid f(U) \mid \leqslant 1$, set $V = U^{\circ\circ}$, then V is polar $\mathbb{K}$-convex and sequential neighbourhood of zero, so $V \in \mathcal{V}$. Suppose, towards a contradiction, that there exists $y \in V$ such that $\mid f(y) \mid > 1$; let $\lambda \in \mathbb{K}$ such that $1 \leqslant \mid \lambda \mid < \mid f(y) \mid$, so for all $x \in U$ we have:

$$\begin{aligned} \mid \left(\frac{1}{\lambda} f\right)(x) \mid &= \frac{1}{\mid \lambda \mid} \mid f(x) \mid \\ &\leqslant \frac{1}{\mid \lambda \mid} \\ &\leqslant 1 \end{aligned}$$

then $\frac{1}{\lambda} f \in U^{\circ}$. But $y \in V$ and $V = U^{\circ\circ}$ so $\mid \left(\frac{1}{\lambda} f\right)(y) \mid \leqslant 1$ or $\mid f(y) \mid \leqslant \mid \lambda \mid$ which is not the case. Therefore $\mid f(V) \mid \leqslant 1$. Hence, $f \in \left(E,\tau^{ps}\right)'$ since V is a neighborhood of zero for τ^{ps}. Therefore $\left(E,\tau^{ps}\right)' = E^s$. □

Proposition 4.6. *τ^{ps} coincides with the topology of uniform convergence on the sequentially τ-equicontinuous subsets of E^s.*

Proof. Let ϱ be the topology of uniform convergence on the sequentially τ-equicontinuous subsets of E^s. A fundamental system of neighborhoods of zero for ϱ is formed by all the A° where A is sequentially τ-equicontinuous in E^s. Let A be a sequentially τ-equicontinuous set in E^s, then A° is $\mathbb{K}$-convex and polar. If $(x_n)_n$ is a sequence which converges to zero in (E,τ), then the sequence $\sup\limits_{f\in A} \left(\mid f(x_n) \mid\right)_n$ converges to zero, from which there is $n_0 \in \mathbb{N}$ such that:

$$\begin{aligned} (\forall n \geqslant n_0) \quad \sup_{f\in A} \mid f(x_n) \mid \leqslant 1 &\Rightarrow (\forall f \in A) \quad (\forall n \geqslant n_0) \quad \mid f(x_n) \mid \leqslant 1 \\ &\Rightarrow (\forall n \geqslant n_0) \quad x_n \in A^{\circ}. \end{aligned}$$

Therefore A° is a sequential neighborhood of zero. Hence, A° is a neighborhood of zero for τ^{ps}. Thus, $\tau^{ps} \geqslant \varrho$.

Let V be a $\mathbb{K}$-convex, polar and sequential neighborhood of zero in (E,τ). Set $A=V^{\circ}$, so $A^{\circ}=V$. We show that A is sequentially τ-equicontinuous. Let $(x_n)_n$ be a sequence that converges to zero in (E,τ), then for all $\varepsilon>0$ there is $\lambda\in\mathbb{K}$ such that $0<\mid\lambda\mid\leqslant\varepsilon$. Therefore, the sequence $\left(\frac{1}{\lambda}x_n\right)_n$ converges to zero in (E,τ), from which there is $n_0\in\mathbb{N}$ such that

$$(\forall n\geqslant n_0)\quad \frac{1}{\lambda}x_n\in V\Rightarrow(\forall f\in A)\quad(\forall n\geqslant n_0)\quad\mid f\left(\frac{1}{\lambda}x_n\right)\mid\leqslant 1$$
$$\Rightarrow(\forall n\geqslant n_0)\quad\sup_{f\in A}\mid f(x_n)\mid\leqslant\mid\lambda\mid\leqslant\varepsilon.$$

Hence $\sup\limits_{f\in A}(\mid f(x_n)\mid)_n$ converges to zero, so A is sequentially τ-equicontinuous. Consequently, $V=A^{\circ}$ is a neighborhood of zero in E for the topology ϱ. Therefore $\varrho\geqslant\tau^{ps}$. Thus, $\varrho=\tau^{ps}$. □

Given a locally $\mathbb{K}$-convex space (E,τ), we denote by E^b the space of all linear forms on E which are bounded on the $\tau-$bounded sets of E. Let τ^0 be the topology on E^b of uniform convergence over the $\tau-$null sequences of E. τ^0 admits as the basis of neighborhood of zero the sets A° where

$$A=\left\{(x_n)_n\subset E:\lim_{n\to+\infty}x_n=0\ in\ (E,\tau)\right\}.$$

Lemma 4.7. *Let H be a $\mathbb{K}$-convex subset of E^s. Then:*

$$(\forall x\in E)\quad p_{H^{\circ}}(x)=\sup_{f\in H}\mid f(x)\mid.$$

Proof. We know that for all $x\in E$, $p_{H^{\circ}}(x)=\inf\{\mid\lambda\mid: x\in\lambda H^{\circ}\}$ and

$$\{x\in E: p_{H^{\circ}}(x)<1\}\subset H^{\circ}\subset\{x\in E: p_{H^{\circ}}(x)\leqslant 1\}.$$

The result is trivially verified if $x=0$. Let $x\in E\setminus\{0\}$, then for all $f\in H$ and all $\lambda\in\mathbb{K}$ such that $x\in\lambda H^{\circ}$ we have:

$$\lambda\in\mathbb{K}\setminus\{0\},\ \frac{1}{\lambda}x\in H^{\circ}\Rightarrow\mid f\left(\frac{1}{\lambda}x\right)\mid\leqslant 1$$
$$\Rightarrow\mid f(x)\mid\leqslant\mid\lambda\mid.$$

Therefore,

$$\sup_{f\in H}\mid f(x)\mid\leqslant\inf\{\mid\lambda\mid: x\in\lambda H^{\circ}\}$$

To obtain a contradiction, we suppose that $\sup\limits_{f\in H}\mid f(x)\mid<p_{H^{\circ}}(x)$. Then, there is $\mu\in\mathbb{K}$ such that:

$$\sup_{f\in H}\mid f(x)\mid\leqslant\mid\mu\mid<p_{H^{\circ}}(x).$$

Therefore,

$$\begin{aligned}(\forall f \in H) \quad \mid f(x) \mid \leqslant \mid \mu \mid &\Rightarrow \mid f\left(\frac{1}{\mu}x\right) \mid \leqslant 1 \\ &\Rightarrow \frac{1}{\mu}x \in H^{\circ} \\ &\Rightarrow x \in \mu H^{\circ} \\ &\Rightarrow p_{H^{\circ}}(x) \leqslant \mid \mu \mid .\end{aligned}$$

which is not the case. Hence, $p_{H^{\circ}}(x) = \sup\limits_{f \in H} \mid f(x) \mid$. □

Lemma 4.8. *If H is sequentially τ-equicontinuous in E^s, then H° is a sequential neighborhood of zero and $p_{H^{\circ}}$ is sequentially τ-equicontinuous.*

Proof. Let $(x_n)_n$ be a sequence which converges to zero in (E, τ), then the sequence $\left(\sup\limits_{f \in H} \mid f(x_n) \mid\right)_n$ converges to zero in $\mathbb{R}^{+}$, from which there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $\sup\limits_{f \in H} \mid f(x_n) \mid \leqslant 1$. Consequently, for all $n \geqslant n_0$, $x_n \in H^{\circ}$.
Furthermore, for all $\varepsilon > 0$, there exists $\lambda \in K$ such that $0 < \mid \lambda \mid \leqslant \varepsilon$. The sequence $\left(\frac{1}{\lambda}x_n\right)_n$ converges to zero in (E, τ) from which there is $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$, $\frac{1}{\lambda}x_n \in H^{\circ}$. Therefore, for all $n \geqslant n_0$ $p_{H^{\circ}}\left(\frac{1}{\lambda}x_n\right) \leqslant 1$, or $\forall n \geqslant n_0$ $p_{H^{\circ}}(x_n) \leqslant \mid \lambda \mid \leqslant \varepsilon$. Thus, the sequence $(p_{H^{\circ}}(x_n))_n$ converges to zero. □

Next theorem will characterize the sequentially τ-equicontinuous subsets. Let us recall that a sequence $\{x_1, x_2, \ldots, x_k\}$ in a locally $\mathbb{K}$-convex space E is t-orthogonal with respect to semi-norm p, such that $t \in]0, 1]$, if for each $n \in \mathbb{N}$ and $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{K}$ we have

$$p\left(\sum_{i=1}^{k} \lambda_i x_i\right) \geqslant t \max_{1 \leqslant i \leqslant k} \left(p(\lambda_i x_i)\right)$$

Theorem 4.9. *Let H be an absolutely $\mathbb{K}$-convex subset of E^s, then the following assertions are equivalent:*

(i) *H is sequentially τ-equicontinuous;*

(ii) *H is τ^0-compactoid.*

Proof. *(i)* $\Rightarrow$ *(ii)* Assume that H be sequentially τ-equicontinuous. Let $A = \{x_n \in E : n \in \mathbb{N}\}$, $\lim\limits_{n \to +\infty} x_n = 0$ in (E, τ). Then, the sequence $(p_{H^{\circ}} \mid f(x_n) \mid)_n$ converges to zero. Set $p = p_{H^{\circ}}$, then p is a $n.a.$ polar semi-norm (Lemma 4.7) which is sequentially τ-continuous (Lemma 4.8). Therefore, for all $x \in E$, $p(x) = \sup\limits_{f \in H} \mid f(x_n) \mid$ (Lemma 4.7). Let $\mu \in \mathbb{K}$ such that $0 < \mid \mu \mid < 1$, hence there exists $N \in \mathbb{N}$ such that for all $n > N$, $p(x_n) \leqslant\mid \mu \mid^2$. Set $F = [x_0, x_1, ..., x_N]$. Let $B = \{z_1, z_2, ..., z_k\}$ be a $\mid \mu \mid$-orthogonal basis of (F, p). We can assume that $\mid \mu \mid\leqslant p(z_k) \leqslant p(z_{k-1}) \leqslant ... \leqslant p(z_1) \leqslant 1$. Let $B^{'} = \{g_1, g_2, ..., g_k\}$ be the dual basis of B, that is to say for all $(i,j) \in (\{1,2,...,k\})^2$, $g_i \in F^{'}$ and $g_i(z_j) = \delta_{ij}$.

Let $x = \sum_{i=1}^{k} \lambda_i z_i$, where $(\lambda_i)_{1\leqslant i\leqslant k} \subset \mathbb{K}$, an element of F, then

$$p(x) \geqslant\mid \mu \mid \max_{1\leqslant i\leqslant k} (\mid \lambda_i \mid p(z_i)) \geqslant \max_{1\leqslant i\leqslant k} \left(\mid \lambda_i \mu^2 \mid\right).$$

Therefore, for all $i \in \{1,2,...,k\}$, $\mid g_i(x) \mid=\mid \lambda_i \mid$, from which $\mid g_i(x) \mid\leqslant \frac{1}{|\mu^2|} p(x)$. Since p is polar, there is $f_i \in E^{'}$ such that $f_{i|F} = g_i$ and for all $x \in E$ $\mid f_i(x) \mid\leqslant \frac{1}{|\mu^2|} p(x)$ [18, Proposition 5.6], and since p is sequentially $\tau-$continuous and polar, we have for all $i \in \{1,2,...,k\}$, $f_i \in E^{ps} \subset E^s$ [18].

Let $f \in H$. Set $h = \sum_{i=1}^{k} f(z_i) f_i$ and $g = f - h$. Then, for all $x \in F$

$$\begin{aligned} h(x) &= \sum_{i=1}^{k} f(z_i) f_i(x) \\ &= \sum_{i=1}^{k} f(z_i) \lambda_i \\ &= f\left(\sum_{i=1}^{k} \lambda_i z_i\right) \\ &= f(x). \end{aligned}$$

Therefore $h = f$ over F and $\mid f \mid\leqslant p$ because $f \in H$, from which for all $i \in \{1,2,...,k\}$, $\mid f(z_i) \mid\leqslant p(z_i) \leqslant 1$. Hence, $h \in \Gamma(f_1, f_2, ..., f_k)$. Furthermore, for all $n > N$, we have $\mid f(x_n) \mid\leqslant p(x_n) \leqslant 1$, and for all $i \in \{1,2,...,k\}$, $\mid f_i(x_n) \mid\leqslant \frac{1}{|\mu^2|} p(x_n) \leqslant 1$, from which

$$\begin{aligned} for\ all\ n > N, \quad \mid h(x_n) \mid &=\mid \sum_{i=1}^{k} f(z_i) f_i(x_n) \mid \\ &\leqslant \max_{1\leqslant i\leqslant k} (\mid f(z_i) \mid\mid f_i(x) \mid) \\ &\leqslant 1. \end{aligned}$$

Then,

$$\begin{aligned} for\ all\ n > N \quad \mid g(x_n) \mid &=\mid f(x_n) - h(x_n) \mid \\ &\leqslant max(\mid f(x_n) \mid, \mid h(x_n) \mid) \\ &\leqslant 1. \end{aligned}$$

Therefore, for all $n \in \mathbb{N}$, $\mid g(x_n) \mid \leqslant 1$ which implies that $g \in A^\circ$. Hence, $f \in \Gamma(f_1, f_2, ..., f_k) + \mu A^\circ$ $(f = g + h)$. Thus, $H \subset \Gamma(f_1, f_2, ..., f_k) + \mu A^\circ$. Which shows that H is τ^0-compactoid.

$(ii) \Rightarrow (i)$ Let $(x_n)_n$ be a convergent sequence to zero in (E, τ) and set $A = \{x_n \in E : n \in \mathbb{N}\}$. Let $\varepsilon > 0$ and $\mu \in \mathbb{K}$ such that $0 <\mid \mu \mid \leqslant \varepsilon$, A° is a neighbourhood of zero in (E, τ°). Let $(f_i)_{1 \leqslant i \leqslant k} \subset E^s$ such that

$$H \subset \Gamma(f_1, f_2, ..., f_k) + \mu A^\circ$$

We have for all $i \in \{1, 2, ..., k\}$ the sequence $(f_i(x_n))_n$ converges to zero. Therefore, there exists $N \in \mathbb{N}$ such that for all $n > N$ and all $i \in \{1, 2, ..., k\}$, $\mid f_i(x_n) \mid \leqslant \mid \mu \mid$. Let $f \in H$, then there exists $(\lambda_i)_{1 \leqslant i \leqslant k} \subset K$ such that for all $i \in \{1, 2, ..., k\}$, $\mid \lambda_i \mid \leqslant 1$ and there is $g \in A^\circ$ such that $f = \sum_{i=1}^{k} \lambda_i f_i + \mu g$.

$$\begin{aligned} for\ all\ n > N, \quad \mid f(x_n) \mid &\leqslant max \left\{ \max_{1 \leqslant i \leqslant k} (\mid \lambda_i \mid\mid f_i(x_n) \mid, \mid \mu g(x_n) \mid) \right\} \\ &\leqslant \mid \mu \mid \end{aligned}$$

from which for all $n > N$, $\sup_{f \in H} \mid f(x_n) \mid \leqslant \mid \mu \mid \leqslant \varepsilon$.

Therefore the sequence $\left(\sup_{f \in H} \mid f(x_n) \mid \right)_n$ converges to zero, and consequently H is sequentially τ-equicontinuous. □

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