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New type of timelike $(1,3)$-Bertrand curves in Minkowski space-time

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Abstract. In the present paper, we define the notion of $(1,3)$-V Bertrand curves in $\mathbb{E}_1^4$. Then we find the necessary and sufficient conditions for timelike curves in $\mathbb{E}_1^4$ to be $(1,3)$-V Bertrand curves. Finally we give some related examples.

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1. Introduction

Many studies have been done on the general theorem of curves in Euclidean space (or more generally in a Riemannian manifold). In the light of these studies, we have deep knowledge on its local geometry as well as its global geometry. One of the most important problems in Euclidean spaces is the characterization of curves. There are two well-known methods for solving this problem. One of them is to figure out the relationship between the Frenet vectors of the curves [10] and the other is to determine the shape and size of a regular curve by using its curvatures k_1 (or $\varkappa$) and k_2 (or τ).

In 1845, Saint Venant [14] put forward the question whether the principal normal of a curve is the principal normal of another curve on the surface generated by the principal normal of the given one. Bertrand gave an answer to this question in the paper published in 1850 [2]. He proved that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship calculated with constant coefficients should exist between the first and second curvatures of the given original curve. In other words, we have $\lambda k_1 + \mu k_2 = 1$, $\lambda, \mu \in R$ where k_1 and k_2 is denot by the first and second curvatures of a given curve, respectively. Since 1850, after the paper of Bertrand, the pairs of curves like this have been called Conjugate Bertrand Curves, or more commonly Bertrand Curves [10].

When considering the properties of Bertrand curves in Euclidean n-space, either k_2 or k_3 turns out to be zero, which means that Bertrand curves in $\mathbb{E}^n$ $(n > 3)$ are degenerate curves [13]. This result is restated by Matsuda and

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Yorozu [11]. They proved that there was not any special Bertrand curve in $\mathbb{E}^n$ $(n > 3)$ and defined a new kind, which is called $(1,3)$-type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many researchers in a Minkowski 3-space and Minkowski space-time (see [1, 7, 9, 15, 18, 17, 16]) as well as in Euclidean space.

Recently, in [4], the author defined V-Bertrand curves in the Euclidean 3-space. Following [4], the authors in [5] defined $(1,3)$-V Bertrand in Euclidean 4-space $\mathbb{E}^4$.

In this paper, we consider timelike $(1,3)$-V Bertrand curves in Minkowski space-time $\mathbb{E}^4_1$. Since the plane spanned by the principal normal vector and the second binormal vector of a timelike curve in $\mathbb{E}^4_1$ is a spacelike plane, the $(1,3)$-V Bertrand mate curve of a timelike curve can be a timelike curve, spacelike curve and Cartan null curve. For all cases, we give the necessary and sufficient conditions for a timelike curve to be a $(1,3)$-V Bertrand curve in $\mathbb{E}^4_1$ and we give some related examples. Finally we show that if a timelike curve in $\mathbb{E}^4_1$ is a $(1,3)$-N Bertrand curve or $(1,3)$-B_2 Bertrand curve, then the Bertrand mate curve of the given curve cannot be a Cartan null curve.

2. Preliminaries

The Minkowski space-time $\mathbb{E}^4_1$ is the Euclidean 4-space $\mathbb{E}^4$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a rectangular coordinate system of $\mathbb{E}^4_1$. Recall that a vector $v \in \mathbb{E}^4_1 \backslash \{0\}$ can be spacelike if $g(v, v) > 0$, timelike if $g(v, v) < 0$ and null (lightlike) if $g(v, v) = 0$. In particular, the vector $v = 0$ is said to be spacelike. The norm of a vector v is given by $||v|| = \sqrt{|g(v, v)|}$. Two vectors v and w are said to be orthogonal, if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}^4_1$, can locally be spacelike, timelike or null (lightlike), if all its velocity vectors $\alpha'(s)$ are respectively spacelike, timelike or null ([12]).

A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$ ([3], [12]). On the other hand, a non-null curve α is parametrized by the arclength parameter s if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\{T, N, B_1, B_2\}$ be the moving Frenet frame along a curve α in $\mathbb{E}^4_1$, consisting of the tangent, the principal normal, the first binormal and the second binormal vector field respectively.

If α is a spacelike or a timelike curve whose the Frenet frame $\{T, N, B_1, B_2\}$ contains only non-null vector fields, the Frenet equations are given by ([8])

$$(2.1) \qquad \begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2\kappa_1 & 0 & 0 \\ -\epsilon_1\kappa_1 & 0 & \epsilon_3\kappa_2 & 0 \\ 0 & -\epsilon_2\kappa_2 & 0 & -\epsilon_1\epsilon_2\epsilon_3\kappa_3 \\ 0 & 0 & -\epsilon_3\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix},$$

where $g(T, T) = \epsilon_1$, $g(N, N) = \epsilon_2$, $g(B_1, B_1) = \epsilon_3$, $g(B_2, B_2) = \epsilon_4$, $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 =$

-1, $\epsilon_i \in \{-1, 1\}$, $i \in \{1, 2, 3, 4\}$. In particular, the following conditions hold:

$$g(T, N) = g(T, B_1) = g(T, B_2) = g(N, B_1) = g(N, B_2) = g(B_1, B_2) = 0.$$

If α is a null Cartan curve, the Cartan Frenet equations are given by ([3])

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \kappa_2 & 0 & -\kappa_1 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ -\kappa_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.2}$$

where the first curvature $\kappa_1(s) = 0$, if $\alpha(s)$ is a null straight line or $\kappa_1(s) = 1$ in all other cases. In this case, the following conditions hold:

$$g(T, T) = g(B_1, B_1) = 0, \quad g(N, N) = g(B_2, B_2) = 1,$$

$$g(T, N) = g(T, B_2) = g(N, B_1) = g(N, B_2) = g(B_1, B_2) = 0, \quad g(T, B_1) = 1.$$

3. On $(1,3)$-V Bertrand curves in $\mathbb{E}_1^4$

In this section, we give the definition of $(1,3)$-V Bertrand curve in $\mathbb{E}_1^4$ and we find the necessary and sufficient conditions for curves in $\mathbb{E}_1^4$ to be $(1,3)$-V Bertrand curves.

Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and curvatures $\kappa_1, \kappa_2, \kappa_3$. Consider the vector field V given by

$$V(s) = a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s)$$

where $a(s)$, $b(s)$, $c(s)$, $d(s)$ are functions on I. Then we can define the unit speed curve

$$\gamma(s) = \int_0^s V(u)\, du,$$

which is called the integral curve of $V(s)$ ([6]).

Definition 3.1. Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and curvatures $\kappa_1, \kappa_2, \kappa_3$. For $(e(s), f(s)) \neq (0, 0)$, we can define the curve

$$\beta^*(h(s)) = \int_0^s V(u)\, du + e(s)N(s) + f(s)B_2(s)$$

with the Frenet frame $\{T^*, N^*, B_1^*, B_2^*\}$ and curvatures $\kappa_1^*, \kappa_2^*, \kappa_3^*$. Then β is called a $(1,3)$-V Bertrand curve if there exist a curve $\beta^*(h(s))$ such that the plane spanned by $\{N, B_2\}$ coincides with the plane spanned by$\{N^*, B_2^*\}$. Here the curve β^* is called the $(1,3)$-V Bertrand mate curve of the curve β.

For $a(s) = 1$, $b(s) = c(s) = d(s) = 0$, let us call the curve β a $(1,3)$-T Bertrand curve or $(1,3)$-Bertrand curve defined by Matsuda and Yorozu in [11].

For $b(s) = 1$, $a(s) = c(s) = d(s) = 0$, let us call the curve β a $(1,3)$-N Bertrand curve.

For $c(s) = 1$, $a(s) = b(s) = d(s) = 0$, let us call the curve β a $(1,3)$-B_1 Bertrand curve.

For $d(s) = 1$, $a(s) = b(s) = c(s) = 0$, let us call the curve β a $(1,3)$-B_2 Bertrand curve.

Remark 3.2. Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a timelike $(1,3)$-V Bertrand curve and β^* be the $(1,3)$-V Bertrand mate curve of the curve β in $\mathbb{E}_1^4$. Then β^* may be a timelike curve, a spacelike curve with timelike B_1^* or a Cartan null curve.

In the following theorem, we consider β^* as a timelike curve and a spacelike curve with timelike B_1^*.

Theorem 3.3. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a timelike curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_2, \kappa_3$. Then, the curve β is a $(1,3)$-V Bertrand curve if and only if one of the following conditions holds:*
(i) there exist functions a, b, c, d, e, f satisfying

$$b + e' = 0, \quad d + f' = 0, \quad c + e\kappa_2 - f\kappa_3 = 0, \quad a + e\kappa_1 \neq 0. \tag{3.1}$$

In this case, there exists a homothety map between β and β^*. So β^* is a timelike curve.
(ii) there exist functions a, b, c, d, e, f and constant real numbers v, μ satisfying

$$b + e' = 0, \quad d + f' = 0, \quad c + e\kappa_2 - f\kappa_3 \neq 0, \tag{3.2}$$

$$a + e\kappa_1 = v(c + e\kappa_2 - f\kappa_3) \tag{3.3}$$

$$\mu\kappa_3 = v\kappa_1 - \kappa_2 \tag{3.4}$$

$$\kappa_1\kappa_2(1 + v^2) - v(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \neq 0. \tag{3.5}$$

Here if $v^2 > 1$, then β^* is a timelike curve; if $v^2 < 1$, β^* is a spacelike curve with timelike B_1^*.

Proof. We assume that $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ is a timelike $(1,3)$-V Bertrand curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_1, \kappa_3$, and β^* is the $(1,3)$-V Bertrand mate curve of β parametrized by arc-length s^* with the Frenet frame $\{T^*, N^*, B_1^*, B_2^*\}$ and curvatures $\kappa_1^*, \kappa_2^*, \kappa_3^*$. Then, we can write the curve β^* as follows

$$\begin{aligned} \beta^*(s^*) &= \beta^*(h(s)) = \int (a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s))\, ds \\ &\quad + e(s)N(s) + f(s)B_2(s) \end{aligned} \tag{3.6}$$

for all $s \in I$, where $a(s)$, $b(s)$, $c(s)$, $d(s)$, $e(s)$ and $f(s)$ are C^∞−functions on I.

Here assume that β^* is a timelike curve. Differentiating (3.6) with respect to s, we get

$$T^*h' = (a + e\kappa_1)T + (b + e')\,N + (c + e\kappa_2 - f\kappa_3)B_1 + (d + f')\,B_2. \tag{3.7}$$

By taking the scalar product of (3.7) with N and B_2, respectively, we have

$$b + e' = 0 \quad \text{and} \quad d + f' = 0. \tag{3.8}$$

Substituting (3.8) in (3.7), we find

$$T^*h' = (a + e\kappa_1)T + (c + e\kappa_2 - f\kappa_3)B_1. \tag{3.9}$$

By taking the scalar product of (3.9) with itself, we obtain

$$(h')^2 = (a + e\kappa_1)^2 - (c + e\kappa_2 - f\kappa_3)^2 \tag{3.10}$$

If we denote

$$\delta = \frac{a + e\kappa_1}{h'} \quad \text{and} \quad \gamma = \frac{c + e\kappa_2 - f\kappa_3}{h'}, \tag{3.11}$$

we get

$$T^* = \delta T + \gamma B_1 \tag{3.12}$$

Differentiating (3.12) with respect to s, we have

$$h'\kappa_1^* N^* = \delta' T + (\delta\kappa_1 - \gamma\kappa_2)N + \gamma' B_1 + \gamma\kappa_3 B_2. \tag{3.13}$$

By taking the scalar product of (3.13) with T and B_1, respectively, we get

$$\delta' = 0 \quad \text{and} \quad \gamma' = 0 \tag{3.14}$$

Case 1. Assume that $\gamma = 0$. Then we obtain

$$c + e\kappa_2 - f\kappa_3 = 0 \quad \text{and} \quad a + e\kappa_1 \neq 0.$$

Case 2. Assume that $\gamma \neq 0$. Then we find

$$c + e\kappa_2 - f\kappa_3 \neq 0 \tag{3.15}$$

and

$$a + e\kappa_1 = v(c + e\kappa_2 - f\kappa_3) \tag{3.16}$$

where $v = \delta/\gamma$. Substituting (3.14) in (3.13), we get

$$h'\kappa_1^* N^* = (\delta\kappa_1 - \gamma\kappa_2)N + \gamma\kappa_3 B_2. \tag{3.17}$$

By taking the scalar product of (3.17) with itself, we obtain

$$(h')^2(\kappa_1^*)^2 = (\delta\kappa_1 - \gamma\kappa_2)^2 + \gamma^2\kappa_3^2. \tag{3.18}$$

Substituting (3.11) in (3.18), we find

$$(h')^2(\kappa_1^*)^2 = \frac{(c + e\kappa_2 - f\kappa_3)^2}{(h')^2}[(v\kappa_1 - \kappa_2)^2 + \kappa_3^2]. \tag{3.19}$$

Substituting (3.16) in (3.10), we have

$$(h')^2 = (c + e\kappa_2 - f\kappa_3)^2[v^2 - 1], \tag{3.20}$$

where $v^2 > 1$. Substituting (3.20) in (3.19), we get

$$(h')^2(\kappa_1^*)^2 = \frac{1}{v^2 - 1}[(v\kappa_1 - \kappa_2)^2 + \kappa_3^2]. \tag{3.21}$$

If we denote

$$\lambda_1 = \frac{(\delta\kappa_1 - \gamma\kappa_2)}{h'\kappa_1^*} \quad \text{and} \quad \lambda_2 = \frac{\gamma\kappa_3}{h'\kappa_1^*}, \tag{3.22}$$

we get

$$N^* = \lambda_1 N + \lambda_2 B_2. \tag{3.23}$$

Differentiating (3.23) with respect to s, we find

$$h'\kappa_1^* T^* + h'\kappa_2^* B_1^* = \kappa_1\lambda_1 T + \lambda_1' N + (\lambda_1\kappa_2 - \lambda_2\kappa_3)B_1 + \lambda_2' B_2. \tag{3.24}$$

By taking the scalar product of (3.24) with N and B_2, respectively, we obtain

$$\lambda_1' = 0 \text{ and } \lambda_2' = 0 \tag{3.25}$$

Substituting (3.11) in (3.22), we get

$$\lambda_1 = \frac{(c + e\kappa_2 - f\kappa_3)(v\kappa_1 - \kappa_2)}{(h')^2\kappa_1^*} \tag{3.26}$$

and

$$\lambda_2 = \frac{\kappa_3(c + e\kappa_2 - f\kappa_3)}{(h')^2\kappa_1^*}. \tag{3.27}$$

From (3.26) and (3.27), since $\lambda_2 \neq 0$, we have

$$\mu\kappa_3 = v\kappa_1 - \kappa_2, \tag{3.28}$$

where $\mu = \lambda_1/\lambda_2$. Substituting (3.25) in (3.24), we find

$$-h'\kappa_1^* T^* + h'\kappa_2^* B_1^* = \kappa_1\lambda_1 T + (\lambda_1\kappa_2 - \lambda_2\kappa_3)B_1. \tag{3.29}$$

From (3.9) and (3.29), we obtain

$$h'\kappa_2^* B_1^* = A(s)T + B(s)B_1 \tag{3.30}$$

where

$$A(s) = \frac{(c + e\kappa_2 - f\kappa_3)}{(h')^2\kappa_1^*(v^2-1)}[\kappa_1\kappa_2(1+v^2) - v(\kappa_1^2+\kappa_2^2+\kappa_3^2)] \tag{3.31}$$

and

$$B(s) = \frac{v(c + e\kappa_2 - f\kappa_3)}{(h')^2\kappa_1^*(v^2-1)}[\kappa_1\kappa_2(1+v^2) - v(\kappa_1^2+\kappa_2^2+\kappa_3^2)]. \tag{3.32}$$

Since $f'\kappa_2^* B_1^* \neq 0$, we get

$$\kappa_1\kappa_2(1+v^2) - v(\kappa_1^2+\kappa_2^2+\kappa_3^2) \neq 0. \tag{3.33}$$

Conversely, assume that $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ is a curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_1, \kappa_3$. Firstly, assume that the condition (i) holds for functions a, b, c, d, e, f. Then, we can define the curve β^* as

$$\begin{aligned}\beta^*(s^*) \quad &= \int (a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s))\, ds \\ &\quad + e(s)N(s) + f(s)B_2(s).\end{aligned} \tag{3.34}$$

Differentiating (3.34) with respect to s and using the Frenet formulae (2.1), we find

$$\frac{d\beta^*}{ds} = (a + e\kappa_1)T. \tag{3.35}$$

From (3.35), we have

$$h' = \frac{ds^*}{ds} = \left\|\frac{d\beta^*}{ds}\right\| = \varepsilon_1(a + e\kappa_1) > 0$$

where $\varepsilon_1 = sgn\,(a + e\kappa_1)$. Then we can easily obtain

$$T^* = \varepsilon_1 T, \quad N^* = \varepsilon_1\varepsilon_2 N, \quad B_1^* = \varepsilon_1\varepsilon_2\varepsilon_3 B_1, \quad B_2^* = \varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4 B_2$$

and

$$\kappa_1^* = \frac{\varepsilon_2\kappa_1}{h'}, \quad \kappa_2^* = \frac{\varepsilon_3\kappa_2}{h'}, \quad \kappa_3^* = \frac{\varepsilon_4\kappa_3}{h'},$$

where $\varepsilon_i = sgn\,(\kappa_{i-1})$ for $i \in \{2,3,4\}$. Therefore, the curve β is a $(1,3)$-V Bertrand curve.

Now assume that the condition (ii) holds. Then, we can define the curve β^* as

$$\begin{aligned}\beta^*(s^*) \quad &= \int (a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s))\, ds \\ &\quad + e(s)N(s) + f(s)B_2(s).\end{aligned} \tag{3.36}$$

Differentiating (3.36) with respect to s and using the Frenet formulae (2.1), we find

$$\frac{d\beta^*}{ds} = (a + e\kappa_1)T + (c + e\kappa_2 - f\kappa_3)B_1. \tag{3.37}$$

From (3.37) and (3.3), we get

$$\frac{d\beta^*}{ds} = (c + e\kappa_2 - f\kappa_3)[vT + B_1]. \tag{3.38}$$

From (3.38), we have

$$h' = \frac{ds^*}{ds} = \left\| \frac{d\beta^*}{ds} \right\| = \varepsilon_1(c + e\kappa_2 - f\kappa_3)\sqrt{v^2 - 1} > 0 \tag{3.39}$$

where $\varepsilon_1 = sgn\,(c + e\kappa_2 - f\kappa_3)$. Now, by rewriting (3.38), we obtain

$$T^* h' = (c + e\kappa_2 - f\kappa_3)[vT + B_1]. \tag{3.40}$$

Substituting (3.39) in (3.40), we find

$$T^* = \varepsilon_1(v^2 - 1)^{-\frac{1}{2}}[vT + B_1]. \tag{3.41}$$

From (3.41), we get $g(T^*, T^*) = -1$. Differentiating (3.41) with respect to s, we find

$$\frac{dT^*}{ds^*} = \frac{\varepsilon_1(v^2 - 1)^{-\frac{1}{2}}}{h'}[(v\kappa_1 - \kappa_2)N + \kappa_3 B_2]. \tag{3.42}$$

Using (3.42), we have

$$\kappa_1^* = \left\| \frac{dT^*}{ds^*} \right\| = \frac{\varepsilon_2 \kappa_3 \sqrt{\mu^2 + 1}}{h' \sqrt{v^2 - 1}} \tag{3.43}$$

where $\varepsilon_2 = sgn\,(\kappa_3)$. From (3.42) and (3.43), we have

$$N^* = \frac{1}{\kappa_1^*}\frac{dT^*}{ds^*} = \frac{\varepsilon_1 \varepsilon_2}{\sqrt{\mu^2 + 1}}[\mu N + B_2]. \tag{3.44}$$

Using (3.44), we get $g(N^*, N^*) = 1$. Differentiating (3.44) with respect to s, we find

$$\frac{dN^*}{ds^*} = \frac{\varepsilon_1 \varepsilon_2}{h' \sqrt{\mu^2 + 1}} \left(\mu\kappa_1 T + (\mu\kappa_2 - \kappa_3)\, B_1 \right). \tag{3.45}$$

From (3.41), (3.43) and (3.45), we find

$$\frac{dN^*}{ds^*} - \kappa_1^* T^* = \frac{P(s)}{R(s)}T + \frac{Q(s)}{R(s)}B_1 \tag{3.46}$$

where

$$(3.47) \qquad \begin{array}{rcl} P(s) &=& \varepsilon_1\varepsilon_2\left[\kappa_1\kappa_2(1+v^2)-v(\kappa_1^2+\kappa_2^2+\kappa_3^2)\right]\neq 0 \\ Q(s) &=& \varepsilon_1\varepsilon_2 v\left[\kappa_1\kappa_2(1+v^2)-v(\kappa_1^2+\kappa_2^2+\kappa_3^2)\right]\neq 0 \\ R(s) &=& h'(v^2-1)\left(\mu^2+1\right)\kappa_3\neq 0 \end{array}$$

Now we can define κ_2^* as

$$(3.48) \qquad \kappa_2^* = \frac{\left|\kappa_1\kappa_2(1+v^2)-v(\kappa_1^2+\kappa_2^2+\kappa_3^2)\right|}{\varepsilon_2 h'\sqrt{v^2-1}\sqrt{\mu^2+1}\kappa_3} > 0.$$

Next, we can define the unit vector B_1^* as

$$(3.49) \qquad B_1^* = \frac{1}{\kappa_2^*}[\frac{dN^*}{ds^*}-\kappa_1^*T^*] = \frac{\varepsilon_1\varepsilon_3}{\sqrt{v^2-1}}[T+vB_1].$$

where $\varepsilon_3 = sgn\left(\kappa_1\kappa_2(1+v^2)-v(\kappa_1^2+\kappa_2^2+\kappa_3^2)\right)$. From (3.49), we have $g(B_1^*,B_1^*)=1$. Thus, we can define the unit vector B_2^* as

$$(3.50) \qquad B_2^* = \frac{\varepsilon_1\varepsilon_2}{\sqrt{\mu^2+1}}[N-\mu B_2].$$

From (3.50), we get $g(B_2^*,B_2^*)=1$. Lastly, we can define κ_3^* as

$$\kappa_3^* = \left\langle \frac{dB_1^*}{ds^*}, B_2^*\right\rangle = \frac{-\varepsilon_2\varepsilon_3\kappa_1(v^2-1)}{h'\sqrt{v^2-1}\sqrt{\mu^2+1}} \neq 0.$$

Then we get that $\text{span}\{\mathrm{N},\mathrm{B}_2\} = \text{span}\{\mathrm{N}^*,\mathrm{B}_2^*\}$. Thus the curve β is a $(1,3)$-V Bertrand curve.

The case when β^* is a spacelike curve with timelike B_1^* can be done similarly for $\gamma \neq 0$. □

As applications result of the above theorem, we get the following corollaries.

Corollary 3.4. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a timelike curve parametrized by arc length s with Frenet frame $\{T,N,B_1,B_2\}$ and non-zero curvatures $\kappa_1,\kappa_2,\kappa_3$. Then, the curve β is a $(1,3)$-T Bertrand curve if and only if one of the following conditions holds:*
(i) there exist real numbers e, f satisfying

$$e\kappa_2 - f\kappa_3 = 0, \quad 1+e\kappa_1 \neq 0.$$

In this case, there exists a homothety map between β and β^*. So β^* is a timelike curve.
(ii) there exist constant real numbers e, f, v, μ satisfying

$$e\kappa_2 - f\kappa_3 \neq 0, \quad 1+e\kappa_1 = v(e\kappa_2 - f\kappa_3), \quad \mu\kappa_3 = v\kappa_1-\kappa_2,$$

$$\kappa_1\kappa_2(1+v^2)-v(\kappa_1^2+\kappa_2^2+\kappa_3^2)\neq 0.$$

Here if $v^2>1$, then β^* is a timelike curve; if $v^2<1$, β^* is a spacelike curve with timelike B_1^*.

Corollary 3.5. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a timelike curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_2, \kappa_3$. Then, the curve β is a $(1,3)$-N Bertrand curve if and only if one of the following conditions holds:*
(i) there exist a real number f satisfying $s\kappa_2 + f\kappa_3 = 0$, for $s \neq 0$. In this case, there exists a homothety map between β and β^*. So β^* is a timelike curve.
(ii) there exist constant real numbers f, v, μ satisfying

$$s\kappa_2 + f\kappa_3 \neq 0, \quad s\kappa_1 = v(s\kappa_2 + f\kappa_3), \quad \mu\kappa_3 = v\kappa_1 - \kappa_2,$$

$$\kappa_1\kappa_2(1+v^2) - v(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \neq 0.$$

Here if $v^2 > 1$, then β^* is a timelike curve; if $v^2 < 1$, β^* is a spacelike curve with timelike B_1^*.

Example 3.6. (i) Let us choose $f = 1$, $v = \sqrt{2}$, $\mu = 0$, $\kappa_1 = 1$, $\kappa_2 = \sqrt{2}$, $\kappa_3 = -s/\sqrt{2}$ where $s \neq 0$. Then it is clear that the conditions in (ii) of Corollary 3.5 are satisfied. As a result, the timelike curve with curvatures $\kappa_1 = 1$, $\kappa_2 = \sqrt{2}$, $\kappa_3 = -s/\sqrt{2}$, is a $(1,3)$-N Bertrand curve in $\mathbb{E}_1^4$ and the B_2-Bertrand mate curve is a timelike curve.
(ii) Let us choose $f = 1$, $v = 1/\sqrt{2}$, $\mu = 0$, $\kappa_1 = \sqrt{2}$, $\kappa_2 = 1$, $\kappa_3 = s$ where $s \neq 0$. Then it is clear that the conditions in (ii) of Corollary 3.5 are satisfied. As a result, the timelike curve with curvatures $\kappa_1 = \sqrt{2}$, $\kappa_2 = 1$, $\kappa_3 = s$, is a $(1,3)$-N Bertrand curve in $\mathbb{E}_1^4$ and the B_2-Bertrand mate curve is a spacelike curve.

Corollary 3.7. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a timelike curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_2, \kappa_3$. Then, the curve β is $(1,3)$-B_1 Bertrand curve if and only if one of the following conditions holds:*
(i) there exist non-zero real numbers e, f satisfying $1 + e\kappa_2 - f\kappa_3 = 0$. In this case, there exists a homothety map between β and β^*. So β^* is a timelike curve.
(ii) there exist constant real numbers e, f, v, μ satisfying

$$1 + e\kappa_2 - f\kappa_3 \neq 0, \quad e\kappa_1 = v(1 + e\kappa_2 - f\kappa_3), \quad \mu\kappa_3 = v\kappa_1 - \kappa_2,$$

$$\kappa_1\kappa_2(1+v^2) - v(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \neq 0.$$

Here if $v^2 > 1$, then β^* is a timelike curve; if $v^2 < 1$, β^* is a spacelike curve with timelike B_1^*.

Example 3.8. (Example for $(1,3)$-B_1 Bertrand) Let us consider the timelike curve with the equation

$$\beta(s) = \left(\sqrt{2}\sinh s, \sqrt{2}\cosh s, \sin s, \cos s\right).$$

The Frenet Frame of β is given by

$$\begin{aligned} T(s) &= \left(\sqrt{2}\cosh s, \sqrt{2}\sinh s, \cos s, -\sin s\right), \\ N(s) &= \frac{\sqrt{3}}{3}\left(\sqrt{2}\sinh s, \sqrt{2}\cosh s, -\sin s, -\cos s\right), \\ B_1(s) &= \left(-\cosh s, -\sinh s, -\sqrt{2}\cos s, \sqrt{2}\sin s\right), \\ B_2(s) &= \frac{\sqrt{3}}{3}\left(\sinh s, \cosh s, \sqrt{2}\sin s, \sqrt{2}\cos s\right). \end{aligned}$$

The curvatures of β are $k_1(s) = \sqrt{3}$, $k_2(s) = 2\sqrt{6}/3$, $k_3(s) = 1/\sqrt{3}$. By taking $e = 1$, $f = \sqrt{3}$, $v = 3/\left(2\sqrt{2}\right)$, $\mu = 1/\left(2\sqrt{2}\right)$ in (ii) of Corollary 3.7, we obtain

$$\begin{aligned} \beta^*(s) &= \int_0^s B_1(u)\,du + N(s) + \sqrt{3}B_2(s) \\ &= \left(\frac{\sqrt{2}}{\sqrt{3}}\sinh s, \frac{\sqrt{2}}{\sqrt{3}}\cosh s, -\frac{1}{\sqrt{3}}\sin s, -\frac{1}{\sqrt{3}}\cos s\right) \end{aligned}$$

with the curvatures $\kappa_1^* = 3$, $\kappa_2^* = 2\sqrt{2}$, $\kappa_3^* = 1$ and the Frenet frame

$$\begin{aligned} T^* &= \left(\sqrt{2}\cosh s, \sqrt{2}\sinh s, -\cos s, \sin s\right), \\ N^* &= \left(\frac{\sqrt{2}}{\sqrt{3}}\sinh s, \frac{\sqrt{2}}{\sqrt{3}}\cosh s, \frac{1}{\sqrt{3}}\sin s, \frac{1}{\sqrt{3}}\cos s\right), \\ B_1^* &= \left(-\cosh s, -\sinh s, \sqrt{2}\cos s, -\sqrt{2}\sin s\right), \\ B_2^* &= \left(\frac{1}{\sqrt{3}}\sinh s, \frac{1}{\sqrt{3}}\cosh s, -\frac{\sqrt{2}}{\sqrt{3}}\sin s, -\frac{\sqrt{2}}{\sqrt{3}}\cos s\right). \end{aligned}$$

It can be seen that

$$\begin{aligned} N^* &= \frac{1}{3}N + \frac{2\sqrt{2}}{3}B_2, \\ B_2^* &= \frac{2\sqrt{2}}{3}N - \frac{1}{3}B_2 \end{aligned}$$

which implies that β is $(1,3)$-B_1 Bertrand curve and β^* is timelike $(1,3)$-B_1 Bertrand mate curve of β.
(ii) taking $e = \sqrt{6}/2$, $f = \sqrt{3}/2$, $v = 3\sqrt{2}/5$, $\mu = -\sqrt{2}/5$ in (ii) of Corollary 3.7, we obtain

$$\begin{aligned} \beta^*(s) &= \int_0^s B_1(u)\,du + \frac{\sqrt{6}}{2}N(s) + \frac{\sqrt{3}}{2}B_2(s) \\ &= \left(\frac{1}{2}\sinh s, \frac{1}{2}\cosh s, -\sqrt{2}\sin s, -\sqrt{2}\cos s\right) \end{aligned}$$

with the curvatures $\kappa_1^* = 6/7$, $\kappa_2^* = 8\sqrt{2}/21$, $\kappa_3^* = 2/3$ and the Frenet frame

$$\begin{aligned}
T^* &= \left(\frac{1}{\sqrt{7}}\cosh s, \frac{1}{\sqrt{7}}\sinh s, -\frac{2\sqrt{2}}{\sqrt{7}}\cos s, \frac{2\sqrt{2}}{\sqrt{7}}\sin s\right),\\
N^* &= \left(\frac{1}{3}\sinh s, \frac{1}{3}\cosh s, \frac{2\sqrt{2}}{3}\sin s, \frac{2\sqrt{2}}{3}\cos s\right),\\
B_1^* &= \left(-\frac{2\sqrt{2}}{\sqrt{7}}\cosh s, -\frac{2\sqrt{2}}{\sqrt{7}}\sinh s, \frac{1}{\sqrt{7}}\cos s, -\frac{1}{\sqrt{7}}\sin s\right),\\
B_2^* &= \left(-\frac{2\sqrt{2}}{3}\sinh s, -\frac{2\sqrt{2}}{3}\cosh s, \frac{1}{3}\sin s, \frac{1}{3}\cos s\right).
\end{aligned}$$

It can be seen that

$$\begin{aligned}
N^* &= -\frac{\sqrt{2}}{3\sqrt{3}}N + \frac{5}{3\sqrt{3}}B_2,\\
B_2^* &= -\frac{5}{3\sqrt{3}}N - \frac{\sqrt{2}}{3\sqrt{3}}B_2,
\end{aligned}$$

which implies that β is $(1,3)$-B_1 Bertrand curve and β^* is spacelike $(1,3)$-B_1 Bertrand mate curve of β.

Corollary 3.9. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a timelike curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_2, \kappa_3$. Then, the curve β is $(1,3)$-B_2 Bertrand curve if and only if one of the following conditions holds:*
(i) there exist a real number $e \neq 0$ satisfying $e\kappa_2 + s\kappa_3 = 0$. In this case, there exists a homothety map between β and β^*. So β^* is a timelike curve.
(ii) there exist constant real numbers e, v, μ satisfying

$$e\kappa_2 + s\kappa_3 \neq 0, \quad e\kappa_1 = v(e\kappa_2 + s\kappa_3), \quad \mu\kappa_3 = v\kappa_1 - \kappa_2,$$
$$\kappa_1\kappa_2(1+v^2) - v(\kappa_1^2 + \kappa_2^2 + \kappa_3^2) \neq 0.$$

Here if $v^2 > 1$, then β^* is a timelike curve; if $v^2 < 1$, β^* is a spacelike curve with timelike B_1^*.

Example 3.10. (i) Let us choose $e = 1$, $v = \sqrt{2}$, $\mu = 0$, $\kappa_1 = s$, $\kappa_2 = \sqrt{2}s$, $\kappa_3 = -1/\sqrt{2}$, where $s \neq 0$. Then it is clear that the conditions in (ii) of Corollary 3.9 are satisfied. As a result, the timelike curve with curvatures $\kappa_1 = s$, $\kappa_2 = \sqrt{2}s$, $\kappa_3 = -1/\sqrt{2}$ is a $(1,3)$-B_2 Bertrand curve in $\mathbb{E}_1^4$ and the B_2-Bertrand mate curve is a timelike curve.
(ii) Let us choose $e = 1$, $v = 1/\sqrt{2}$, $\mu = 0$, $\kappa_1 = s$, $\kappa_2 = s/\sqrt{2}$, $\kappa_3 = 1/\sqrt{2}$ where $s \neq 0$. Then it is clear that the conditions in (ii) of Corollary 3.9 are satisfied. As a result, the timelike curve with curvatures $\kappa_1 = s$, $\kappa_2 = s/\sqrt{2}$, $\kappa_3 = 1/\sqrt{2}$, is a $(1,3)$-B_2 Bertrand curve in $\mathbb{E}_1^4$ and the B_2-Bertrand mate curve is a spacelike curve.

In the following theorem, we consider β^* as a Cartan null curve.

Theorem 3.11. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}^4_1$ be a timelike curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_2, \kappa_3$. Then, the curve β is a $(1,3)$-V Bertrand curve if and only if there exist functions a, b, c, d, e, f and constant real numbers γ, μ satisfying*

$$b + e' = 0, \quad d + f' = 0, \quad c + e\kappa_2 - f\kappa_3 \neq 0, \tag{3.51}$$

$$c + e\kappa_2 - f\kappa_3 = m\,(a + e\kappa_1)\,, \quad \mu\kappa_3 = m\kappa_1 - \kappa_2 \tag{3.52}$$

$$|a + e\kappa_1| = \gamma^2 \left|\kappa_3\sqrt{\mu^2+1}\right|, \quad |\mu\kappa_1| \neq |\mu\kappa_2 - \kappa_3|\,, \tag{3.53}$$

where $m = \pm 1$. Here β^* is a Cartan null curve.

Proof. We assume that $\beta : I \subset \mathbb{R} \to \mathbb{E}^4_1$ is timelike $(1,3)$-V Bertrand curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_1, \kappa_3$, and β^* is Cartan null $(1,3)$-V Bertrand mate curve of β parametrized by arc-length s^* with the Frenet frame $\{T^*, N^*, B_1^*, B_2^*\}$ and curvatures $\kappa_1^*, \kappa_2^*, \kappa_3^*$. Then, we can write the curve β^* as follows

$$\begin{aligned} \beta^*(s^*) &= \beta^*(h(s)) = \int \left(a(s)T(s) + b(s)N(s) + c(s)B_1(s) + d(s)B_2(s)\right) ds \\ &\quad + e(s)N(s) + f(s)B_2(s) \end{aligned} \tag{3.54}$$

for all $s \in I$ where $a(s)$, $b(s)$, $c(s)$, $d(s)$, $e(s)$ and $f(s)$ are $C^\infty-$functions on I. Differentiating (3.54) with respect to s, we get

$$T^*h' = (a + e\kappa_1)T + (b + e')\,N + (c + e\kappa_2 - f\kappa_3)B_1 + (d + f')\,B_2. \tag{3.55}$$

By taking the scalar product of (3.55) with N and B_2, respectively, we have

$$b + e' = 0 \quad \text{and} \quad d + f' = 0. \tag{3.56}$$

Substituting (3.56) in (3.55), we find

$$T^*h' = (a + e\kappa_1)T + (c + e\kappa_2 - f\kappa_3)B_1 \tag{3.57}$$

which leads to

$$(a + e\kappa_1)^2 = (c + e\kappa_2 - f\kappa_3)^2 \tag{3.58}$$

or

$$\begin{aligned} c + e\kappa_2 - f\kappa_3 &\neq 0, \\ c + e\kappa_2 - f\kappa_3 &= m\,(a + e\kappa_1)\,, \end{aligned}$$

where $m = \pm 1$. If we denote $\gamma = (a + e\kappa_1)\,/h'$, we get

$$T^* = \gamma\,(T + mB_1) \tag{3.59}$$

Differentiating (3.59) with respect to s, we have

$$h'N^* = \gamma'T + \gamma(\kappa_1 - m\kappa_2)N + m\gamma'B_1 + \gamma m\kappa_3 B_2. \tag{3.60}$$

By taking the scalar product of (3.60) with T, B_1 and itself, respectively, we get $\gamma' = 0$ and

$$(h')^2 = \gamma^2\left((\kappa_1 - m\kappa_2)^2 + \kappa_3^2\right). \tag{3.61}$$

If we denote

$$\lambda_1 = \frac{\gamma(\kappa_1 - m\kappa_2)}{h'} \quad \text{and} \quad \lambda_2 = \frac{\gamma m\kappa_3}{h'}, \tag{3.62}$$

we get

$$N^* = \lambda_1 N + \lambda_2 B_2. \tag{3.63}$$

Differentiating (3.63) with respect to s, we find

$$h'\kappa_2^* T^* - h'B_1^* = \kappa_1\lambda_1 T + \lambda_1' N + (\lambda_1\kappa_2 - \lambda_2\kappa_3)B_1 + \lambda_2' B_2. \tag{3.64}$$

By taking the scalar product of (3.64) with N and B_2, respectively, we obtain

$$\lambda_1' = 0 \quad \text{and} \quad \lambda_2' = 0 \tag{3.65}$$

Since $\lambda_2 \neq 0$, we have

$$\mu\kappa_3 = m\kappa_1 - \kappa_2, \tag{3.66}$$

where $\mu = \lambda_1/\lambda_2$. From 3.61 and 3.66, we get

$$(h')^2 = \gamma^2\kappa_3^2\left(\mu^2 + 1\right)$$

or

$$(a + e\kappa_1)^2 = \gamma^4\kappa_3^2\left(\mu^2 + 1\right),$$

which leads to

$$|a + e\kappa_1| = \gamma^2\left|\kappa_3\sqrt{\mu^2 + 1}\right|.$$

Substituting (3.65) in (3.64), we find

$$h'\kappa_2^* T^* - h'B_1^* = \kappa_1\lambda_1 T + (\lambda_1\kappa_2 - \lambda_2\kappa_3)B_1. \tag{3.67}$$

By taking the scalar product of (3.64) with itself, we get

$$-2(h')^2\kappa_2^* = -(\kappa_1\lambda_1)^2 + (\lambda_1\kappa_2 - \lambda_2\kappa_3)^2 \neq 0. \tag{3.68}$$

From (3.62) and (3.68), we have $|\mu\kappa_1| \neq |\mu\kappa_2 - \kappa_3|$.

Conversely, assume that $\beta : I \subset \mathbb{R} \to \mathbb{E}^4$ is a curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_1, \kappa_3$.

Assume that the conditions (3.51), (3.52) and (3.53) hold. Then, we can define the curve β^* as

$$\begin{aligned} \beta^*(s^*) &= \int \left(a(s)T(s)+b(s)N(s)+c(s)B_1(s)+d(s)B_2(s)\right)ds \\ &\quad +e(s)N(s)+f(s)B_2(s). \end{aligned} \tag{3.69}$$

Differentiating (3.69) with respect to s, we find

$$\frac{d\beta^*}{ds} = (a+e\kappa_1)\left(T+mB_1\right). \tag{3.70}$$

Differentiating (3.70) with respect to s, we obtain

$$\frac{d^2\beta^*}{ds^2} = (a+e\kappa_1)'\left(T+mB_1\right)+(a+e\kappa_1)m\kappa_3\left(\mu N+B_2\right). \tag{3.71}$$

By taking the scalar product of (3.64) with itself, we get

$$\left(h'\right)^4 = \left(a+e\kappa_1\right)^2\kappa_3^2\left(\mu^2+1\right)$$

or

$$h' = m_1m_2\gamma\kappa_3\sqrt{\mu^2+1}, \tag{3.72}$$

where $m_1 = sgn\left(\gamma\right)$ and $m_2 = sgn\left(\kappa_3\right)$. From 3.72 and 3.70, we have

$$T^* = \frac{a+e\kappa_1}{h'}\left(T+mB_1\right) = m_1\gamma\left(T+mB_1\right). \tag{3.73}$$

Here $g\left(T^*,T^*\right)=0$. Differentiating (3.70) with respect to s, we find

$$N^* = \frac{m_2m}{\sqrt{\mu^2+1}}\left(\mu N+B_2\right)$$

and

$$\frac{dN^*}{ds^*} = \frac{m_2m}{h'\sqrt{\mu^2+1}}\left(\mu\kappa_1T+\left(\mu\kappa_2-\kappa_3\right)B_1\right).$$

Then we can obtain κ_2^* as

$$\kappa_2^* = \frac{-\mu^2\kappa_1^2+\left(\mu\kappa_2-\kappa_3\right)^2}{-2\left(h'\right)^2\left(\mu^2+1\right)} \neq 0.$$

Now we can choose

$$\begin{aligned} B_1^* &= \frac{m_1m}{2\gamma}\left(-mT+B_1\right), \\ B_2^* &= \frac{m_2m}{\sqrt{\mu^2+1}}\left(-N+\mu B_2\right). \end{aligned}$$

Then we can obtain κ_3^* as

$$\kappa_3^* = \frac{m_2m\kappa_1}{\gamma h'\sqrt{\mu^2+1}} \neq 0.$$

Thus β^* is a Cartan curve and $(1,3)$-V Bertrand mate curve of β. This completes the proof. □

Corollary 3.12. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a timelike curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_2, \kappa_3$. Then, the curve β is a $(1,3)$-T Bertrand curve if and only if there exist real numbers e, f, γ, μ satisfying*

$$e\kappa_2 - f\kappa_3 \neq 0, \quad e\kappa_2 - f\kappa_3 = m\left(1 + e\kappa_1\right), \quad \mu\kappa_3 = m\kappa_1 - \kappa_2$$

$$\left|1 + e\kappa_1\right| = \gamma^2 \left|\kappa_3\sqrt{\mu^2 + 1}\right|, \quad \left|\mu\kappa_1\right| \neq \left|\mu\kappa_2 - \kappa_3\right|,$$

where $m = \pm 1$. Here β^* is a Cartan null curve.

Corollary 3.13. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a timelike curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_2, \kappa_3$. Then the curve β is a $(1,3)$-N Bertrand curve if and only if*

$$\left|\kappa_1\right| = \left|\kappa_2\right| \quad \text{and} \quad \left|s\kappa_1\right| = \gamma^2 \left|\kappa_3\right|$$

for a real number γ. Here the Bertrand mate curve β^* can be written as

$$\beta^*(s) = \int B_1(s)ds - sN\left(s\right).$$

Proof. If we take $a = c = d = 0$ and $b = 1$, for real numbers f, μ, we have

$$s\kappa_2 + f\kappa_3 = m\kappa_1 s, \quad \mu\kappa_3 = m\kappa_1 - \kappa_2,$$

where $m = \pm 1$. From the first equation, we find

$$f\kappa_3 = s\left(m\kappa_1 - \kappa_2\right) = s\mu\kappa_3,$$

which implies that $f = \mu = 0$. Thus we can easily obtain that $\left|\kappa_1\right| = \left|\kappa_2\right|$ and $\left|s\kappa_1\right| = \gamma^2\left|\kappa_3\right|$. $\square$

Corollary 3.14. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a timelike curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_2, \kappa_3$. Then, the curve β is a $(1,3)$-B_1 Bertrand curve if and only if there exist real numbers e, f, γ, μ satisfying*

$$1 + e\kappa_2 - f\kappa_3 \neq 0, \quad 1 + e\kappa_2 - f\kappa_3 = me\kappa_1, \quad \mu\kappa_3 = m\kappa_1 - \kappa_2$$

$$\left|e\kappa_1\right| = \gamma^2 \left|\kappa_3\sqrt{\mu^2 + 1}\right|, \quad \left|\mu\kappa_1\right| \neq \left|\mu\kappa_2 - \kappa_3\right|,$$

where $m = \pm 1$. Here β^* is a Cartan null curve.

Example 3.15. For the same curve β in Example 3.8, we find $\mu = 3m - 2\sqrt{2}$, which does not satisfy the condition $\left|\mu\kappa_1\right| \neq \left|\mu\kappa_2 - \kappa_3\right|$. So β is not a $(1,3)$-B_1 Bertrand curve.

Example 3.16. Let us consider the timelike curve with the equation

$$\beta(s)=\left(\frac{2\sqrt{3}}{3}\sinh\left(\sqrt{3}s\right),\frac{2\sqrt{3}}{3}\cosh\left(\sqrt{3}s\right),\sqrt{3}\sin s,-\sqrt{3}\cos s\right).$$

The Frenet Frame of β is given by

$$\begin{aligned}
T(s)&=\left(2\cosh\left(\sqrt{3}s\right),2\sinh\left(\sqrt{3}s\right),\sqrt{3}\cos s,-\sqrt{3}\sin s\right),\\
N(s)&=\left(\frac{2}{\sqrt{5}}\sinh\left(\sqrt{3}s\right),\frac{2}{\sqrt{5}}\cosh\left(\sqrt{3}s\right),-\frac{1}{\sqrt{5}}\sin s,\frac{1}{\sqrt{5}}\cos s\right),\\
B_1(s)&=\left(-\sqrt{3}\cosh\left(\sqrt{3}s\right),\sqrt{3}\sinh\left(\sqrt{3}s\right),-2\cos s,-2\sin s\right),\\
B_2(s)&=\left(\frac{1}{\sqrt{5}}\sinh\left(\sqrt{3}s\right),\frac{1}{\sqrt{5}}\cosh\left(\sqrt{3}s\right),\frac{2}{\sqrt{5}}\sin s,-\frac{2}{\sqrt{5}}\cos s\right).
\end{aligned}$$

The curvatures of β are $k_1(s)=\sqrt{15}$, $k_2(s)=8/\sqrt{5}$, $k_3(s)=1/\sqrt{5}$. For $e=\sqrt{5}$ and $f=9\sqrt{5}-5\sqrt{15}$, we can define the curve β^* as

$$\begin{aligned}
\beta^*(s)&=\int_0^s B_1(u)\,du+\sqrt{5}N(s)+\left(9\sqrt{5}-5\sqrt{15}\right)B_2\\
&=\left(A\sinh\left(\sqrt{3}s\right),A\cosh\left(\sqrt{3}s\right),-\sqrt{3}A\sin s,\sqrt{3}A\cos s\right),
\end{aligned}$$

where $A=10-5\sqrt{3}$. The curvatures are given as

$$\kappa_1^*=1,\qquad \kappa_2^*=\frac{3+2\sqrt{3}}{30},\kappa_3^*=\frac{2+\sqrt{3}}{10}.$$

The Frenet frame is given as

$$\begin{aligned}
T^*&=\frac{\sqrt{3A}}{2}\left(\cosh\left(\sqrt{3}s\right),\sinh\left(\sqrt{3}s\right),-\cos s,-\sin s\right),\\
N^*&=\left(\frac{\sqrt{3}}{2}\sinh\left(\sqrt{3}s\right),\frac{\sqrt{3}}{2}\cosh\left(\sqrt{3}s\right),\frac{1}{2}\sin s,-\frac{1}{2}\cos s\right),\\
B_1^*&=\frac{1}{\sqrt{20\sqrt{3}-30}}\left(-\cosh\left(\sqrt{3}s\right),-\sinh\left(\sqrt{3}s\right),-\cos s,-\sin s\right),\\
B_2^*&=\left(-\frac{1}{2}\sinh\left(\sqrt{3}s\right),-\frac{1}{2}\cosh\left(\sqrt{3}s\right),\frac{\sqrt{3}}{2}\sin s,-\frac{\sqrt{3}}{2}\cos s\right).
\end{aligned}$$

It can be seen that

$$\begin{aligned}
N^*&=\frac{2\sqrt{15}-\sqrt{5}}{10}N+\frac{2\sqrt{5}+\sqrt{15}}{10}B_2,\\
B_2^*&=-\frac{2\sqrt{5}+\sqrt{15}}{10}N+\frac{2\sqrt{15}-\sqrt{5}}{10}B_2,
\end{aligned}$$

which implies that β is a $(1,3)$-B_1 Bertrand curve and β^* is a Cartan null $(1,3)$-B_1 Bertrand mate curve of β.

Corollary 3.17. *Let $\beta : I \subset \mathbb{R} \to \mathbb{E}_1^4$ be a timelike curve parametrized by arc length s with Frenet frame $\{T, N, B_1, B_2\}$ and non-zero curvatures $\kappa_1, \kappa_2, \kappa_3$. If the curve β is a $(1,3)$-B_2 Bertrand curve, then the Bertrand mate curve β^* can not be a Cartan null curve.*

Proof. If we take $a = b = c = 0$ and $d = 1$, for real numbers e, μ, we have

$$e\kappa_2 + s\kappa_3 = me\kappa_1, \quad \mu\kappa_3 = m\kappa_1 - \kappa_2$$

where $m = \pm 1$. Here we find $s = e\mu$, which contradicts that e and μ are constant. □

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