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$RC-$Class on some fixed point theorems for multivalued monotone mappings in ordered uniform spaces

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Abstract. In this paper, we use the concepts of $RC-$class function which was introduced by A. H. Ansari in [8] and define a new order relation with $RC-$class function. Then we prove some new fixed point and coupled fixed point theorems in ordered uniform spaces.

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1. Introduction

Considerable literature of fixed point theory dealing with contractive or contractive type mappings (e.g. [1, 2, 3, 4, 6, 5, 7, 9, 10, 8, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21, 22, 23, 8, 13]). Some of them are about fixed point and coupled fixed point theorems in partially ordered metric spaces [5, 9, 10, 11, 12, 17, 23]. Aamri and El Moutawakil have presented the concept of an $E-$distance function on uniform spaces [2]. I. Altun and M. Imdad have defined a partial order relation in uniform spaces using the concept of an $E-$distance function [7]. In this work, we use the relation on uniform spaces and we give $RC-$class function on some fixed point theorems for multivalued monotone mappings in ordered uniform spaces. Now, we will talk about some relevant concepts in uniform spaces. We term a pair (X, ϑ) to be a uniform space. The uniform space consist of a $X \neq \emptyset$ with a uniformity ϑ with a filter on $X \times X$ which includes the diagonal $\Delta = \{(x, x) : x \in X\}$. If $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$ then x and y are said to be V-close. Also a sequence $\{x_n\}$ in X, is said to be a Cauchy sequence with regard to uniformity ϑ if for any $V \in \vartheta$, there exists $N \geq 1$ such that x_n and x_m are V-close for $m, n \geq N$. An uniformity ϑ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}$ when V runs over ϑ.

A uniform space (X, ϑ) is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to diagonal Δ of X i.e. $(x, y) \in V$ for $V \in \vartheta$ implies $x = y$. Notice that Hausdorffness of the topology induced by the uniformity

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guarantees the uniqueness of limit of a sequence in uniform spaces. An element of uniformity ϑ is said to be symmetrical if $V = V^{-1} = \{(y,x) : (x,y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x,y) \in W$ then x and y are both W and V-close and then one may assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space (X,ϑ), they are naturally interpreted with respect to the topological space $(X,\tau(\vartheta))$.

2. Preliminaries

We will talk about definitions and lemmas in the continuation of this work.

Definition 2.1 ([2])**.** Let (X,ϑ) be a Hausdorff uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be an E-distance if

(p_1) For any $V \in \vartheta$ there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ for some $z \in X$, imply $(x,y) \in V$,

(p_2) $p(x,y) \le p(x,z) + p(z,y)$, $\forall x,y,z \in X$.

The following lemma embodies some useful properties of E-distance.

Lemma 2.2 ([1, 2])**.** *Let (X,ϑ) be a Hausdorff uniform space and p be an E-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x,y,z \in X$, the following holds:*

(a) If $p(x_n,y) \le \alpha_n$ and $p(x_n,z) \le \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x,y) = 0$ and $p(x,z) = 0$, then $y = z$.

(b) If $p(x_n,y_n) \le \alpha_n$ and $p(x_n,z) \le \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to z.

(c) If $p(x_n,x_m) \le \alpha_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in (X,ϑ).

Let (X,ϑ) be a uniform space equipped with E-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

Definition 2.3 ([1, 2])**.** Let (X,ϑ) be a Hausdorff uniform space and p be an E-distance on X. Then

(i) X said to be S-complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} p(x_n,x) = 0$,

(ii) X is said to be p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$,

(iii) $f : X \to X$ is p-continuous if $\lim_{n\to\infty} p(x_n,x) = 0$ implies

$$\lim_{n\to\infty} p(fx_n,fx) = 0,$$

(iv) $f : X \to X$ is $\tau(\vartheta)$-continuous if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$ implies $\lim_{n\to\infty} fx_n = fx$ with respect to $\tau(\vartheta)$.

Remark 2.4 ([2]). Let (X, ϑ) be a Hausdorff uniform space and let $\{x_n\}$ be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Then Lemma 1 (b) gives that $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ which shows that S-completeness implies p-Cauchy completeness.

Lemma 2.5 ([13]). *Let (X, ϑ) be a Hausdorff uniform space, p be E-distance on X and $\varphi : X \to \mathbb{R}$. Define the relation " $\preceq$ " on X as follows;*

$$x \preceq y \Leftrightarrow x = y \text{ or } p(x, y) \le \varphi(x) - \varphi(y).$$

Then " $\preceq$ " is a (partial) order on X induced by φ.

In September 2014 the concepts of $RC-$class and $LC-$class for Caristi's fixed point theorem (see Definition 2.6 and 2.10) were introduced by A. H. Ansari in [8].

Definition 2.6. Let $F : \mathbb{R}_1^2 \to \mathbb{R}, \mathbb{R}_1 \subset \mathbb{R}$ be a function. F is said to be an $RC-$class if F is continuous and satisfies

$$\begin{aligned} F(s,t) &\ge 0 \Longrightarrow s \ge t \\ F(t,t) &= 0 \\ s &\le t \Longrightarrow F(e,s) \ge F(e,t) \\ t &\le e \le s \Longrightarrow F(s,e) + F(e,t) \le F(s,t) \\ \exists g &: \mathbb{R} \to \mathbb{R},\ F(g(s), g(t)) \ge 0 \Longrightarrow s \le t \end{aligned}$$

where $s, t, e \in \mathbb{R}$.

In the following, you can see some examples of $RC-$class functions.

Example 2.7. For $n \in \mathbb{N}$ and $a > 1$,

$$\begin{array}{ccc} F(s,t) = s - t & & g(t) = -t \\ F(s,t) = \frac{s-t}{1+t} & , & g(t) = \frac{1}{t} - 1 \\ F(s,t) = s^{2n+1} - t^{2n+1} & , & g(t) = -t \\ F(s,t) = a^s - a^t & , & g(t) = -t \\ F(s,t) = a^s - a^t + t - s & , & g(t) = -t \\ F(s,t) = e^{s^{2n+1} - t^{2n+1}} - 1 & , & g(t) = -t \\ F(s,t) = e^{s-t} - 1 & , & g(t) = -t \end{array}$$

Remark 2.8. $F(s,t) = e^{s-t} - 1 \ge s - t,\ s \ge t.$

Remark 2.9. $|\varphi(x_m) - \varphi(x_n)| < \varepsilon \Longrightarrow F(\varphi(x_m), \varphi(x_n)) \to 0$

Definition 2.10. We say that $\mathcal{H}$: $\mathbb{R}^+ \to \mathbb{R}^+$ is an $LC-$class function if $\mathcal{H}$ is a continuous and increasing function such that $\mathcal{H}(t) > 0, t > 0, \mathcal{H}(0) = 0$ and

$$\mathcal{H}(s + t) \le \mathcal{H}(s) + \mathcal{H}(t).$$

and

$$x \le y \Longrightarrow \mathcal{H}(x) \le \mathcal{H}(y)$$

Example 2.11. For $a > 1, m > 0$ and $n \in \mathbb{N}$

$$\begin{aligned}\mathcal{H}(t) &= 1 - a^{-t}, \\ \mathcal{H}(t) &= mt \\ \mathcal{H}(t) &= m\sqrt[n]{t} \\ \mathcal{H}(t) &= \log_a 1 + t \\ \mathcal{H}(t) &= \log_a 1 + \sqrt[n]{t}\end{aligned}$$

are some examples of $LC-$class function.

Remark 2.12. $\mathcal{H}(t) = \log_a 1 + t \leq t$, $a > \ln a$.

Lemma 2.13. *Let (X, ϑ) be a Hausdorff uniform space, p be E-distance on X and $\varphi : X \to \mathbb{R}$ be an one to one function. Define the relation " $\preceq$ " on X as follows;*

$$x \preceq y \Leftrightarrow x = y \text{ or } p(x, y) \leq F(\varphi(x), \varphi(y)).$$

where F is RC−class. Then " $\preceq$ " is a (partial) order on X induced by φ.

3. The Fixed Point Theorems of Multivalued mappings

Lemma 3.1. *Let (X, ϑ) be a Hausdorff uniform space and p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function which is bounded below and " $\preceq$ " the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping, $[x, +\infty) = \{y \in X : x \preceq y\}$ and $M = \{x \in X \mid T(x) \cap [x, +\infty) \neq \emptyset\}$. Suppose that:*

(i) T is upper semi-continuous, that is $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in T(x_0)$;

(ii) $M \neq \emptyset$;

(iii) for each $x \in M$, $T(x) \cap M \cap [x, +\infty) \neq \emptyset$.

Then T has a fixed point x^* and there exists a sequence $\{x_n\}$ with

$$x_{n-1} \preceq x_n \in T(x_{n-1}), n = 1, 2, 3, ...$$

such that $x_n \to x^*$. Moreover, if φ is lower semi-continuous, then $x_n \preceq x^*$ for all n.

Proof. By the condition (ii), take $x_0 \in M$. From (iii), there exist $x_1 \in T(x_0) \cap M$ and $x_0 \preceq x_1$. Again from (iii), there exist $x_2 \in T(x_1) \cap M$ and thus $x_1 \preceq x_2$.

Continuing this procedure we get a sequence $\{x_n\}$ satisfying

$$x_{n-1} \preceq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, ...$$

So by the definition of " $\preceq$ ", we have $...\varphi(x_2) \leq \varphi(x_1) \leq \varphi(x_0)$ i.e. the sequence $\{\varphi(x_n)\}$ is a non-increasing sequence in $\mathbb{R}$. Since φ is bounded from below, $\{\varphi(x_n)\}$ is convergent and hence it is Cauchy i.e. for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\varphi(x_m) - \varphi(x_n)| < \varepsilon$. Since $x_n \preceq x_m$ and by Remark 3, we have $x_n = x_m$ or

$$\begin{aligned} p(x_n, x_m) &\leq F(\varphi(x_n), \varphi(x_m)) \\ &< \varepsilon \end{aligned}$$

which shows that (in view of Lemma 1 (c)) that $\{x_n\}$ is p-Cauchy sequence. By the p-Cauchy completeness of X, $\{x_n\}$ converges to x^*. Since T is upper semi-continuous, $x^* \in T(x^*)$.

Moreover, when φ is lower semi-continuous, for each n

$$\begin{aligned} p(x_n, x^*) &= \lim_{m\to\infty} p(x_n, x_m) \\ &\leq \lim_{m\to\infty} \sup F(\varphi(x_n), \varphi(x_m)) \\ &= F(\varphi(x_n), \lim_{m\to\infty} \inf \varphi(x_m)) \\ &\leq F(\varphi(x_n), \varphi(x^*)). \end{aligned}$$

So $x_n \preceq x^*$, for all n. □

Similarly we can prove the following.

Theorem 3.2. *Let (X, ϑ) be a Hausdorff uniform space and p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function which is bounded above and " $\preceq$ " the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping, $(-\infty, x] = \{y \in X : y \preceq x\}$ and $M = \{x \in X \mid T(x) \cap (-\infty, x] \neq \emptyset\}$. Suppose that:*

(i) T is upper semi-continuous, that is $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in T(x_0)$;

(ii) $M \neq \emptyset$;

(iii) for each $x \in M$, $T(x) \cap M \cap (-\infty, x] \neq \emptyset$.

Then T has a fixed point x^* and there exists a sequence $\{x_n\}$ with

$$x_{n-1} \succeq x_n \in T(x_{n-1}), n = 1, 2, 3, ...$$

such that $x_n \to x^*$. Moreover, if φ is upper semi-continuous, then $x^* \preceq x_n$ for all n.

Corollary 3.3. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function which is bounded below and " $\preceq$ " the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $[x, +\infty) = \{y \in X : x \preceq y\}$. Suppose that:*

(i) T is upper semi-continuous, that is $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in T(x_0)$;

(ii) T satisfies the monotonic condition: for any x, $y \in X$ with $x \preceq y$ and any $u \in T(x)$, there exists $v \in T(y)$ such that $u \preceq v$;

(iii) there exists an $x_0 \in X$ such that $T(x_0) \cap [x_0, +\infty) \neq \emptyset$.

Then T has a fixed point x^* and there exists a sequence $\{x_n\}$ with

$$x_{n-1} \preceq x_n \in T(x_{n-1}), n = 1, 2, 3, ...$$

such that $x_n \to x^*$. Moreover, if φ is lower semi-continuous, then $x_n \preceq x^*$ for all n.

Proof. By (iii), $x_0 \in M = \{x \in X : T(x) \cap [x, +\infty) \neq \emptyset\}$. For $x \in M$, take $y \in T(x)$ and $x \preceq y$. By the monotonicity of T, there exists $z \in T(y)$ such that $y \preceq z$. So $y \in M$, and $T(x) \cap M \cap [x, +\infty) \neq \emptyset$. The conclusion follows from Theorem 1. $\square$

Corollary 3.4. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function which is bounded above and " $\preceq$ " the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $(-\infty, x] = \{y \in X : y \preceq x\}$. Suppose that:*

(i) T is upper semi-continuous;

(ii) T satisfies the monotonic condition; for any $x, y \in X$ with $x \preceq y$ and any $v \in T(y)$, there exists $u \in T(x)$ such that $u \preceq v$;

(iii) there exists an $x_0 \in X$ such that $T(x_0) \cap (-\infty, x_0] \neq \emptyset$.

Then T has a fixed point x^* and there exists a sequence $\{x_n\}$ with

$$x_{n-1} \succeq x_n \in T(x_{n-1}), n = 1, 2, ...$$

such that $x_n \to x^*$. Moreover, if φ is upper semi-continuous, then $x_n \succeq x^*$ for all n.

Corollary 3.5. *Let (X, ϑ) be a Hausdorff uniform space and p be an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function which is bounded below and " $\preceq$ " the order introduced by φ. Let X be also a p-Cauchy complete space, $f : X \to X$ be a map and $M = \{x \in X : x \preceq f(x)\}$. Suppose that:*

(i) f is $\tau(\vartheta)$-continuous;

(ii) $M \neq \emptyset$;

(iii) for each $x \in M$, $f(x) \in M$.

Then f has a fixed point x^* and the sequence

$$x_{n-1} \preceq x_n = f(x_{n-1}), n = 1, 2, 3, ..$$

converges to x^*. Moreover, if φ is lower semi-continuous, then $x_n \preceq x^*$ for all n.

Corollary 3.6. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function which is bounded above, and " $\preceq$ " the order introduced by φ. Let X be also a p-Cauchy complete space, $f : X \to X$ be a map and $M = \{x \in X : x \succeq f(x)\}$. Suppose that:*

(i) f is $\tau(\vartheta)$-continuous;

(ii) $M \neq \emptyset$;

(iii) for each $x \in M$, $f(x) \in M$.

Then f has a fixed point x^* and the sequence

$$x_{n-1} \succeq x_n = f(x_{n-1}),\ n = 1, 2, 3, ...$$

converges to x^*. Moreover, if φ is upper semi-continuous, then $x_n \succeq x^*$ for all n.

Corollary 3.7. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function which is bounded below, and " $\preceq$ " the order introduced by φ. Let X be also a p-Cauchy complete space, $f : X \to X$ be a map and $M = \{x \in X : x \succeq f(x)\}$. Suppose that:*
(i) f is $\tau(\vartheta)$-continuous;
(ii) f is monotone increasing, that is for $x \preceq y$ we have $f(x) \preceq f(y)$;
(iii) there exists an x_0, with $x_0 \preceq f(x_0)$.
Then f has a fixed point x^* and the sequence

$$x_{n-1} \preceq x_n = f(x_{n-1}),\, n = 1, 2, 3, ...$$

converges to x^*. Moreover, if φ is lower semi-continuous, then $x_n \preceq x^*$ for all n.

Example 3.8. Let $A = \{a, b, c\}$ and $\vartheta = \{V \subset A \times A : \Delta \subset V\}$. Define $p : A \times A \to \mathbb{R}^+$ as $p(x, x) = 0$ for all $x \in A$, $p(a, b) = p(b, a) = 2$, $p(a, c) = p(c, a) = 1$ and $p(b, c) = p(c, b) = 3$. By the definition of ϑ, $\bigcap\limits_{V \in \vartheta} V = \Delta$ and this shows that the uniform space (A, ϑ) is a Hausdorff uniform space. Furthermore, $p(a, b) \leq p(a, c) + p(c, b)$, $p(a, c) \leq p(a, b) + p(b, c)$ and $p(b, c) \leq p(b, a) + p(a, c)$ for $a, b, c \in A$ and thus p is an E-distance on A. Next define $\varphi : A \to \mathbb{R}$, $\varphi(a) = 3$, $\varphi(b) = 2$, $\varphi(c) = 1$. Since $p(a, c) = p(c, a) = 1 \leq \varphi(a) - \varphi(c)$, therefore $a \preceq c$. But as $p(b, a) = p(a, b) = 2 \nleq |\, \varphi(a) - \varphi(b)\, |$ therefore $a \npreceq b$ and $b \npreceq a$. Again, $b \npreceq c$ and $c \npreceq b$ which show that this ordering is partial and hence (A, ϑ) is a partially ordered uniform space. Define $g : A \to A$ as $g(a) = a$, $g(b) = b$ and $g(c) = c$, then we can verify that all conditions of Corollary 5 are satisfied and g has a fixed point. Notice that $p(g(a), g(b)) = p(a, b)$. This shows that g is neither E-contractive nor E expansive, therefore the results of [2] are not applicable in the context of this example.

Corollary 3.9. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function which is bounded above and " $\preceq$ " the order introduced by φ. Let X be also a p-Cauchy complete space and $f : X \to X$ be a map. Suppose that:*
(i) f is $\tau(\vartheta)$-continuous;
(ii) f is monotone increasing, that is, for $x \preceq y$ we have $f(x) \preceq f(y)$;
(iii) there exists an x_0 with $x_0 \succeq f(x_0)$.
Then f has a fixed point x^* and the sequence

$$x_{n-1} \succeq x_n = f(x_{n-1}),\ n = 1, 2, 3, ...$$

converges to x^*. Moreover, if φ is upper semi-continuous, then $x_n \succeq x^*$ for all n.

Theorem 3.10. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one and continuous function bounded below and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $[x, +\infty) = \{y \in X : x \preceq y\}$. Suppose that:*

(i) T satisfies the monotonic condition: for each $x \preceq y$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $u \preceq v$;

(ii) $T(x)$ is compact for each $x \in X$;

(iii) $M = \{x \in X : T(x) \cap [x, +\infty) \neq \emptyset\} \neq \emptyset$.

Then T has a fixed point x_0.

Proof. We shall prove that M has a maximal element. Let $\{x_v\}_{v\in\Lambda}$ be a totally ordered subset in M, where Λ is a directed set. For $v, \mu \in \Lambda$ and $v \leq \mu$, one has $x_v \preceq x_\mu$, which implies that $\varphi(x_v) \geq \varphi(x_\mu)$ for $v \leq \mu$. Since φ is bounded below, $\{\varphi(x_v)\}$ is a convergence net in $\mathbb{R}$. So it is a Cauchy net i.e. for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $v \leq \mu$ we have $|\varphi(x_\mu) - \varphi(x_v)| < \varepsilon$. By the p-Cauchy completeness of X, let x_v converge to z in X.

For given $\mu \in \Lambda$, from Remark 3

$p(x_\mu, z) = \lim\limits_{v} p(x_\mu, x_v) \leq \lim\limits_{v} F(\varphi(x_\mu), \varphi(x_v)) = F(\varphi(x_\mu), \varphi(x_z))$. So $x_\mu \preceq z$ for all $\mu \in \Lambda$.

For $\mu \in \Lambda$, by the condition (i), for each $u_\mu \in T(x_\mu)$, there exists a $v_\mu \in T(z)$ such that $u_\mu \preceq v_\mu$. By the compactness of $T(z)$, there exists a convergence subnet $\{v_{\mu'}\}$ of $\{v_\mu\}$. Suppose that $\{v_{\mu'}\}$ converges to $w \in T(z)$. Take Λ' such that $\mu' \geq \Lambda'$ implies $u_\mu \preceq v_\mu \preceq v_{\mu'}$.

We have

$p(u_\mu, w) = \lim\limits_{\mu'} p(u_\mu, v_{\mu'}) \leq \lim\limits_{\mu'} F(\varphi(u_\mu), \varphi(v_{\mu'})) = F(\varphi(u_\mu), \varphi(w))$.

So $u_\mu \preceq w$ for all μ and

$$p(z, w) = \lim_{\mu} p(u_\mu, w) \leq \lim_{\mu} F(\varphi(u_\mu), \varphi(w)) = F(\varphi(z), \varphi(w)).$$

So $z \preceq w$ and this gives that $z \in M$. Hence we have proven that $\{x_\mu\}$ has an upper bound in M.

By Zorn's Lemma, there exists a maximal element x_0 in M. By the definition of M, there exists a $y_0 \in T(x_0)$ such that $x_0 \preceq y_0$. By the condition (i), there exists a $z_0 \in T(y_0)$ such that $y_0 \preceq z_0$. Hence $y_0 \in M$. Since x_0 is a maximal element in M, it follows that $y_0 = x_0$ and $x_0 \in T(x_0)$. So x_0 is a fixed point of T. □

Theorem 3.11. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one and continuous function bounded above and "$\preceq$" the order introduced by φ. Let X be also a p-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $(-\infty, x] = \{y \in X : y \preceq x\}$. Suppose that*

(i) T satisfies the following condition; for each $x \preceq y$ and $v \in T(x)$, there exists $u \in T(y)$ such that $u \preceq v$;

(ii) $T(x)$ is compact for each $x \in X$;

(iii) $M = \{x \in X : T(x) \cap (-\infty, x] \neq \emptyset\} \neq \emptyset$.
Then T has a fixed point.

Corollary 3.12. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one continuous function bounded below and " $\preceq$ " the order introduced by φ. Let X be also a p-Cauchy complete space and $f : X \to X$ be a map. Suppose that;*
(i) f is monotone increasing, that is for $x \preceq y$, $f(x) \preceq f(y)$;
(ii) there is an $x_0 \in X$ such that $x_0 \preceq f(x_0)$.
Then f has a fixed point.

Corollary 3.13. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one continuous function bounded above and " $\preceq$ " the order introduced by φ. Let X be also a p-Cauchy complete space and $f : X \to X$ be a map. Suppose that;*
(i) f is monotone increasing, that is for $x \preceq y$, $f(x) \preceq f(y)$;
(ii) there is an $x_0 \in X$ such that $x_0 \succeq f(x_0)$.
Then f has a fixed point.

4. The Coupled Fixed Point Theorems of Multivalued Mappings

Definition 4.1. An element $(x, y) \in X \times X$ is called a coupled fixed point of the multivalued mapping $T : X \times X \to 2^X$ if $x \in T(x, y)$, $y \in T(y, x)$.

Theorem 4.2. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function bounded below and " $\preceq$ " be the order in X introduced by φ. Let X be also a p-Cauchy complete space, $T : X \times X \to 2^X$ be a multivalued mapping, $[x, +\infty) = \{y \in X : x \preceq y\}$, $(-\infty, y] = \{x \in X : x \preceq y\}$, and $M = \{(x, y) \in X \times X : x \preceq y,\ T(x, y) \cap [x, +\infty) \neq \emptyset$ and $T(y, x) \cap (-\infty, y] \neq \emptyset\}$. Suppose that:*
(i) T is upper semi-continuous, that is, $x_n \in X$, $y_n \in X$ and $z_n \in T(x_n, y_n)$, with $x_n \to x_0$, $y_n \to y_0$ and $z_n \to z_0$ implies $z_0 \in T(x_0, y_0)$;
(ii) $M \neq \emptyset$;
(iii) for each $(x, y) \in M$, there is $(u, v) \in M$ such that $u \in T(x, y) \cap [x, +\infty)$ and $v \in T(y, x) \cap (-\infty, y]$
Then T has a coupled fixed point (x^*, y^*) i.e. $x^* \in T(x^*, y^*)$ and $y^* \in T(y^*, x^*)$. Also there exist two sequences $\{x_n\}$ and $\{y_n\}$ with

$$x_{n-1} \preceq x_n \in T(x_{n-1}, y_{n-1}),\ y_{n-1} \succeq y_n \in T(y_{n-1}, x_{n-1}),\ n = 1, 2, 3, ...$$

such that $x_n \to x^*$ and $y_n \to y^*$.

Proof. By the condition (ii), take $(x_0, y_0) \in M$. From (iii), there exist $(x_1, y_1) \in M$ such that $x_1 \in T(x_0, y_0)$, $x_0 {\preceq} x_1$ and $y_1 \in T(y_0, x_0)$, $y_1 \preceq y_0$. Again from (iii), there exist $(x_2, y_2) \in M$ such that $x_2 \in T(x_1, y_1)$, $x_1 {\preceq} x_2$ and $y_2 \in T(y_1, x_1)$, $y_2 \preceq y_1$.

Continuing this procedure we get two sequences $\{x_n\}$ and $\{y_n\}$ satisfying $(x_n, y_n) \in M$ and

$$x_{n-1} \preceq x_n \in T(x_{n-1}, y_{n-1}), \ n = 1, 2, ...$$

and

$$y_{n-1} \succeq y_n \in T(y_{n-1}, x_{n-1}), \ n = 1, 2, ...$$

So

$$x_0 \preceq x_1 \preceq ... \preceq x_n \preceq ... \preceq y_n \preceq ... \preceq y_2 \preceq y_1.$$

Hence

$$\varphi(x_0) \geq \varphi(x_1) \geq ... \geq \varphi(x_n) \geq ... \geq \varphi(y_n) \geq ... \geq \varphi(y_1) \geq \varphi(y_0).$$

From this we get that $\varphi(x_n)$ and $\varphi(y_n)$ are convergent sequences. By the definition of " $\preceq$ " as in the proof of Theorem 1, it is easy to prove that $\{x_n\}$ and $\{y_n\}$ are p-Cauchy sequences. Since X is p-Cauchy complete, let $\{x_n\}$ converge to x^* and $\{y_n\}$ converge to y^*. Since T is upper semi-continuous, $x^* \in T(x^*, y^*)$ and $y^* \in T(y^*, x^*)$. Hence (x^*, y^*) is a coupled fixed point of T. □

Corollary 4.3. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function bounded below, and " $\preceq$ " be the order in X introduced by φ. Let X be also a p-Cauchy complete space, $f : X \times X \to X$ be a mapping and $M = \{(x, y) \in X \times X : x \preceq y$ and $x \preceq f(x, y)$ and $f(x, y) \preceq y\}$. Suppose that;*

(i) f is $\tau(\vartheta)$-continuous;

(ii) $M \neq \emptyset$;

(iii) for each $(x, y) \in M$, $x \preceq f(x, y)$ and $f(y, x) \preceq y$.

Then f has a coupled fixed point (x^*, y^*), i.e. $x^* = f(x^*, y^*)$ and $y^* = f(y^*, x^*)$ and there exist two sequences $\{x_n\}$ and $\{y_n\}$ with $x_{n-1} \preceq x_n = f(x_{n-1}, y_{n-1})$, $y_{n-1} \succeq y_n = f(y_{n-1}, x_{n-1})$, $n = 1, 2, ...$ such that $x_n \to x^*$ and $y_n \to y^*$.

Corollary 4.4. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one function bounded below, and " $\preceq$ " be the order in X introduced by φ. Let X be also a p-Cauchy complete space, $f : X \times X \to X$ be a mapping and $M = \{(x, y) \in X \times X : x \preceq y$ and $x \preceq f(x, y)$ and $f(x, y) \preceq y\}$. Suppose that;*

(i) f is $\tau(\vartheta)$-continuous;

(ii) $M \neq \emptyset$;

(iii) f is mixed monotone, that is for each $x_1 \preceq x_2$ and $y_1 \succeq y_2$, $f(x_1, y_1) \preceq f(x_2, y_2)$.

Then f has a coupled fixed point (x^*, y^*) and there exist two sequences $\{x_n\}$ and $\{y_n\}$ with $x_{n-1} \preceq x_n = f(x_{n-1}, y_{n-1})$, $y_{n-1} \succeq y_n = f(y_{n-1}, x_{n-1})$, $n = 1, 2, ...$ such that $x_n \to x^*$ and $y_n \to y^*$.

Theorem 4.5. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be an one to one continuous function, and " $\preceq$ " be the order in X introduced by φ. Let X be also a p-Cauchy complete space, $T : X \times X \to 2^X$ be a multivalued mapping, $[x, +\infty) = \{y \in X : x \preceq y\}$, $(-\infty, y] = \{x \in X : x \preceq y\}$, and $M = \{(x, y) \in X \times X : x \preceq y$, $T(x, y) \cap [x, +\infty) \neq \emptyset$ and $T(y, x) \cap (-\infty, y] \neq \emptyset\}$. Suppose that;*

(i) T is mixed monotone, that is for $x_1 \preceq y_1$, $x_2 \succeq y_2$ and $u \in T(x_1, y_1)$, $v \in T(y_1, x_1)$, there exist $w \in T(x_2, y_2)$, $z \in T(y_2, x_2)$ such that $u \preceq w$, $v \succeq z$;

(ii) $M \neq \emptyset$;

(iii) $T(x, y)$ is compact for each $(x, y) \in X \times X$.

Then T has a coupled fixed point.

Proof. By (ii), there exists $(x_0, y_0) \in M$ with $x_0 \preceq y_0$, $T(x_0, y_0) \cap [x_0, +\infty) \neq \emptyset$ and $T(y_0, x_0) \cap (-\infty, y_0] \neq \emptyset$. Let $C = \{(x, y) : x_0 \preceq x,\ y \preceq y_0,\ T(x, y) \cap [x, +\infty) \neq \emptyset$ and $T(y, x) \cap (-\infty, y] \neq \emptyset\}$. Then $(x_0, y_0) \in C$. Define the order relation " $\preceq$ " in C by

$$(x_1, y_1) \preceq (x_2, y_2) \Leftrightarrow x_1 \preceq x_2,\ y_2 \preceq y_1.$$

It is easy to prove that $(C, \preceq)$ becomes an ordered space.

We shall prove that C has a maximal element. Let $\{x_v, y_v\}_{v \in \Lambda}$ be a totally ordered subset in C, where Λ is a directed set. For $v, \mu \in \Lambda$ and $v \leq \mu$, one has $(x_v, y_v) \preceq (x_\mu, y_\mu)$. So $x_v \preceq x_\mu$ and $y_\mu \preceq y_v$, which implies that

$$\varphi(x_0) \geq \varphi(x_v) \geq \varphi(x_\mu) \geq \varphi(y_0)$$

and

$$\varphi(y_0) \leq \varphi(y_\mu) \leq \varphi(y_v) \leq \varphi(x_0)$$

for $v \leq \mu$.

Since $\{\varphi(x_v)\}$ and $\{\varphi(y_v)\}$ are convergence nets in $\mathbb{R}$. From

$$p(x_v, x_\mu) \leq F(\varphi(x_v), \varphi(x_\mu)) \text{ and } p(y_\mu, y_v) \leq F(\varphi(y_\mu), \varphi(y_v)),$$

we get that $\{x_v\}$ and $\{y_v\}$ are p-Cauchy nets in X. By the p-Cauchy completeness of X, let x_v converge to x^* and y_v converge to y^* in X. For given $\mu \in \Lambda$,

$$\begin{aligned} p(x_\mu, x^*) &= \lim_v p(x_\mu, x_v) \leq \lim_v F(\varphi(x_\mu), \varphi(x_v)) = F(\varphi(x_\mu), \varphi(x^*)) \\ p(y_\mu, y^*) &= \lim_v p(y_\mu, y_v) \leq \lim_v F(\varphi(y_v), \varphi(y_\mu)) = F(\varphi(y_v), \varphi(y^*)). \end{aligned}$$

So $x_0 \preceq x_\mu \preceq x^*$ and $y_\mu \succeq y^* \succeq y_0$ for all $\mu \in \Lambda$.

For $\mu \in \Lambda$, by the condition (i), for each $u_\mu \in T(x_\mu, y_\mu)$ with $x_\mu \preceq u_\mu$ and $v_\mu \in T(y_\mu, x_\mu)$ with $v_\mu \preceq y_\mu$, there exist $w_\mu \in T(x^*, y^*)$ and $z_\mu \in T(y^*, x^*)$ such that $u_\mu \preceq w_\mu$ and $v_\mu \succeq z_\mu$. By the compactness of $T(x^*, y^*)$ and $T(y^*, x^*)$, there exist convergence subnets $\{w_{\mu'}\}$ of $\{w_\mu\}$ and $\{z_{\mu'}\}$ of $\{z_\mu\}$. Suppose that

$\{w_{\mu'}\}$ converges to $w \in T(x^*, y^*)$ and $\{z_{\mu'}\}$ converges to $z \in T(y^*, x^*)$. Take Λ', such that $\mu' \geq \Lambda'$ implies $u_\mu \preceq v_\mu \preceq v_{\mu'}$. We have

$$\begin{aligned} p(u_\mu, w) &= \lim_{\mu'} p(u_\mu, u_{\mu'}) \leq \lim_{\mu'} F(\varphi(u_\mu), \varphi(u_{\mu'})) = F(\varphi(u_\mu), \varphi(w)) \\ p(z, v_\mu) &= \lim_{\mu'} p(v_{\mu'}, v_\mu) \leq \lim_{\mu'} F(\varphi(v_{\mu'}), \varphi(v_\mu)) = F(\varphi(z), \varphi(v_\mu)). \end{aligned}$$

So $x_\mu \preceq u_\mu \preceq w$ and $z \preceq v_\mu \preceq y_\mu$ for all μ. Also

$$\begin{aligned} p(x^*, w) &= \lim_{\mu'} p(x_{\mu'}, u_{\mu'}) \leq \lim_{\mu'} F(\varphi(x_{\mu'}), \varphi(u_{\mu'})) = F(\varphi(x^*), \varphi(w)) \\ p(z, y^*) &= \lim_{\mu'} p(v_{\mu'}, y_{\mu'}) \leq \lim_{\mu'} F(\varphi(v_{\mu'}), \varphi(y_{\mu'})) = F(\varphi(z), \varphi(y^*)). \end{aligned}$$

So $x^* \preceq w$ and $z \preceq y^*$, this gives that $(x^*, y^*) \in C$. Hence we have proven that $\{x_\mu, y_\mu\}_{\mu \in \Lambda}$ has an upper bound in C.

By Zorn's lemma, there exists a maximal element $(\bar{x}, \bar{y})$ in C. By the definition of C, there exist $\bar{u} \in T\left(\bar{x}, \bar{y}\right)$, $\bar{v} \in T\left(\bar{y}, \bar{x}\right)$, such that $x_0 \preceq \bar{u}$, $\bar{v} \preceq y_0$ and $\bar{x} \preceq \bar{u}$, $\bar{v} \preceq \bar{y}$. By the condition (i) there exist $\bar{w} \in T(\bar{u}, \bar{v})$, $\bar{z} \in T(\bar{v}, \bar{u})$ such that $x_0 \preceq \bar{u} \preceq \bar{w}$ and $\bar{z} \preceq \bar{v} \preceq y_0$. Hence $\left(\bar{u}, \bar{v}\right) \in C$ and $\left(\bar{x}, \bar{y}\right) \preceq \left(\bar{u}, \bar{v}\right)$. Since $\left(\bar{x}, \bar{y}\right)$ is a maximal element in C, it follows that $\left(\bar{x}, \bar{y}\right) = \left(\bar{u}, \bar{v}\right)$, and it follows that $\bar{x} = \bar{u} \in T(\bar{x}, \bar{u})$ and $\bar{y} = \bar{v} \in T(\bar{y}, \bar{x})$. So $\left(\bar{x}, \bar{y}\right)$ is a coupled fixed point of T. □

Corollary 4.6. *Let (X, ϑ) be a Hausdorff uniform space, p an E-distance on X, $\varphi : X \to \mathbb{R}$ be a continuous function, and " $\preceq$ " be the order in X introduced by φ. Let X be also a p-Cauchy complete space and $f : X \times X \to X$ be a mapping. Suppose that;*

(i) f is mixed monotone, that is for $x_1 \preceq y_1$, $x_2 \succeq y_2$ and $f(x_1, y_1) \preceq f(y_2, x_2)$;

(ii) there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $f(y_0, x_0) \preceq y_0$.

Then f has a coupled fixed point.

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