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Oscillation theorems for higher order nonlinear functional dynamic equations with unbounded neutral coefficients on time scales

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Abstract. In this paper, we will establish some oscillation criteria for the even-order nonlinear functional dynamic equations with unbounded neutral coefficients on time scales

$$\left(r(t)\left(v^{\Delta^{n-1}}(t)\right)^{\beta}\right)^{\Delta}+f\left(t,\left(u\circ h\right)(t)\right)=0,\quad \text{for all } J_{t_0},$$

on a time scale $\mathbb{T}$, with $\sup\mathbb{T}=\infty$, where β is a quotient of odd integer, such as $\beta>0$, with $J_{t_0}=[t_0,\infty)\cap\mathbb{T}$, and

$$v(t):=u(t)+\sum\nolimits_{i=1}^{i=k}p_i(t)\left(u\circ\eta_i\right)(t),\quad \text{for all } J_{t_0},$$

where n is an integer, such as $n\geq 1$. This study aims to present some new sufficient conditions for the oscillatory of solutions to a class of even-order nonlinear functional dynamic equations.

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1. Introduction

The theory of time scales was introduced by Hilger [14] in order to unify, extend and generalize ideas from discrete calculus, quantum calculus and continuous calculus to arbitrary time scale calculus. The books on the subjects of time scale, that is, measure chain, by Bohner and Peterson [5, 6], summarize and organize much of time scale calculus.

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The theory of oscillations is an important branch of the applied theory of dynamic equations related to the study of oscillatory phenomena in technology and natural and social sciences. In recent years, there has been much research activity concerning the oscillation of solutions of various dynamic equations on time scales.

Motivated by the above articles, in this article we are interested in the oscillation of solutions of the even-order nonlinear delay dynamic equation

$$\left(r(t)\left(v^{\Delta^{n-1}}(t)\right)^{\beta}\right)^{\Delta}+f\left(t,(u\circ h)(t)\right)=0,\quad \text{for all } J_{t_0}, \tag{1.1}$$

on a time scale $\mathbb{T}$, with $\sup\mathbb{T}=\infty$, where $J_{t_0}=[t_0,\infty)\cap\mathbb{T}$ and β is a quotient of odd integer, such as $\beta>0$, with $t_0\in\mathbb{T}$, and

$$v(t):=u(t)+\sum\nolimits_{i=1}^{i=k}p_i(t)\left(u\circ\eta_i\right)(t),\quad \text{for all } t\in J_{t_0}, \tag{1.2}$$

where n is an integer, such as $k\geq 1$. Since we are interested in oscillation, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above and is a time scale interval of the form $J_{t_0}:=[t_0,\infty)\cap\mathbb{T}$. Equation (1.1) will be studied under the following assumptions:

(C_1) The function $f:J_{t_0}\times\mathbb{R}\to\mathbb{R}$ such that $f\in\mathcal{C}\left(J_{t_0}\times\mathbb{R},\mathbb{R}\right)$, $uf\left(t,u\right)>0$, for all $(t,u)\in J_{t_0}\times\mathbb{R}-\{0\}$ and there is $q\in\mathcal{C}\left(J_{t_0},[0,\infty)\right)$ such that

$$u^{-\beta}f\left(t,u\right)\geq q\left(t\right),\quad \text{for all } \left(t,u\right)\in J_{t_0}\times\mathbb{R}-\left\{0\right\}.$$

(C_2) $r,\{p_i\}_{i\in\{1,..,k\}},\{\eta_i\}_{i\in\{1,..,k\}},h\in\mathcal{C}\left(J_{t_0},[0,\infty)\right)$, such as $\{\eta_i\}_{i\in\{1,..,k\}}$ are strictly increasing, h is non-increasing,

$$\int_{t_0}^{\infty}r^{-1/\beta}(s)\Delta s=\infty,\quad \lim_{t\longrightarrow\infty}\eta_i(t)=\lim_{t\longrightarrow\infty}h(t)=\infty,\ \text{for } i\in\left\{1,..,k\right\},$$

and

$$\eta_i\left(t\right)\leq\eta_k\left(t\right),\quad \text{for } i\in\left\{1,..,k\right\},\quad t\in J_{t_0} \tag{1.3}$$

(C_3) The function η_k and h satisfies either

$$\eta_k(t)\geq h(t),\quad \text{for}\quad t\in J_{t_0} \tag{1.4}$$

or

$$\eta_k(t)\leq h(t),\quad \text{for}\quad t\in J_{t_0} \tag{1.5}$$

By a solution of (1.1) we mean a nontrivial real-valued function $u\in\mathcal{C}^n\left(J_{T_u},\mathbb{R}\right)$, $T_u\in J_{t_0}$ which satisfies (1.1) on J_{T_u}. The solutions vanishing in some neighbourhood of infinity will be excluded from our consideration. A solution u of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

Recently, there has been an increasing interest in obtaining sufficient conditions for oscillation and nonoscillation of solutions of various dynamic equations on time scales, we refer the reader to the articles [1, 2, 3, 4, 7, 9, 11, 16, 17, 18, 20, 23, 24, 25, 26] and the references cited therein. Zhang et al [27] studied a class of second-order nonlinear delay dynamic equations of neutral

$$\left[r\left(t\right)\left(z^{\Delta}\left(t\right)\right)^{\alpha}\right]^{\Delta}+q\left(t\right)f\left(x\left(\delta\left(t\right)\right)\right)=0, \qquad t\in J_{t_0}.$$

where $\alpha\geq 1$ is a ratio of odd integers and $z\left(t\right)=x\left(t\right)-p\left(t\right)x\left(\tau\left(t\right)\right)$. Grace et al [12] studied oscillation of fourth-order delay differential equations

$$\left(r_3(r_2(r_1y^{'})^{'})^{'}\right)^{'}(t)+q\left(t\right)y\left(\tau\left(y\right)\right)=0, \qquad t\in j_0,$$

under the assumption

$$\int_{t_0}^{\infty}\frac{dt}{r_i\left(t\right)}<\infty, \qquad i=1,2,3$$

J. Džurina et al [8] studied oscillation for the second-order non-canonical delay differential equations

$$\left(r\left(t\right)\left(y^{'}\left(t\right)\right)^{\gamma}\right)^{'}+q\left(t\right)y^{\gamma}\left(\tau\left(y\right)\right)=0, \qquad t\in I_0,$$

under the condition

$$\int_{t_0}^{\infty}r^{-1/\gamma}\left(t\right)dt<\infty.$$

O. Özdemir [21], studied oscillation for the second-order half-linear functional dynamic equations on time scales

$$\left(r(t)\left(y^{\Delta}(t)\right)^{\beta}\right)^{\Delta}+\sum\nolimits_{i=1}^{i=n}q_i\left(t\right)x^{\beta}\left(h_i\left(t\right)\right)=0, \quad t\in J_{t_0},$$

on a time scale $\mathbb{T}$, where $n\geq 1$ is an integer, β is a quotient of odd integer and

$$y(t):=x(t)+p_1\left(t\right)x\left(\tau_1\left(t\right)\right)+p_2\left(t\right)x\left(\tau_2\left(t\right)\right).$$

So far, there are any results on oscillatory of (1.1). Hence the aim of this paper is to give some oscillation criteria for this equation.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. For $t\in\mathbb{T}$, we define the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$ and the backward jump operator $\rho:\mathbb{T}\to\mathbb{T}$ by $\rho(t):=\sup\{s\in\mathbb{T}:s<t\}$. Then, one defines the graininess function $\mu:\mathbb{T}\to[0,+\infty[$

by $\mu(t) = \sigma(t) - t$. If $\sigma(t) > t$, then we say that t is right-scattered; if $\rho(t) < t$, then t is left-scattered. Moreover, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. If $\mathbb{T}$ has a left-scattered maximum m, then we define $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $u : \mathbb{T} \to \mathbb{R}$, then $u^\sigma : \mathbb{T} \to \mathbb{R}$ is given by $u^\sigma(t) = u(\sigma(t))$ for all $t \in \mathbb{T}$.

Let $u : \mathbb{T} \to \mathbb{R}$ be a real valued function on a time scale $\mathbb{T}$. Then, for $t \in \mathbb{T}^k$, we define $u^\Delta(t)$ to be the number, if one exists, such that for all $\varepsilon > 0$, there is a neighborhood $\mathcal{V}$ of t such that for all $s \in \mathcal{V}$,

$$\left|u^\sigma(t) - u(s) - u^\Delta(t)(\sigma(t) - s)\right| \leq \varepsilon \left|\sigma(t) - s\right|.$$

We say that u is delta differentiable on $\mathbb{T}$ provided $u^\Delta(t)$ exists for all $t \in \mathbb{T}^k$. We will make use of the following product and quotient rules for the derivative of the product uv and the quotient $\frac{u}{v}$ (where $vv^\sigma \neq 0$) of two differentiable functions u and v

$$(uv)^\Delta = u^\Delta v^\sigma + uv^\Delta, \quad \text{and} \quad \left(\frac{u}{v}\right)^\Delta = \frac{u^\Delta v - uv^\Delta}{vv^\sigma}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ will be called rd-continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, we write $f \in \mathcal{C}_{rd}(\mathbb{T}) = \mathcal{C}_{rd}(\mathbb{T}, \mathbb{R})$.

For a, $b \in \mathbb{T}$, and for a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\,\Delta t = f(b) - f(a).$$

An integration by parts formula reads

$$\int_a^b f(t)\,g^\Delta(t)\,\Delta t = \left[f(t)\,g(t)\right]_a^b - \int_a^b f^\Delta(t)\,g^\sigma(t)\,\Delta t.$$

and the improper integrals are defined in the usual way by

$$\int_a^\infty f(t)\,\Delta t = \lim_{b\to\infty} \int_a^b f(t)\,\Delta t.$$

For more on the calculus on time scales, we refer the reader to the books [5, 6].

3. Auxiliary result

The following auxiliary results may play a major role throughout the proofs of our main results.

For simplification, we note

$$\mathcal{D} := \{(t,s) \in \mathbb{T} : t \geq s \geq t_0\} \quad \text{and} \quad \mathcal{D}_0 := \{(t,s) \in \mathbb{T} : t > s \geq t_0\}.$$

Definition 3.1. [22]The function $H \in \mathcal{C}_{rd}(\mathcal{D}, \mathbb{R})$ is said to belong to the function class $\mathcal{P}$ if *i)* $H(t,t)=0$, for $t \geq t_0$ and $H(t,s)>0$ on $\mathcal{D}_0$,

ii) H has a nonpositive rd-continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ on $\mathcal{D}_0$ and there exists a function $\eta \in \mathcal{C}_{rd}(\mathbb{T},\mathbb{R})$, such that

$$H^{\Delta_s}(t,s)+H(t,s)\frac{\eta^{\Delta}(s)}{\eta(\sigma(s))}=\frac{h(t,s)}{\eta(\sigma(s))}H^{1/\gamma}(t,s).$$

Recall that the generalized polynomial on time scale is given by:

$$h_0\left(t,t\right)_0=1 \quad \text{and} \quad h_{n+1}\left(t,t_0\right)=\int_{t_0}^{t} h_n\left(s,t_0\right)\Delta s, \quad \text{for } n\in\mathbb{N}, \quad t\in\mathbb{T}.$$

Lemma 3.2 (Kiguarde's Lemma)**.** *[10, Theorem 2.2]. Let* $n\in\mathbb{N}$*,* $f\in\mathcal{C}^n_{rd}(\mathbb{T},\mathbb{R})$ *and* $\sup\mathbb{T}=\infty$*. Suppose that* f *is either positive or negative and* f^{Δ^n} *is not identically zero and is either nonnegative or nonpositive on* J_{t_0} *for some* $t_0\in\mathbb{T}$*. Then there exist* $t_1\in J_{t_0}$*,* $m\in[0,n)_{\mathbb{Z}}$ *such that* $(-1)^{n-m}f\left(t\right)f^{\Delta^n}\left(t\right)\geq 0$ *holds for all* $t\in J_{t_1}$ *with*

(1) $f\left(t\right)f^{\Delta^j}\left(t\right)\geq 0$ *holds for all* $t\in J_{t_1}$ *and* $j\in[0,m)_{\mathbb{Z}}$*,*

(2) $(-1)^{m+j}f\left(t\right)f^{\Delta^j}\left(t\right)\geq 0$ *holds for all* $t\in J_{t_1}$ *and* $j\in[m,n)_{\mathbb{Z}}$*.*

Lemma 3.3. *[10, Lemma 2.3] Let* $\sup\mathbb{T}=\infty$ *and* $f\in\mathcal{C}^n_{rd}(\mathbb{T},\mathbb{R})$*, with* $n\geq 2$*. Moreover, suppose that Kiguarde's Lemma 3.2 holds with* $m\in[1,n)_{\mathbb{Z}}$ *and* $f^{\Delta^n}\leq 0$ *on* $\mathbb{T}$*. Then there exists a sufficiently large* $t_1\in\mathbb{T}$ *such that*

$$f^{\Delta}(t)\geq h_{m-1}(t,t_1)f^{\Delta^m}(t), \qquad \text{for all } t\in J_{t_1}.$$

Corollary 3.4. *[10, Corollary 2.4]Assume that the conditions of Lemma 3.3 hold. Then*

$$f(t)\geq h_m(t,t_1)f^{\Delta^m}(t), \qquad \text{for all } t\in J_{t_1}.$$

Lemma 3.5. *[15]If* $n\in\mathbb{N}$*,* $\sup\mathbb{T}=\infty$ *and* $f\in\mathcal{C}^n_{rd}(J_{t_0},\mathbb{R})$ *then the following statements are true.*

(1) $\liminf_{t\to\infty} f^{\Delta^n}\left(t\right)>0$ *implies* $\lim_{t\to\infty} f^{\Delta^k}\left(t\right)=\infty$*, for all* $k\in m\in[1,n)_{\mathbb{Z}}$*.*

(2) $\limsup_{t\to\infty} f^{\Delta^n}\left(t\right)<0$ *implies* $\lim_{t\to\infty} f^{\Delta^k}\left(t\right)=-\infty$*, for all* $k\in m\in[1,n)_{\mathbb{Z}}$*.*

Next, we need the following lemma see [13].

Lemma 3.6. *[13] If* A *and* B *are nonnegative and* $\gamma>0$*, then*

$$\lambda AB^{\lambda-1}-A^{\lambda}\leq(\lambda-1)B^{\lambda}, \tag{3.1}$$

where equality holds if and only if $A=B$*.*

4. Oscillation Results

In this section, we establish some sufficient conditions which guarantee that every solution u of (1.1) oscillates on J_{t_0}. We give the main results and for simplification, we note

$$E(t,t_1) := \left(\frac{1}{r(t)}\int_t^\infty \psi(s,t_1)\,q(s)\varphi_k^\beta(h(s))\Delta s\right)^{\frac{1}{\beta}},$$
$$\varphi_k(t) := \frac{1}{p_k\left(\eta_k^{-1}(t)\right)}\left[1-\frac{1}{p_k\left(\eta_k^{-1}(t)\right)}-\sum_{i=1}^{i=k-1}\frac{p_i(\eta_k^{-1}(t))}{\left(p_k\circ\eta_k^{-1}\circ\eta_i\circ\eta_k^{-1}\right)(t)}\right],$$
$$\psi(t,t_1) := \begin{cases}\dfrac{\left(\eta_k^{-1}\circ h\right)(t)-t_1}{t-t_1}, & \text{if } \eta_k(t)\geq h(t),\\ 1, & \text{if } \eta_k(t)<h(t),\end{cases},$$
$$\delta_+(t) := \max\{\delta(t),0\}.$$

Lemma 4.1. *Assume that conditions* (C_1)-(C_2) *hold. Suppose that* u *is an eventually positive solution of* (1.1). *Then there exists* $t_1\in J_{t_0}$ *sufficiently large, such that*

$$v^\Delta(t)>0,\quad v^{\Delta^{n-1}}(t)>0\quad \textit{and}\quad \left(r\left[v^{\Delta^{n-1}}\right]^\beta\right)^\Delta(t)<0,$$

for $t\in J_{t_1}$.

Proof. Let u be an eventually positive solution of 1.1. Then there exists a $t_1\in J_{t_0}$ such that $(u\circ h)(t)>0$ and $v(t)>0$ for $t\in J_{t_1}$. From (C_1), we have

$$\left(r\left[v^{\Delta^{n-1}}\right]^\beta\right)^\Delta(t)<0,\quad \text{for } t\in J_{t_1}.$$

Thus, $t\to r\left[v^{\Delta^{n-1}}\right]^\beta$ is decreasing on J_{t_1}. We claim that $t\to r\left[v^{\Delta^{n-1}}\right]^\beta>0$, for $t\in J_{t_1}$. If not, then there exist a $t_2\in J_{t_1}$, such that

$$r(t)\left[v^{\Delta^{n-1}}(t)\right]^\beta\leq r(t_2)\left[v^{\Delta^{n-1}}(t_2)\right]^\beta=-c<0,\quad \text{for } t\in J_{t_2}.$$

Integrating the above inequality from t_2 to t, we obtain

$$v^{\Delta^{n-2}}(t)\leq v^{\Delta^{n-2}}(t_2)-c^{\frac{1}{\beta}}\int_{t_2}^t (r(s))^{-\frac{1}{\beta}}\Delta s<0,\quad \text{for } t\in J_{t_2}.$$

which implies that $\lim_{t\to\infty} v^{\Delta^{n-2}}(t)=-\infty$. By Lemma 3.5, we obtain $\lim_{t\to\infty} v(t)=-\infty$, which is a contradiction. From Lemma 3.2, there exists an integer $m\in\{1,3,...,n-1\}$ such that (1) and (2) hold on J_{t_2}, clearly $v^\Delta(t)>0$, for $t\in J_{t_2}$. □

Lemma 4.2. *Let u be an eventually positive solution of* (1.1), *then*

$$u(t) \geq \frac{\left(v\circ\eta_k^{-1}\right)(t)}{p_k\left(\eta_k^{-1}(t)\right)} - \frac{\left(v\circ\eta_k^{-1}\right)(t)}{\left[p_k\left(\eta_k^{-1}(t)\right)\right]^2} - \sum_{i=1}^{i=k-1}\frac{p_i(\eta_k^{-1}(t))}{p_k(\eta_k^{-1}(t))}\frac{\left(v\circ\eta_k^{-1}\circ\eta_i\circ\eta_k^{-1}\right)(t)}{\left(p_k\circ\eta_k^{-1}\circ\eta_i\circ\eta_k^{-1}\right)(t)}. \tag{4.1}$$

Proof. Let u be an eventually positive solution of 1.1. From the definition of v, we get

$$p_k(t)\left(u(\eta_k(t)\right) := v(t) - u(t) - \sum_{i=1}^{i=k-1} p_i(t)\left(u\circ\eta_i\right)(t), \quad \text{for all } t\in J_{t_0},$$

and so

$$p_k(\eta_k^{-1}(t))u(t) := v(\eta_k^{-1}(t)) - u(\eta_k^{-1}(t)) - \sum_{i=1}^{i=k-1} p_i(\eta_k^{-1}(t))\left(u\circ\eta_i\right)(\eta_k^{-1}(t)),$$

Repeating the same process, we have

$$u(t) = \frac{1}{p_k(\eta_k^{-1}(t))}\{v(\eta_k^{-1}(t)) - u(\eta_k^{-1}(t)) - \sum_{i=1}^{i=k-1} p_i(\eta_k^{-1}(t))\left(u\circ\eta_i\right)(\eta_k^{-1}(t)).$$

As above we see that

$$u(t) \leq \frac{\left(v\circ\eta_k^{-1}\right)(t)}{\left(p_k\circ\eta_k^{-1}\right)(t)},$$

then

$$\left(u\circ\eta_i\circ\eta_k^{-1}\right)(t) \leq \frac{\left(v\circ\eta_k^{-1}\circ\eta_i\circ\eta_k^{-1}\right)(t)}{\left(p_k\circ\eta_k^{-1}\circ\eta_i\circ\eta_k^{-1}\right)(t)}, \quad i\in\{1,2,...,k\},$$

which yields

$$u(t) \geq \frac{\left(v\circ\eta_k^{-1}\right)(t)}{p_k\left(\eta_k^{-1}(t)\right)} - \frac{\left(v\circ\eta_k^{-1}\right)(t)}{\left[p_k\left(\eta_k^{-1}(t)\right)\right]^2} - \sum_{i=1}^{i=k-1}\frac{p_i(\eta_k^{-1}(t))}{p_k(\eta_k^{-1}(t))}\frac{\left(v\circ\eta_k^{-1}\circ\eta_i\circ\eta_k^{-1}\right)(t)}{\left(p_k\circ\eta_k^{-1}\circ\eta_i\circ\eta_k^{-1}\right)(t)}$$

Thus, (4.1) holds. This completes the proof. □

Lemma 4.3. *Let u be an eventually positive solution of 1.1. Then*

$$\left(r(t)\left(v^{\Delta^{n-1}}(t)\right)^\beta\right)^\Delta \leq -q(t)\varphi_k^\beta(h(t))v^\beta(\eta_k^{-1}(h(t)), \quad \textit{for } t\in J_{t_1}. \tag{4.2}$$

Proof. Let u be an eventually positive solution of 1.1 on J_{t_0}. It follows that the functions $(\eta_i)_{i\in[1,n]_{\mathbb{Z}}}$ are strictly increasing on J_{t_1} and by (1.3), we have that

$$\left(\eta_k^{-1}\circ\eta_i\circ\eta_k^{-1}\right)(t) \leq \eta_k^{-1}(t) \quad \text{for } i\in\{1,2,...,k\}, \quad t\in J_{t_0}.$$

Since v is strictly increasing, we obtain

$$\left(v\circ\eta_k^{-1}\circ\eta_i\circ\eta_k^{-1}\right)(t)<\left(v\circ\eta_k^{-1}\right)(t)\quad \text{for } i\in\{1,2,...,k\},\quad t\in J_{t_0}. \tag{4.3}$$

Substituting (4.3) in (4.1), we have

$$\begin{aligned} u(t) &\geq \frac{\left(v\circ\eta_k^{-1}\right)(t)}{p_k\left(\eta_k^{-1}(t)\right)}\left[1-\frac{1}{p_k\left(\eta_k^{-1}(t)\right)}-\sum_{i=1}^{i=k-1}\frac{p_i(\eta_k^{-1}(t))}{\left(p_k\circ\eta_k^{-1}\circ\eta_i\circ\eta_k^{-1}\right)(t)}\right] \\ &\geq \varphi_k(t)\,v(\eta_k^{-1}(t)),\quad \text{for } t\in J_{t_1}. \end{aligned}$$

From the above inequality, we obtain

$$u(h(t))\geq\varphi_k(h(t))v(\eta_k^{-1}(h(t)),\quad \text{for } t\in J_{t_1}. \tag{4.4}$$

Substituting (4.4) in (1.1), we get

$$\left(r(t)\left(v^{\Delta^{n-1}}(t)\right)^{\beta}\right)^{\Delta}\leq -q(t)\varphi_k^{\beta}(h(t))v^{\beta}(\eta_k^{-1}(h(t)),\quad \text{for } t\in J_{t_1}.$$

Thus, 4.2 holds. This completes the proof. □

Theorem 4.4. *Assume that conditions* (C_1)-(C_3), *and* (1.5) *hold. If there exists a positive function* $\tau\in\mathcal{C}_{rd}^1(J_{t_0},\mathbb{R})$, *such that for all sufficiently large* $t_1\in J_{t_0}, t_2\in J_{t_2}$,

$$\limsup_{t\longrightarrow\infty}\int_{t_2}^{t}\left[\tau(s)\,q(s)\varphi_k^{\beta}(h(s))-\frac{r(s)\left[\tau^{\Delta}(s)\right]_+^{\beta+1}}{(\beta+1)^{\beta+1}E(s,t_1)\,\tau^{\beta}(s)}\right]\Delta s=\infty. \tag{4.5}$$

Then any solution of (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution u on J_{t_0}. We may assume without loss of generality that there exists $t_1\in J_{t_0}$ such that

$$u(t)>0,\quad (u\circ h)(t)>0,\quad \text{and}\quad (u\circ\eta_i)(t)>0,\ \text{for } t\in J_{t_1},\ i\in\{1,2,..,k\}.$$

Integrating equation (4.2) from $t\in J_{t_1}$ to ∞, we have

$$v^{\Delta^{n-1}}(t)\geq\left(\frac{1}{r(t)}\int_t^{\infty}q(s)\varphi_k^{\beta}(h(s))v^{\beta}(\eta_k^{-1}(h(s))\Delta s\right)^{\frac{1}{\beta}},\quad \text{for } t\in J_{t_1} \tag{4.6}$$

From (1.5), we have

$$\left(\eta_k^{-1}\circ h\right)(s)\geq s\geq t,\quad \text{for } t\in J_{t_1}.$$

Since $v^{\Delta}(t)>0$ for $t\in J_{t_1}$, we obtain

$$v\left(\left(\eta_k^{-1}\circ h\right)(s)\right)\geq v(s)\geq v(t),\quad \text{for } t\in J_{t_1} \tag{4.7}$$

As above we see that

$$\begin{aligned} v^{\Delta^{n-1}}(t) &\geq v(t)\left(\frac{1}{r(t)}\int_t^{\infty} q(s)\varphi_k^{\beta}(h(s))\Delta s\right)^{\frac{1}{\beta}} \\ &= v(t)\,E(t,t_1), \quad \text{for } t\in J_{t_1} \end{aligned} \tag{4.8}$$

Define the function ω by:

$$\omega(t) := \tau(t)\frac{r(t)\left(v^{\Delta^{n-1}}(t)\right)^{\beta}}{v^{\beta}(t)} > 0, \quad \text{for all } t\in J_{t_1}. \tag{4.9}$$

From (4.9) and (4.2), we get

$$\begin{aligned} \omega^{\Delta}(t) &\leq \tau^{\Delta}(t)\frac{r(\sigma(t))\left(v^{\Delta^{n-1}}(\sigma(t))\right)^{\beta}}{v^{\beta}(\sigma(t))} - \tau(t)r(\sigma(t))\frac{\left(v^{\Delta^{n-1}}(\sigma(t))\right)^{\beta}\left(v^{\beta}(t)\right)^{\Delta}}{v^{\beta}(t)v^{\beta}(\sigma(t))} \\ &\quad -\tau(t)\left(q(t)\varphi_k^{\beta}(h(t))\frac{v^{\beta}(\eta_k^{-1}(h(t)))}{v^{\beta}(t)}\right). \end{aligned} \tag{4.10}$$

Substituting (4.7) in (4.10), we have

$$\begin{aligned} \omega^{\Delta}(t) &\leq \tau^{\Delta}(t)r^{\sigma}(t)\frac{\left(v^{\Delta^{n-1}}(\sigma(t))\right)^{\beta}}{v^{\beta}(\sigma(t))} \\ &\quad -\tau(t)r^{\sigma}(t)\frac{\left(v^{\Delta^{n-1}}(\sigma(t))\right)^{\beta}\left(v^{\beta}(t)\right)^{\Delta}}{[v(t)v^{\sigma}(t)]^{\beta}} - \tau(t)q(t)\varphi_k^{\beta}(h(t)) \\ &\leq \frac{\tau^{\Delta}(t)}{\tau^{\sigma}(t)}\omega^{\sigma}(t) - \frac{\tau(t)}{\tau^{\sigma}(t)}\frac{\left(v^{\beta}(t)\right)^{\Delta}}{v^{\beta}(t)}\omega^{\sigma}(t) - \tau(t)q(t)\varphi_k^{\beta}(h(t)). \end{aligned}$$

By Pötzsche's chain rule [5, Theorem 1.90] and (4.8), we obtain

$$\omega^{\Delta}(t) \leq \frac{\tau^{\Delta}(t)}{\tau^{\sigma}(t)}\omega^{\sigma}(t) - \beta E(t)\frac{\tau(t)}{\tau^{\sigma}(t)}\frac{v^{\Delta^{n-1}}(t)}{v(t)}\omega^{\sigma}(t) - \tau(t)q(t)\varphi_k^{\beta}(h(t)).$$

Using the fact that $t\to v$ is increasing on J_{t_1} and $t\to r(t)\left(v^{\Delta^{n-1}}(t)\right)^{\beta}$ is decreasing on J_{t_1}, we find

$$\begin{aligned} \left(\frac{\omega(t)}{\tau(t)}\right)^{1/\beta} &= \frac{r^{1/\beta}(t)v^{\Delta^{n-1}}(t)}{v(t)} \geq \frac{r^{1/\beta}(\sigma(t))v^{\Delta^{n-1}}(\sigma(t))}{v(t)} \\ &\geq \left(\frac{\omega(\sigma(t))}{\tau(\sigma(t))}\right)^{1/\beta}, \quad \text{for all } t\in J_{t_1}. \end{aligned}$$

Substituting (4.9) in above inequality, we get

$$\begin{aligned} \omega^{\Delta}(t) &\leq \frac{\tau^{\Delta}(t)}{\tau^{\sigma}(t)}\omega^{\sigma}(t) - \beta E(t,t_1)\left(\frac{\tau^{\beta}(t)}{r(t)\,\tau^{\beta+1}(\sigma(t))}\right)^{\frac{1}{\beta}}\omega^{1+\frac{1}{\beta}}(\sigma(t)) \\ &\quad -\tau(t)q(t)\varphi_k^{\beta}(h(t)). \end{aligned} \tag{4.11}$$

If we apply Lemma 3.6 , we see that
(4.12)
$$\frac{\tau^{\Delta}(t)}{\tau^{\sigma}(t)}\omega^{\sigma}(t)-\beta\left[\frac{\tau^{\beta}(t)}{r(t)\tau^{\beta+1}(\sigma(t))}\right]^{\frac{1}{\beta}}E(t,t_1)\,\omega^{\frac{\beta+1}{\beta}}(\sigma(t))\leq\frac{1}{(\beta+1)^{\beta+1}}\frac{r(t)\left(\tau^{\Delta}(t)\right)^{\beta+1}}{E(t,t_1)\,\tau^{\beta}(t)}.$$

Using (4.12) in (4.11), we obtain

$$\omega^{\Delta}(t)\leq\frac{1}{(\beta+1)^{\beta+1}}\frac{r(t)\left(\tau^{\Delta}(t)\right)^{\beta+1}}{E(t,t_1)\,\tau^{\beta}(t)}-\tau(t)q(t)\varphi_k^{\beta}(h(t)).$$

Integrating the last inequality from t_2 to t, we have

$$\int_{t_2}^{t}\left[\tau(s)\,q(s)\varphi_k^{\beta}(h(s))-\frac{r(s)\left[\tau^{\Delta}(s)\right]_{+}^{\beta+1}}{(\beta+1)^{\beta+1}E(s,t_1)\,\tau^{\beta}(s)}\right]\Delta s\leq\omega(t_2),$$

which contradicts (4.5). This completes the proof. □

Theorem 4.5. *Assume that conditions* (C_1)-(C_3), *and* (1.4) *hold. If there exists a positive function* $\theta\in\mathcal{C}_{rd}^{1}(J_{t_0},\mathbb{R})$, *such that for all sufficiently large* $t_1\in J_{t_0}, t_2\in J_{t_2}$,
(4.13)
$$\limsup_{t\longrightarrow\infty}\int_{t_2}^{t}\left[\theta(s)\,q(s)\varphi_k^{\beta}(h(s))\psi(s,t_1)-\frac{r(s)\left[\theta^{\Delta}(s)\right]_{+}^{\beta+1}}{(\beta+1)^{\beta+1}E(s,t_1)\,\theta^{\beta}(s)}\right]\Delta s=\infty.$$

Then any solution of (1.1) *is oscillatory.*

Proof. Suppose that (1.1) has a nonoscillatory solution u on J_{t_0}. We may assume without loss of generality that there exists $t_1\in J_{t_0}$ such that

$$u(t)>0,\quad (u\circ h)(t)>0,\quad\text{and}\quad (u\circ\eta_i)(t)>0,\ \text{for } t\in J_{t_1},\ i\in\{1,2,..,k\}.$$

From Lemma 3.3 we have

$$v(t)\geq v^{\Delta}(t)(t-t_1),\quad\text{for } t\in J_{t_1},$$

Hence $t\to\dfrac{v}{t-t_1}$ is a nonincreasing function on J_{t_2} and from (1.4), we have

$$\left(\eta_k^{-1}\circ h\right)(t)\leq t,\quad\text{for } t\in J_{t_1},$$

then

$$\frac{v\left(\eta_k^{-1}\circ h\right)(t)}{\left(\eta_k^{-1}\circ h\right)(t)-t_1}\geq\frac{v(t)}{t-t_1},\quad\text{for } t\in J_{t_2}. \tag{4.14}$$

By (4.6) and as above inequality, we get

$$\begin{aligned} v^{\Delta^{n-1}}(t) &\geq \left(\frac{1}{r(t)}\int_t^\infty \frac{\left(\eta_k^{-1}\circ h\right)(s)-t_1}{s-t_1} q(s)\varphi_k^\beta(h(s))v(s)\,\Delta s\right)^{\frac{1}{\beta}} \\ &\geq v(t)\left(\frac{1}{r(t)}\int_t^\infty \frac{\left(\eta_k^{-1}\circ h\right)(s)-t_1}{s-t_1} q(s)\varphi_k^\beta(h(s))\Delta s\right)^{\frac{1}{\beta}} \\ &= v(t)\,E(t,t_1), \quad \text{for } t\in J_{t_2} \end{aligned}$$

Substituting (4.14) in (4.10), we get

$$\begin{aligned} \omega^\Delta(t) \leq\; & \frac{\theta^\Delta(t)}{\theta^\sigma(t)}\omega^\sigma(t) - \beta E(t,t_1)\left(\frac{\theta^\beta(t)}{r(t)\,\theta^{\beta+1}(\sigma(t))}\right)^{\frac{1}{\beta}} \omega^{1+\frac{1}{\beta}}(\sigma(t)) \\ & -\theta(t)q(t)\varphi_k^\beta(h(t))\psi(t,t_1). \end{aligned}$$

The rest of proof is similar to that of Theorem 4.4. □

Corollary 4.6. *Assume that conditions* (C_1)-(C_3) *hold, such that for all sufficiently large* $t_1\in J_{t_0}$, $t_2\in J_{t1}$,

$$\limsup_{t\longrightarrow\infty}\int_{t_2}^t q(s)\varphi_k^\beta(h(s))\psi(s,t_1)\,\Delta s=\infty. \tag{4.15}$$

Then any solution of (1.1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 4.4, if we put $\tau(t)=1$ in Equation (4.5), we find Equation (4.15). □

Corollary 4.7. *Assume that conditions* (C_1)-(C_3) *hold. If there exists a positive function* $\tau\in\mathcal{C}^1_{rd}(J_{t_0},\mathbb{R})$ *such that* (4.5) *holds, then any solution of is oscillatory or convergent.*

Theorem 4.8. *Assume that conditions* (C_1)-(C_3), *and* (1.4) *hold. If there exists a positive function* $\tau\in\mathcal{C}^1_{rd}(J_{t_0},\mathbb{R}^+)$ *and* H, $h\in\mathcal{C}_{rd}(\mathcal{D},\mathbb{R})$, *where* H *belongs to the class* $\mathcal{P}$, *such that for all sufficiently large* $t_1\in J_{t_0}$, $t_2\in J_{t_2}$,

(4.16)

$$\limsup_{t\longrightarrow\infty}\frac{1}{H(t,t_2)}\int_{t_2}^t\left[H(t,s)q(s)\varphi_k^\beta(h(s))\psi(s,t_1)-\frac{r(s)\left[\tau^\Delta(s)\right]_+^{\beta+1}}{(\beta+1)^{\beta+1}E(s,t_1)\tau^\beta(s)}\right]\Delta s=\infty,$$

then any solution of (1.1) *is oscillatory.*

5. Examples

In this section, we give an example to illustrate our main result.

Example 5.1. Consider the neutral differential equation

$$\sum\nolimits_{i=0}^{i=k} \left[e^{t}u\left(t+i\right)\right]^{(n)}+tx\left(t+2k\right)=0, \quad t\geq 0. \tag{5.1}$$

Here, $\mathbb{T}=\mathbb{R}$, $\beta=1$, $r\left(t\right)=1$, $n,k\in\mathbb{N}$, $\eta_{i}\left(t\right)=t+i$, $p_{i}\left(t\right)=e^{t}$, for all $i\in\{1,2,...,k\}$, $h\left(t\right)=t+k$ and $f\left(t,x\right)=tx$. Then $q\left(t\right)=t$ and the hypotheses $(C_1)-(C_2)$, (1.5) hold. On the other hand, we see that

$$\eta_{i}^{-1}\left(t\right)=t-i, \quad \text{for all } i\in\left\{1,2,...,k\right\},$$

As $\eta_k(t)\leq h(t)$, then $\psi\left(t,t_1\right)=1$, on the other hand, we have

$$\begin{aligned}\varphi_{k}\left(t\right) &= e^{-t+k}\left[1-e^{-t+k}-\frac{1-e^{1-k}}{e-1}\right]\\ &\geq \lambda e^{k-t}, \text{ for } t \text{ large enough}\end{aligned}$$

where, $\lambda\in(0,1)$, then

$$E(t,t_1)\geq\lambda\int_{t}^{\infty}se^{-s}ds\geq\lambda\in\left(0,1\right).$$

Let $\tau\left(t\right)=1$, for $t\geq0$. Thus, (4.5) holds. By Theorem 4.4, equation (5.1) is oscillatory.

Example 5.2. Consider the neutral differential equation

$$\left[\sqrt[3]{x(t)+\sum\nolimits_{i=1}^{i=k-1}u(t+i)+2^{t+k}u\left(t+k\right)}\right]^{\Delta}+\sqrt[3]{x\left(t\right)}=0, \quad t\in\mathbb{N}. \tag{5.2}$$

Here, $\mathbb{T}=\mathbb{N}$, $\beta=\frac{1}{3}$, $r\left(t\right)=1$, $n,k\in\mathbb{N}$, $p_{i}\left(t\right)=1$, for all $i\in\{1,2,...,k-1\}$, $p_{k}\left(t\right)=2^{t+k}$, $\eta_{i}\left(t\right)=t+i$, for all $i\in\{1,2,...,k\}$, $h\left(t\right)=t$ and $f\left(t,x\right)=\sqrt[3]{x}$. Then $q\left(t\right)=1$ and the hypotheses (C_1)-(C_1), (1.4) hold. Therefore $\eta_{i}^{-1}\left(t\right)=t-i$, for all $i\in\{1,2,...,k\}$. As $\eta_k(t)\geq h(t)$, then

$$\psi\left(t,t_1\right)=\frac{t-k-t_1}{t-t_1}\geq\frac{t}{2}, \quad \text{for } t \text{ large enough}$$

then

$$\begin{aligned}\varphi_{k}\left(t\right) &= \frac{1}{2^{t}}\left(1-\frac{1}{2^{t}}-\sum\nolimits_{i=1}^{i=k-1}\frac{1}{2^{t-k+i}}\right)\\ &\geq \lambda 2^{-t}, \quad \text{for } t \text{ large enough}\end{aligned}$$

where $\lambda\in(0,1)$, then

$$E(t,t_1)\geq\left(\frac{\lambda}{2}\int_{t}^{\infty}s2^{-\frac{s}{3}}\Delta s\right)^{3}\geq\frac{\lambda^{3}t^{3}2^{-t}}{8} \quad \text{for } t \text{ large enough}$$

Let $\theta\left(t\right)=\left(\sqrt[3]{2}\right)^{t}$, for $t\in\mathbb{N}$. Thus, (4.13) holds. By Theorem 4.5, equation (5.2) is oscillatory.

6. Conclusion

In this paper, we use Riccati transformation technique to establish some new oscillation results of higher-order nonlinear neutral dynamic equations with damping on time scales. Our results not only unify the oscillation of differential equations and difference equations but also improve the differential equations established in [19, 27], etc.

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