

## Warped-twisted products and Einstein-like manifolds

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**Abstract.** We study warped-twisted product manifolds in the form  ${}_{f_2}M_1 \times_{f_1} M_2$  with warping function  $f_2$  on  $M_2$  and twisting function  $f_1$ . We give a necessary and sufficient condition for a warped-twisted product to be a doubly warped product. We also give some conditions for such manifolds to be a twisted and a base conformal warped product. Moreover, we give some results for Einstein-like warped-twisted product manifolds of different classes.

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### 1. Introduction

Let  $(M_1, g_1)$  and  $(M_2, g_2)$  be Riemannian manifolds and  $f_1$  and  $f_2$  be positive smooth functions on  $M_1 \times M_2$ . Then *doubly twisted product manifold* [11]  $M = {}_{f_2}M_1 \times_{f_1} M_2$  is the product manifold  $M_1 \times M_2$  furnished with the metric tensor  $g = f_2^2 \pi_1^* g_1 + f_1^2 \pi_2^* g_2$ , where  $\pi_i : M_1 \times M_2 \rightarrow M_i$  is canonical projections for  $i = 1, 2$ . Each function  $f_i$  is called a twisting function of  $(M = {}_{f_2}M_1 \times_{f_1} M_2, g)$ .

If the functions  $f_1$  and  $f_2$  in the above, depend only on the points of  $M_1$  and  $M_2$ , respectively, then we get *doubly warped product manifold*  $M = {}_{f_2}M_1 \times_{f_1} M_2$  [5] with the metric  $g$  given by  $g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2$ . Then each function  $f_i$  is called a *warping function* of  $({}_{f_2}M_1 \times_{f_1} M_2, g)$  for  $i = 1, 2$ . If  $f_1 \equiv 1$  or  $f_2 \equiv 1$  in the definition of doubly twisted product manifold, then  $(M_1 \times_{f_1} M_2, g)$  or  $({}_{f_2}M_1 \times M_2, g)$  is called a *twisted product manifold* [2]. In that case  $f_1$  or  $f_2$  is called a *twisting function*.

Moreover,  $f_1 \equiv 1$  or  $f_2 \equiv 1$ , in the definition of doubly warped product manifold, then we get  $(M_1 \times_{f_1} M_2, g)$  *warped product manifold* [1] with warping function  $f_1$ .

In [13], we defined a new subclass of doubly twisted product under the name of *nearly doubly twisted product of type 1*. In this article, we rename of such products as *warped-twisted products*.

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Let  $(f_2 M_1 \times_{f_1} M_2, g)$  be a doubly twisted product manifold. If  $f_2$  depends only on the points of  $M_2$ , then  $(f_2 M_1 \times_{f_1} M_2, g)$  is a warped-twisted product manifold with the metric tensor  $g = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + f_1^2 \pi_2^* g_2$ . In which case,  $f_2$  is called a warping function and  $f_1$  is called a twisting function of  $(f_2 M_1 \times_{f_1} M_2, g)$ . In this case, if the function  $f_1$  depends only on the the points of  $M_2$ , then the warped-twisted product  $f_2 M_1 \times_{f_1} M_2$  becomes a *base conformal warped product* [4]. We say that a warped-twisted product is *non-trivial* if it is neither doubly warped product nor warped product or base conformal warped product.

*Remark 1.1.* Let  $(M_1, g_1)$  and  $(M_2, g_2)$  be pseudo-Riemannian manifolds with Levi-Civita connections  $\nabla^1$  and  $\nabla^2$ , respectively. By usual convenience, we denote the set of lifts of vector fields on  $M_i$  by  $\mathcal{L}(M_i)$  and use the same notation for a vector field and for its lift. On the other hand, each  $\pi_i$  is a positive homothety, so it preserves the Levi-Civita connection. Thus, there is no confusion using the same notation for a connection on  $M_i$  and for its pullback via  $\pi_i$ .

Now, we recall some facts for later use [10].

Let  $(\bar{M}, \bar{g})$  be a Riemannian manifold of dimension  $m$  and  $\bar{\nabla}$  be the Levi-Civita connection on  $\bar{M}$ . Let  $\{e_1, e_2, \dots, e_m\}$  denote the local orthonormal frame field of  $(\bar{M}, \bar{g})$ .

The Ricci tensor  $S$  of  $(\bar{M}, \bar{g})$  is defined by

$$(1.1) \quad S(\bar{U}, \bar{V}) = \sum_{i=1}^m \bar{g}(R(e_i, \bar{U})\bar{V}, e_i)$$

for any vector fields  $\bar{U}, \bar{V}$  on  $\bar{M}$ , where  $R$  is the Riemannian curvature tensor of  $\bar{M}$ .

The scalar curvature  $\tau$  of  $(\bar{M}, \bar{g})$  is given by

$$(1.2) \quad \tau = \sum_{i=1}^m S(e_i, e_i).$$

For a vector field  $\bar{U}$  on  $\bar{M}$ , divergence of  $\bar{U}$  is defined by

$$(1.3) \quad \text{div} \bar{U} = \sum_{i=1}^m \bar{g}(\bar{\nabla}_{e_i} \bar{U}, e_i).$$

For a function  $f \in C^\infty(\bar{M})$  and a vector field  $\bar{U}$  on  $\bar{M}$ , the Hessian tensor of  $f$  is given by

$$(1.4) \quad H^f(\bar{U}) = \bar{\nabla}_{\bar{U}} \nabla f$$

and the Hessian form of  $f$  is given by

$$(1.5) \quad h^f(\bar{U}, \bar{V}) = \bar{g}(H^f(\bar{U}), \bar{V}).$$

For a function  $f \in C^\infty(\bar{M})$ , the Laplacian of  $f$  is defined by

$$(1.6) \quad \Delta f = \text{div} \nabla f.$$

## 2. Warped Twisted Product Manifolds

In this section, we give the covariant derivative formulas, Ricci tensor, Weyl conformal curvature tensor and scalar curvature of a warped-twisted product manifolds in the form  $f_2 M_1 \times_{f_1} M_2$ , where  $f_2$  is a warping function on  $M_2$  and  $f_1$  is a twisting function. We also give a necessary and sufficient condition for such manifolds to be a doubly warped product. We also give some conditions for such manifolds to be a twisted and a base conformal warped product.

*Remark 2.1.* From now on, throughout this article,  $M$  denotes a warped-twisted product manifold in the form  $f_2 M_1 \times_{f_1} M_2$  with warping function  $f_2 \in C^\infty(M_2)$ , twisting function  $f_1$  and endowed with the Riemannian metric  $g$ .

**Lemma 2.2.** *Let  $M$  be a warped-twisted product manifold. Then, we have*

$$(2.1) \quad \nabla_X Y = \nabla_X^1 Y - g(X, Y) \nabla \tilde{l},$$

$$(2.2) \quad \nabla_X U = \nabla_U X = U(\tilde{l})X + X(k)U,$$

$$(2.3) \quad \nabla_U V = \nabla_U^2 V + U(k)V + V(k)U - g(U, V) \nabla k,$$

for  $X, Y \in \mathcal{L}(M_1)$  and  $U, V \in \mathcal{L}(M_2)$ , where  $k = \log f_1$ ,  $l = \log f_2$  and  $\tilde{l} = l \circ \pi_2$  which is the pullback of  $l$  via  $\pi_2$ .

*Proof.* The proof follows from Proposition 1 of [7] with  $X(\tilde{l}) = 0$ , for  $X \in \mathcal{L}(M_1)$ .  $\square$

For  $X, Y \in \mathcal{L}(M_1)$ , we define [7]

$$(2.4) \quad h_1^k(X, Y) = XY(k) - (\nabla_X^1 Y)(k).$$

Then the Hessian form  $h^k$  of  $k$  on  $(M, g)$  satisfies

$$(2.5) \quad h^k(X, U) = XU(k) - X(k)U(l) - X(k)V(k),$$

$$(2.6) \quad h^k(X, Y) = h_1^k(X, Y) - X(l)Y(k) - X(k)Y(l) + g(X, Y)g(\nabla k, \nabla l),$$

where  $U \in \mathcal{L}(M_2)$ .

*Remark 2.3.* From now on, throughout this paper, we denote by  $l$  the pullback of  $l$ , i.e.,  $l = l \circ \pi_2$ .

Now let  $S$  and  $S^i$  be Ricci tensor of  $(M, g)$  and  $(M_i, g_i)$ , respectively. Then we have the following relations:

**Lemma 2.4.** *Let  $M$  be a warped-twisted product manifold. Then, we have*

$$(2.7) \quad \begin{aligned} S(X, Y) &= S^1(X, Y) + h^l(X, Y) - m_2 \left\{ h_1^k(X, Y) + X(k)Y(k) \right\} \\ &\quad - g(X, Y) \left\{ \Delta l + g(\nabla l, \nabla l) \right\}, \end{aligned}$$

$$(2.8) \quad S(X, U) = (1 - m_2)XU(k) + (m_1 + m_2 - 2)X(k)U(l),$$

$$(2.9) \quad \begin{aligned} S(U, V) &= S^2(U, V) + h^k(U, V) + (1 - m_2)h_2^k(U, V) + m_2U(k)V(k) \\ &\quad - g(U, V) \left\{ \Delta k + g(\nabla k, \nabla k) \right\} \\ &\quad - m_1 \left\{ h_2^l(U, V) + U(l)V(l) - U(l)V(k) - U(k)V(l) \right\} \end{aligned}$$

for  $X, Y \in \mathcal{L}(M_1)$  and  $U, V \in \mathcal{L}(M_2)$ .

*Proof.* The above equations can be easily obtained from Proposition 3 of [7] with  $X(l) = 0$ , for  $X \in \mathcal{L}(M_1)$ .  $\square$

Now let  $\tau^1$  and  $\tau^2$  be scalar curvature tensors of  $(M_1, g_1)$  and  $(M_2, g_2)$ , respectively. Let  $\{e_1, \dots, e_{m_1}, e_{m_1+1}, \dots, e_{m_1+m_2}\}$  be an orthonormal basis of  $M$ , where  $\{f_2e_1, \dots, f_2e_{m_1}\}$  is an orthonormal basis of  $M_1$  and  $\{f_1e_{m_1+1}, \dots, f_1e_{m_1+m_2}\}$  is an orthonormal basis of  $M_2$ . Then we have the following relation from Lemma 2.4:

**Lemma 2.5.** *Let  $M$  be a warped-twisted product manifold and let  $\tau$  be the scalar curvature of  $M$ . Then, we have*

$$(2.10) \quad \begin{aligned} \tau &= \frac{\tau^1}{f_2^2} + \frac{\tau^2}{f_1^2} + \tilde{\Delta}_1(l) + \tilde{\Delta}_2(k) - \frac{m_2}{f_2^2}\Delta_1(k) - \frac{m_1}{f_1^2}\Delta_2(l) \\ &\quad + \frac{(1 - m_2)}{f_1^2}\Delta_2(k) - m_2g(P_1\nabla k, P_1\nabla k) - m_1\Delta l - 2m_1g(\nabla l, \nabla l) \\ &\quad - m_2 \left\{ \Delta k + g(\nabla k, \nabla k) \right\} + m_2g(P_2\nabla k, P_2\nabla k) + 2m_1g(P_2\nabla k, \nabla l) \end{aligned}$$

for  $X, Y \in \mathcal{L}(M_1)$  and  $U, V \in \mathcal{L}(M_2)$ , where  $\tilde{\Delta}_1(k) = \sum_{i=1}^{m_1} h^k(e_i, e_i)$ ,

$$\tilde{\Delta}_2(k) = \sum_{i=m_1+1}^{m_1+m_2} h^k(e_i, e_i), \quad \Delta k = \tilde{\Delta}_1(k) + \tilde{\Delta}_2(k) \quad \text{and} \quad \nabla k = P_1\nabla k + P_2\nabla k.$$

**Definition 2.6.** Let  $(\bar{M}, \bar{g})$  be a Riemannian manifold of dimension  $m$ . Then the Weyl conformal curvature tensor field of  $\bar{M}$  is the tensor field  $\mathcal{W}$  of type  $(1, 3)$  defined as

$$(2.11) \quad \begin{aligned} \mathcal{W}(\bar{X}, \bar{Y})\bar{Z} &= R_{\bar{X}\bar{Y}}\bar{Z} \\ &\quad + \frac{1}{m-2} \left\{ S(\bar{X}, \bar{Z})\bar{Y} - S(\bar{Y}, \bar{Z})\bar{X} + \bar{g}(\bar{X}, \bar{Z})Q\bar{Y} - \bar{g}(\bar{Y}, \bar{Z})Q\bar{X} \right\} \\ &\quad - \frac{\tau}{(m-1)(m-2)} \left\{ \bar{g}(\bar{X}, \bar{Z})\bar{Y} - \bar{g}(\bar{Y}, \bar{Z})\bar{X} \right\} \end{aligned}$$

for any vector fields  $\bar{X}, \bar{Y}$  and  $\bar{Z}$  on  $\bar{M}$ , where  $Q$  is the Ricci operator and  $\tau$  is the scalar curvature of  $\bar{M}$  [17].

**Lemma 2.7.** *Let  $M$  be a warped-twisted product manifold. Then, for  $X \in \mathcal{L}(M_1)$  and  $U, V \in \mathcal{L}(M_2)$ , the Weyl conformal curvature tensor  $\mathcal{W}$  satisfies*

$$(2.12) \quad \mathcal{W}(U, V)X = \frac{(m_1 - 1)}{(m_1 + m_2 - 2)} \left\{ XU(k)V - XV(k)U \right\},$$

where  $m_i = \dim M_i$ ,  $i=1,2$ .

*Proof.* Let  $X \in \mathcal{L}(M_1)$  and  $U, V \in \mathcal{L}(M_2)$ . Then we have

$$(2.13) \quad \mathcal{W}(U, V)X = R(U, V)X + \frac{1}{(m_1 + m_2 - 2)} \left\{ S(U, X)V - S(V, X)U \right\},$$

from (2.11), since  $g(U, X) = g(V, X) = 0$ . If we use (2.8) in (2.13), we get (2.12).  $\square$

**Definition 2.8.** Let  $M$  be a warped-twisted product manifold. Then we say that  $M_2$  is *Weyl conformal flat* along  $M_1$  if  $\mathcal{W}(U, V) = 0$  and  $M_1$  is *Weyl conformal flat* along  $M_2$  if  $\mathcal{W}(X, Y) = 0$ , where  $X, Y \in \mathcal{L}(M_1)$  and  $U, V \in \mathcal{L}(M_2)$ .

Now, we are ready to give another main result.

**Theorem 2.9.** *Let  $M$  be a warped-twisted product manifold and  $\dim M_1 > 1$ . Then  $(M, g)$  can be expressed as a doubly warped product manifold  ${}_f M_1 \times_f M_2$  of  $(M_1, g_1)$  and  $(M_2, \tilde{g}_2)$  with warping functions  $f_1$  and  $f$ , if and only if  $M_2$  is Weyl conformal flat on  $M_1$ , where  $\tilde{g}_2 = f^2 g_2$  for some positive smooth function  $f$  on  $M_2$ .*

*Proof.* If  $(M, g)$  is a doubly warped product manifold of  $(M_1, g_1)$  and  $(M_2, \tilde{g}_2)$  with warping functions  $f_1$  and  $f$ , then we have  $V(k) = 0$  for  $V \in \mathcal{L}(M_2)$ . Thus, the assertion comes immediately from (2.12).

Conversely, if  $\mathcal{W}(U, V)X = 0$  for all  $X \in \mathcal{L}(M_1)$  and  $U, V \in \mathcal{L}(M_2)$ , then we have

$$(2.14) \quad XV(k)U - XU(k)V = 0,$$

from (2.12). For linearly independent vector field  $U$  and  $V$ , we deduce that  $XV(k) = XU(k) = 0$  from (2.14). Hence, it follows that  $f_1 = e^k = a(x)b(y)$ , where  $a$  and  $b$  are positive smooth functions on  $M_1$  and  $M_2$ , respectively. Then, we can write the metric  $g$  as  $g = f_2^2(y)g_1 + a^2(x)\tilde{g}_2$ , where  $\tilde{g}_2 = b^2(y)g_2$ . Therefore,  $({}_f M_1 \times_{f_1} M_2, g)$  is a doubly warped product with  $f = a$ .  $\square$

**Definition 2.10.** A vector field  $\bar{V}$  on a Riemannian manifold  $(\bar{M}, \bar{g})$  is called *torse-forming*, if it satisfies [15]

$$(2.15) \quad \bar{\nabla}_{\bar{X}} \bar{V} = \lambda \bar{X} + \mu(\bar{X})\bar{V}$$

for any vector field  $\bar{X}$  on  $\bar{M}$ , where  $\lambda$  is a function,  $\mu$  is a 1-form. If the 1-form  $\mu$  in (2.15) vanishes identically, then the vector field  $\bar{V}$  is called *concircular* [3, 12, 16, 14].

Now, we give another characterization for a warped-twisted product admitting a concircular vector field.

**Proposition 2.11.** *Let  $M$  be a warped-twisted product manifold. Then*

*a) If  $M$  admits a concircular vector field in  $\mathcal{L}(M_1)$ , then  ${}_{f_2}M_1 \times_{f_1} M_2$  is a twisted product.*

*b) If  $M$  admits a concircular vector field in  $\mathcal{L}(M_2)$ , then  ${}_{f_2}M_1 \times_{f_1} M_2$  is a base-conformal warped product.*

*Proof.* Let  $M$  be a warped-twisted product manifold. Then

**a)** Let  $X$  be a concircular vector field in  $\mathcal{L}(M_1)$ , then for any  $V \in \mathcal{L}(M_2)$ , we have

$$(2.16) \quad \nabla_V X = \lambda V$$

from (2.15). By using (2.2), we obtain

$$V(l)X + X(k)V = \lambda V$$

from (2.16). Hence, we get  $V(l) = 0$  for all  $V \in \mathcal{L}(M_2)$ . Which means that the function  $l$  is constant, so  $f_2$  is also constant. Thus,  ${}_{f_2}M_1 \times_{f_1} M_2$  is a twisted product.

**b)** Let  $U$  be a concircular vector field in  $\mathcal{L}(M_2)$ , then for any  $Y \in \mathcal{L}(M_1)$ , we have

$$(2.17) \quad \nabla_Y U = \lambda Y$$

from (2.15). By using (2.2), we obtain

$$U(l)Y + Y(k)U = \lambda Y$$

from (2.17). Hence, we get  $Y(k) = 0$  for all  $Y \in \mathcal{L}(M_1)$ . Which means that the function  $k = \ln f_1$  depends only points of  $M_2$ . Thus,  ${}_{f_2}M_1 \times_{f_1} M_2$  is a base-conformal warped product.  $\square$

### 3. Einstein-Like Warped-Twisted Products

In [8], Gray defined Einstein-like Riemannian manifolds of different classes. Mantica and Shenawy studied Einstein-like warped product manifolds in [9]. Besides this, El-Sayied and etc. [6] consider the Einstein-like doubly warped products and its applications. In this section, we introduce the different classes of Einstein-like warped-twisted product manifolds.

**Class A.** A Riemannian manifold  $(\bar{M}, \bar{g})$  admitting a *cyclic parallel Ricci tensor*, that is,

$$(\bar{\nabla}_{\bar{X}} \bar{S})(\bar{Y}, \bar{Z}) + (\nabla_{\bar{Y}} \bar{S})(\bar{Z}, \bar{X}) + (\bar{\nabla}_{\bar{Z}} \bar{S})(\bar{X}, \bar{Y}) = 0,$$

for any vector fields  $\bar{X}, \bar{Y}, \bar{Z}$  on  $\bar{M}$  is called an *Einstein-like manifold of class  $\mathcal{A}$* . It is noted that the above condition is equivalent to

$$(3.1) \quad (\bar{\nabla}_{\bar{X}} S)(\bar{X}, \bar{X}) = 0,$$

for any vector field  $\bar{X}$  on  $\bar{M}$ . The inheritance property of Class  $\mathcal{A}$  is given by the following result.

**Theorem 3.1.** *Let  $M$  be an Einstein-like warped-twisted product of class  $\mathcal{A}$ . Then*

$$(3.2) \quad \begin{aligned} \text{a)} \quad (M_1, g_1) \text{ is an Einstein-like manifold of class } \mathcal{A} \text{ if and only if} \\ (\nabla_X^1 h^l)(X, X) = m_2 \{ (\nabla_X^1 h_1^k)(X, X) + 2X(k)h_1^k(X, X) \} \\ - 2g(X, X)S(\nabla l, X), \end{aligned}$$

$$(3.3) \quad \begin{aligned} \text{b)} \quad (M_2, g_2) \text{ is an Einstein-like manifold of class } \mathcal{A} \text{ if and only if} \\ (\nabla_U^2 h^k)(U, U) + (1 - m_2)(\nabla_U^2 h_2^k)(U, U) \\ = m_1(\nabla_U^2 h_2^l)(U, U) - 2m_2 h_2^k(U, U)U(k) \\ + 2U(f_1)f_1 g_2(U, U)k^\diamond + g(U, U)U(k^\diamond) \\ + 2m_1 \{ h_2^l(U, U)U(l) - h_2^l(U, U)U(k) \\ - h_2^k(U, U)U(l) \} \\ + 4U(k)S(U, U) - 2g(U, U)S(\nabla k, U) \end{aligned}$$

for  $X \in \mathcal{L}(M_1)$  and  $U \in \mathcal{L}(M_2)$ , where  $k^\diamond = \Delta k + \|\nabla k\|^2$ .

*Proof.* Let  $M$  be an Einstein-like warped-twisted product of class  $\mathcal{A}$ . Then, for any vector field  $X$  on  $M$ , we have  $(\nabla_X S)(X, X) = 0$ , from (3.1).

$$(3.4) \quad \begin{aligned} \text{a)} \quad (M_1, g_1) \text{ is an Einstein-like manifold of class } \mathcal{A} \text{ if and only if} \\ (\nabla_X^1 S^1)(X, X) = 0. \end{aligned}$$

for any vector field  $X$  on  $M_1$ . On the other hand, from (2.1) and (2.7), we have

$$\begin{aligned} (\nabla_X S)(X, X) &= X(S(X, X)) - 2S(\nabla_X X, X) \\ &= X(S^1(X, X) + h^l(X, X) - m_2 \{ h_1^k(X, X) + X(k)X(k) \} - g(X, X)l^\diamond) \\ &\quad - 2S(\nabla_X^1 X - g(X, X)\nabla l, X). \\ &= X(S^1(X, X)) - 2S^1(\nabla_X^1 X, X) + X(h^l(X, X)) - 2h^l(\nabla_X^1 X, X) \\ &\quad - m_2 \{ X(h_1^k(X, X)) - 2h_1^k(X, X) \} - m_2 X(X(k)X(k)) - X(g(X, X)l^\diamond) \\ &\quad + 2m_2 \nabla_X^1 X(k)X(k) + 2g(\nabla_X^1 X, X)l^\diamond + 2g(X, X)S(\nabla l, X). \end{aligned}$$

for  $X \in \mathcal{L}(M_1)$ . Then, we find

$$(3.5) \quad \begin{aligned} 0 &= (\nabla_X S)(X, X) \\ &= (\nabla_X^1 S^1)(X, X) + (\nabla_X^1 h^l)(X, X) \\ &\quad - m_2(\nabla_X^1 h_1^k)(X, X) - 2m_2 X(X(k))X(k) - (\nabla_X^1 g)(X, X)l^\diamond \\ &\quad + 2m_2 \nabla_X^1 X(k)X(k) + 2g(X, X)S(\nabla l, X). \end{aligned}$$

If we use (2.4) and (3.4), we have

$$(3.6) \quad \begin{aligned} 0 &= (\nabla_X S)(X, X) \\ &= (\nabla_X^1 h^l)(X, X) - m_2(\nabla_X^1 h_1^k)(X, X) \end{aligned}$$

$$(3.7) \quad -2m_2 X(k)h_1^k(X, X) + 2g(X, X)S(\nabla l, X).$$

So, (3.2) follows from (3.6).

**b)**  $(M_2, g_2)$  is an Einstein-like manifold of class  $\mathcal{A}$  if and only if

$$(3.8) \quad (\nabla_U^2 S^2)(U, U) = 0$$

for any vector field  $U$  on  $M_2$ . On the other hand, from (2.3) and (2.9), we have

$$\begin{aligned} &(\nabla_U S)(U, U) \\ &= U(S(U, U)) - 2S(\nabla_U U, U) \\ &= U(S^2(U, U) + h^k(U, U) + (1 - m_2)h_2^k(U, U) + m_2U(k)U(k) - g(U, U)k^\diamond) \\ &\quad - m_1U(h_2^l(U, U) + U(l)U(l) - U(l)U(k) - U(k)U(l)) \\ &\quad - 2S(\nabla_U^2 U + 2U(k)U - g(U, U)\nabla k, U) \\ &= U(S^2(U, U)) - 2S^2(\nabla_U^2 U, U) + U(h^k(U, U)) - 2h^k(\nabla_U^2 U, U) \\ &\quad + (1 - m_2)\{U(h_2^k(U, U)) - 2h_2^k(\nabla_U^2 U, U)\} \\ &\quad - m_1\{U(h_2^l(U, U)) - 2h_2^l(\nabla_U^2 U, U)\} + m_2\{U(U(k)U(k)) - 2\nabla_U^2 U(k)U(k)\} \\ &\quad - U(g(U, U))k^\diamond + 2g(\nabla_U^2 U, U)k^\diamond - g(U, U)U(k^\diamond) \\ &\quad - m_1\{U(U(l)U(l)) - 2U(U(l)U(k))\} \\ &\quad - m_1\{-2\nabla_U^2 U(l)U(l) + 2\nabla_U^2 U(l)U(k) + 2\nabla_U^2 U(k)U(l)\} \\ &\quad - 4U(k)S(U, U) + 2g(U, U)S(\nabla k, U) \end{aligned}$$

for  $U \in \mathcal{L}(M_2)$ . Then, we find

$$(3.9) \quad \begin{aligned} 0 &= (\nabla_U S)(U, U) \\ &= (\nabla_U^2 S^2)(U, U) + (\nabla_U^2 h^k)(U, U) \\ &\quad + (1 - m_2)(\nabla_U^2 h_2^k)(U, U) - m_1(\nabla_U^2 h_2^l)(U, U) \\ &\quad + 2m_2h_2^k(U, U)U(k) - (\nabla_U^2 g)(U, U)k^\diamond - g(U, U)U(k^\diamond) \\ &\quad - m_1\{2U(U(l))U(l) - 2U(U(l))U(k) - 2U(U(k))U(l)\} \\ &\quad - m_1\{-2\nabla_U^2 U(l)U(l) + 2\nabla_U^2 U(l)U(k) + 2\nabla_U^2 U(k)U(l)\} \\ &\quad - 4U(k)S(U, U) + 2g(U, U)S(\nabla k, U). \end{aligned}$$

If we use (2.4) and (3.8), we have

$$(3.10) \quad \begin{aligned} 0 &= (\nabla_U^2 h^k)(U, U) + (1 - m_2)(\nabla_U^2 h_2^k)(U, U) \\ &\quad - m_1(\nabla_U^2 h_2^l)(U, U) + 2m_2h_2^k(U, U)U(k) \\ &\quad - 2U(f_1)f_1g_2(U, U)k^\diamond - g(U, U)U(k^\diamond) \\ &\quad - 2m_1\{h_2^l(U, U)U(l) - h_2^l(U, U)U(k) - h_2^k(U, U)U(l)\} \\ &\quad - 4U(k)S(U, U) + 2g(U, U)S(\nabla k, U). \end{aligned}$$



So, (3.3) follows from (3.10). □

For a singly warped product, one can obtain the following result.

**Corollary 3.2.** *Let  $M$  be an Einstein-like singly warped product of class  $\mathcal{A}$ . Then*

**a)**  $(M_1, g_1)$  is an Einstein-like manifold of class  $\mathcal{A}$  if and only if

$$(3.11) \quad (\nabla_X^1 h^l)(X, X) = 0,$$

**b)**  $(M_2, g_2)$  is an Einstein-like manifold of class  $\mathcal{A}$  if and only if

$$(3.12) \quad (\nabla_U^2 h_2^l)(U, U) + 2h_2^l(U, U)U(l) = 0$$

for  $X \in \mathcal{L}(M_1)$  and  $U \in \mathcal{L}(M_2)$ .

**Class  $\mathcal{B}$ .** If the Ricci tensor of a Riemannian manifold  $(\bar{M}, \bar{g})$  is a Codazzi tensor, i.e.,

$$(3.13) \quad (\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{Y}} S)(\bar{X}, \bar{Z}),$$

for any vector fields  $\bar{X}, \bar{Y}, \bar{Z}$  on  $\bar{M}$ , then  $(\bar{M}, \bar{g})$  is called an *Einstein-like manifold of Class  $\mathcal{B}$* . This condition is equivalent to one of the following conditions:

- i) The Riemann tensor of  $(\bar{M}, \bar{g})$  is harmonic, or
- ii) The Weyl conformal tensor of  $(\bar{M}, \bar{g})$  is harmonic and the scalar curvature of  $(\bar{M}, \bar{g})$  is constant.

The factor manifolds are Einstein-like of Class  $\mathcal{B}$  according to the following result.

**Theorem 3.3.** *Let  $M$  be an Einstein-like warped-twisted product of class  $\mathcal{B}$ . Then*

**a)**  $(M_1, g_1)$  is an Einstein-like manifold of class  $\mathcal{B}$  if and only if

$$(3.14) \quad \begin{aligned} & (\nabla_X^1 h^l)(Y, Z) - (\nabla_Y^1 h^l)(X, Z) \\ &= m_2 \{ (\nabla_X^1 h_1^k)(Y, Z) - (\nabla_Y^1 h_1^k)(X, Z) \} \\ & \quad + m_2 \{ h_1^k(X, Z)Y(k) - h_1^k(Y, Z)X(k) \} \\ & \quad - g(X, Z)S(Y, \nabla l) + g(Y, Z)S(X, \nabla l), \end{aligned}$$

**b)**  $(M_2, g_2)$  is an Einstein-like manifold of class  $\mathcal{B}$  if and only if

$$\begin{aligned}
 & (\nabla_U^2 h^k)(V, W) - (\nabla_V^2 h^k)(U, W) \\
 & + (1 - m_2) \{ (\nabla_U^2 h_2^k)(V, W) - (\nabla_V^2 h_2^k)(U, W) \} \\
 & = m_1 \{ (\nabla_U^2 h_2^l)(V, W) - (\nabla_V^2 h_2^l)(U, W) \} \\
 & \quad + \{ 2U(f_1)f_1g_2(V, W) - 2V(f_1)f_1g_2(U, W) \} k^\diamond \\
 & \quad + g(V, W)U(k^\diamond) - g(U, W)V(k^\diamond) \\
 & \quad - m_2 \{ h_2^k(U, W)V(k) - h_2^k(V, W)U(k) \} \\
 (3.15) \quad & + m_1 \{ h_2^l(U, W)V(l) - h_2^l(U, W)U(l) \} \\
 & + m_1 \{ -h_2^l(U, W)V(k) - h_2^l(V, W)U(l) \} \\
 & + m_1 \{ h_2^k(V, W)U(l) + h_2^l(V, W)U(k) \} \\
 & - U(k)S(V, W) + V(k)S(U, W) \\
 & + g(U, W)S(V, \nabla k) - g(V, W)S(U, \nabla k)
 \end{aligned}$$

for  $X, Y, Z \in \mathcal{L}(M_1)$  and  $U, V, W \in \mathcal{L}(M_2)$ , where  $k^\diamond = \triangle k + \|\nabla k\|^2$ .

*Proof.* Let  $M$  be an Einstein-like warped-twisted product of class  $\mathcal{B}$ .

**a)**  $(M_1, g_1)$  is an Einstein-like manifold of class  $\mathcal{B}$  if and only if

$$(3.16) \quad (\nabla_X^1 S^1)(Y, Z) = (\nabla_Y^1 S^1)(X, Z),$$

for any vector fields  $X, Y, Z$  on  $M_1$ . On the other hand, from (2.1) and (2.7), we have

$$\begin{aligned}
 & (\nabla_X S)(Y, Z) \\
 & = X(S(Y, Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z) \\
 & = X(S(Y, Z)) - S(\nabla_X^1 Y - g(X, Y)\nabla l, Z) - S(Y, \nabla_X^1 Z - g(X, Z)\nabla l) \\
 & = X(S^1(Y, Z)) + X(h^l(Y, Z)) - m_2 \{ X(h_1^k(Y, Z)) + X(Y(k)Z(k)) \} \\
 & \quad - X(g(Y, Z))l^\diamond \\
 & \quad - \{ S^1(\nabla_X^1 Y, Z) + h^l(\nabla_X^1 Y, Z) - m_2 \{ h_1^k(\nabla_X^1 Y, Z) + \nabla_X^1 Y(k)Z(k) \} \\
 & \quad - g(\nabla_X^1 Y, Z)l^\diamond \} \\
 & \quad - \{ S^1(Y, \nabla_X^1 Z) + h^l(Y, \nabla_X^1 Z) - m_2 \{ h_1^k(Y, \nabla_X^1 Z) + \nabla_X^1 Z(k)Y(k) \} \\
 & \quad - g(Y, \nabla_X^1 Z)l^\diamond \} \\
 & \quad + g(X, Y)S(\nabla l, Z) + g(X, Z)S(Y, \nabla l)
 \end{aligned}$$

for  $X, Y, Z \in \mathcal{L}(M_1)$ . Then, we find

$$\begin{aligned}
 (\nabla_X S)(Y, Z) &= X(S^1(Y, Z)) - S^1(\nabla_X^1 Y, Z) - S^1(Y, \nabla_X^1 Z) \\
 &\quad + X(h^l(Y, Z)) - h^l(\nabla_X^1 Y, Z) - h^l(Y, \nabla_X^1 Z) \\
 &\quad - m_2\{X(h_1^k(Y, Z)) - h_1^k(\nabla_X^1 Y, Z) - h_1^k(Y, \nabla_X^1 Z)\} \\
 &\quad - \{X(g(Y, Z))l^\diamond - g(\nabla_X^1 Y, Z)l^\diamond - g(Y, \nabla_X^1 Z)l^\diamond\} \\
 &\quad - m_2\{X(Y(k)Z(k)) - \nabla_X^1 Y(k)Z(k) - \nabla_X^1 Z(k)Y(k)\} \\
 &\quad + g(X, Y)S(\nabla l, Z) + g(X, Z)S(Y, \nabla l)
 \end{aligned}$$

If we use (2.4), we have

$$\begin{aligned}
 (\nabla_X S)(Y, Z) &= (\nabla_X^1 S^1)(Y, Z) + (\nabla_X^1 h^l)(Y, Z) - m_2(\nabla_X^1 h_1^k)(Y, Z) \\
 (3.17) \quad &\quad - m_2\{h_1^k(X, Y)Z(k) + h_1^k(X, Z)Y(k)\} \\
 &\quad + g(X, Y)S(\nabla l, Z) + g(X, Z)S(Y, \nabla l).
 \end{aligned}$$

If we interchange  $X$  and  $Y$  in (3.17), we obtain

$$\begin{aligned}
 (\nabla_Y S)(X, Z) &= (\nabla_Y^1 S^1)(X, Z) + (\nabla_Y^1 h^l)(X, Z) - m_2(\nabla_Y^1 h_1^k)(X, Z) \\
 (3.18) \quad &\quad - m_2\{h_1^k(Y, X)Z(k) + h_1^k(Y, Z)X(k)\} \\
 &\quad + g(Y, X)S(\nabla l, Z) + g(Y, Z)S(X, \nabla l).
 \end{aligned}$$

Hence, from (3.13), (3.17), (3.18) and (3.16), we have

$$\begin{aligned}
 0 &= (\nabla_X^1 h^l)(Y, Z) - (\nabla_Y^1 h^l)(X, Z) \\
 (3.19) \quad &\quad - m_2\{(\nabla_X^1 h_1^k)(Y, Z) - (\nabla_Y^1 h_1^k)(X, Z)\} \\
 &\quad - m_2\{h_1^k(X, Z)Y(k) - h_1^k(Y, Z)X(k)\} \\
 &\quad + g(X, Z)S(Y, \nabla l) - g(Y, Z)S(X, \nabla l).
 \end{aligned}$$

So, (3.14) follows from (3.19).

**b)**  $(M_2, g_2)$  is an Einstein-like manifold of class  $\mathcal{B}$  if and only if

$$(3.20) \quad (\nabla_U^2 S^2)(V, W) = (\nabla_V^2 S^2)(U, W),$$

for any vector fields  $U, V, W$  on  $M_2$ . On the other hand, from (2.3) and (2.9), we have

$$\begin{aligned}
 (\nabla_U S)(V, W) &= U(S(V, W)) - S(\nabla_U V, W) - S(V, \nabla_U W) \\
 &= U(S^2(V, W) + h^k(V, W) + (1 - m_2)h_2^k(V, W) + m_2V(k)W(k) \\
 &\quad - g(V, W)k^\diamond) \\
 &\quad - m_1U(h_2^l(V, W) + V(l)W(l) - V(l)W(k) - V(k)W(l)) \\
 &\quad - S(\nabla_U^2 V + U(k)V + V(k)U - g(U, V)\nabla k, W) \\
 &\quad - S(V, \nabla_U^2 W + U(k)W + W(k)U - g(U, W)\nabla k)
 \end{aligned}$$

for  $U, V, W \in \mathcal{L}(M_2)$ . Then, we find

$$\begin{aligned}
(\nabla_U S)(V, W) &= U(S^2(V, W)) + U(h^k(V, W)) + (1 - m_2)U(h_2^k(V, W)) \\
&\quad + m_2 U(V(k)W(k)) - U(g(V, W)k^\diamond) \\
&\quad - m_1 \{U(h_2^l(V, W)) + U(V(l)W(l)) \\
&\quad - U(V(l)W(k)) - U(V(k)W(l))\} \\
&\quad - \{S^2(\nabla_U^2 V, W) + h^k(\nabla_U^2 V, W) + (1 - m_2)h_2^k(\nabla_U^2 V, W) \\
&\quad + m_2 \nabla_U^2 V(k)W(k) - g(\nabla_U^2 V, W)k^\diamond\} \\
&\quad - \{-m_1 \{h_2^l(\nabla_U^2 V, W) + \nabla_U^2 V(l)W(l) \\
&\quad - \nabla_U^2 V(l)W(k) - \nabla_U^2 V(k)W(l)\}\} \\
&\quad - \{S^2(V, \nabla_U^2 W) + h^k(V, \nabla_U^2 W) + (1 - m_2)h_2^k(V, \nabla_U^2 W) \\
&\quad + m_2 \nabla_U^2 W(k)V(k) - g(V, \nabla_U^2 W)k^\diamond\} \\
&\quad - \{-m_1 \{h_2^l(V, \nabla_U^2 W) + \nabla_U^2 W(l)V(l) \\
&\quad - \nabla_U^2 W(k)V(l) - \nabla_U^2 W(l)V(k)\}\} \\
&\quad - 2U(k)S(V, W) - V(k)S(U, W) - W(k)S(V, U)
\end{aligned}$$

If we arrange this equation, we have

$$\begin{aligned}
(\nabla_U S)(V, W) &= U(S^2(V, W)) - S^2(\nabla_U^2 V, W) - S^2(V, \nabla_U^2 W) \\
&\quad + U(h^k(V, W)) - h^k(\nabla_U^2 V, W) - h^k(V, \nabla_U^2 W) \\
&\quad + (1 - m_2) \{U(h_2^k(V, W)) - h_2^k(\nabla_U^2 V, W) - h_2^k(V, \nabla_U^2 W)\} \\
&\quad - m_1 \{U(h_2^l(V, W)) - h_2^l(\nabla_U^2 V, W) - h_2^l(V, \nabla_U^2 W)\} \\
&\quad - \{U(g(V, W))k^\diamond - g(\nabla_U^2 V, W)k^\diamond - g(V, \nabla_U^2 W)k^\diamond\} \\
&\quad - g(V, W)U(k^\diamond) \\
&\quad + m_2 \{U(V(k))W(k) + U(W(k))V(k) - \nabla_U^2 V(k)W(k) \\
&\quad - \nabla_U^2 W(k)V(k)\} \\
&\quad - m_1 \{U(V(l)W(l)) - U(V(l)W(k)) - U(V(k)W(l))\} \\
&\quad - m_1 \{-\nabla_U^2 V(l)W(l) + \nabla_U^2 V(l)W(k) + \nabla_U^2 V(k)W(l)\} \\
&\quad - m_1 \{-\nabla_U^2 W(l)V(l) + \nabla_U^2 W(k)V(l) + \nabla_U^2 W(l)V(k)\} \\
&\quad - 2U(k)S(V, W) - V(k)S(U, W) - W(k)S(V, U) \\
&\quad + g(U, V)S(\nabla k, W) + g(U, W)S(V, \nabla k).
\end{aligned}$$

If we use (2.4), we have

$$\begin{aligned}
 & (\nabla_U S)(V, W) \\
 &= (\nabla_U^2 S^2)(V, W) + (\nabla_U^2 h^k)(V, W) + (1 - m_2)(\nabla_U^2 h_2^k)(V, W) \\
 &\quad - m_1(\nabla_U^2 h_2^l)(V, W) \\
 &\quad - (\nabla_U^2 g)(V, W)k^\diamond - g(V, W)U(k^\diamond) \\
 (3.21) \quad & + m_2\{h_2^k(U, V)W(k) + h_2^k(U, W)V(k)\} \\
 &\quad - m_1\{h_2^l(U, V)W(l) + h_2^l(U, W)V(l) - h_2^l(U, V)W(k)\} \\
 &\quad - m_1\{-h_2^k(U, W)V(l) - h_2^k(U, V)W(l) - h_2^l(U, W)V(k)\} \\
 &\quad - 2U(k)S(V, W) - V(k)S(U, W) - W(k)S(V, U) \\
 &\quad + g(U, V)S(\nabla k, W) + g(U, W)S(V, \nabla k).
 \end{aligned}$$

If we interchange  $U$  and  $V$  in (3.21), we obtain

$$\begin{aligned}
 & (\nabla_V S)(U, W) \\
 &= (\nabla_V^2 S^2)(U, W) + (\nabla_V^2 h^k)(U, W) + (1 - m_2)(\nabla_V^2 h_2^k)(U, W) \\
 &\quad - m_1(\nabla_V^2 h_2^l)(U, W) \\
 &\quad - (\nabla_V^2 g)(U, W)k^\diamond - g(U, W)V(k^\diamond) \\
 (3.22) \quad & + m_2\{h_2^k(V, U)W(k) + h_2^k(V, W)U(k)\} \\
 &\quad - m_1\{h_2^l(V, U)W(l) + h_2^l(V, W)U(l) - h_2^l(V, U)W(k)\} \\
 &\quad - m_1\{-h_2^k(V, W)U(l) - h_2^k(V, U)W(l) - h_2^l(V, W)U(k)\} \\
 &\quad - 2V(k)S(U, W) - U(k)S(V, W) - W(k)S(U, V) \\
 &\quad + g(V, U)S(\nabla k, W) + g(V, W)S(U, \nabla k).
 \end{aligned}$$

Hence, from (3.13), (3.21), (3.22) and (3.20), we obtain

$$\begin{aligned}
 0 &= (\nabla_U^2 h^k)(V, W) - (\nabla_V^2 h^k)(U, W) \\
 &\quad + (1 - m_2)\{(\nabla_U^2 h_2^k)(V, W) - (\nabla_V^2 h_2^k)(U, W)\} \\
 &\quad - m_1\{(\nabla_U^2 h_2^l)(V, W) - (\nabla_V^2 h_2^l)(U, W)\} \\
 &\quad - 2U(f_1)f_1g_2(V, W)k^\diamond + 2V(f_1)f_1g_2(U, W)k^\diamond \\
 &\quad - g(V, W)U(k^\diamond) + g(U, W)V(k^\diamond) \\
 &\quad + m_2\{h_2^k(U, W)V(k) - h_2^k(V, W)U(k)\} \\
 &\quad - m_1\{h_2^l(U, V)W(l) + h_2^l(U, W)V(l) - h_2^l(U, V)W(k)\} \\
 &\quad - m_1\{-h_2^k(U, W)V(l) - h_2^k(U, V)W(l) - h_2^l(U, W)V(k)\} \\
 &\quad + m_1\{-h_2^k(V, W)U(l) - h_2^k(V, U)W(l) - h_2^l(V, W)U(k)\} \\
 &\quad - 2U(k)S(V, W) - V(k)S(U, W) - W(k)S(V, U) \\
 &\quad + 2V(k)S(U, W) + U(k)S(V, W) + W(k)S(U, V) \\
 &\quad - g(V, U)S(\nabla k, W) - g(V, W)S(U, \nabla k).
 \end{aligned}$$

So, (3.15) follows from the above equation.  $\square$

For a singly warped product, one can obtain the following result.

**Corollary 3.4.** *Let  $M$  be an Einstein-like singly warped product of class  $\mathcal{B}$ . Then*

*a)*  $(M_1, g_1)$  is an Einstein-like manifold of class  $\mathcal{B}$  if and only if

$$(3.23) \quad (\nabla_X^1 h^l)(Y, Z) - (\nabla_Y^1 h^l)(X, Z) = 0$$

*b)*  $(M_2, g_2)$  is an Einstein-like manifold of class  $\mathcal{B}$  if and only if

$$(3.24) \quad (\nabla_U^2 h_2^l)(V, W) - (\nabla_V^2 h_2^l)(U, W) = -h_2^l(U, W)V(l) + h_2^l(V, W)U(l)$$

for  $X, Y, Z \in \mathcal{L}(M_1)$  and  $U, V, W \in \mathcal{L}(M_2)$ .

**Class  $\mathcal{P}$ .** If a Riemannian manifold  $(\bar{M}, \bar{g})$  has a parallel Ricci tensor, i.e.,

$$(3.25) \quad (\bar{\nabla}_{\bar{X}} \bar{S})(\bar{Y}, \bar{Z}) = 0$$

for any vector fields  $\bar{X}, \bar{Y}, \bar{Z}$  on  $\bar{M}$ , then  $(\bar{M}, \bar{g})$  are called *Einstein-like manifolds of Class  $\mathcal{P}$* . In this case,  $(\bar{M}, \bar{g})$  are also called *Ricci symmetric* manifolds.

**Theorem 3.5.** *Let  $M$  be an Einstein-like warped-twisted product of class  $\mathcal{P}$ . Then*

*a)*  $(M_1, g_1)$  is an Einstein-like manifold of class  $\mathcal{P}$  if and only if

$$(3.26) \quad \begin{aligned} (\nabla_X^1 h^l)(Y, Z) &= m_2(\nabla_X^1 h_1^k)(Y, Z) \\ &\quad + m_2\{h_1^k(X, Y)Z(k) + h_1^k(X, Z)Y(k) \\ &\quad - g(X, Y)S(\nabla l, Z) - g(X, Z)S(Y, \nabla l), \end{aligned}$$

*b)*  $(M_2, g_2)$  is an Einstein-like manifold of class  $\mathcal{P}$  if and only if

$$(3.27) \quad \begin{aligned} &(\nabla_U^2 h^k)(V, W) + (1 - m_2)(\nabla_U^2 h_2^k)(V, W) \\ &= m_1(\nabla_U^2 h_2^l)(V, W) + 2U(f_1)f_1g_2(V, W)k^\diamond \\ &\quad + g(V, W)U(k^\diamond) - m_2\{h_2^k(U, V)W(k) + h_2^k(U, W)V(k)\} \\ &\quad + m_1\{h_2^l(U, V)W(l) + h_2^l(U, W)V(l) - h_2^l(U, V)W(k)\} \\ &\quad + m_1\{-h_2^k(U, W)V(l) - h_2^k(U, V)W(l) - h_2^l(U, W)V(k)\} \\ &\quad + 2U(k)S(V, W) + V(k)S(U, W) + W(k)S(V, U) \\ &\quad - g(U, V)S(\nabla k, W) - g(U, W)S(V, \nabla k) \end{aligned}$$

for  $X, Y, Z \in \mathcal{L}(M_1)$  and  $U, V, W \in \mathcal{L}(M_2)$ , where  $k^\diamond = \Delta k + \|\nabla k\|^2$ .

*Proof.* Let  $M$  be an Einstein-like warped-twisted product of class  $\mathcal{P}$ .

*a)*  $(M_1, g_1)$  is an Einstein-like manifold of class  $\mathcal{P}$  if and only if

$$(3.28) \quad (\nabla_X^1 S^1)(Y, Z) = 0$$

for any vector fields  $X, Y, Z$  on  $M_1$ . On the other hand, from (3.17), we have

$$(3.29) \quad \begin{aligned} (\nabla_X S)(Y, Z) &= (\nabla_X^1 S^1)(Y, Z) + (\nabla_X^1 h^l)(Y, Z) - m_2(\nabla_X^1 h_1^k)(Y, Z) \\ &\quad - m_2\{h_1^k(X, Y)Z(k) + h_1^k(X, Z)Y(k)\} \\ &\quad + g(X, Y)S(\nabla l, Z) + g(X, Z)S(Y, \nabla l) \end{aligned}$$

for  $X, Y, Z \in \mathcal{L}(M_1)$ . Thus, using (3.25) and (3.28) in (3.29), we obtain (3.26).

**b)**  $(M_2, g_2)$  is an Einstein-like manifold of class  $\mathcal{P}$  if and only if

$$(3.30) \quad (\nabla_U^2 S^2)(V, W) = 0$$

for any vector fields  $U, V, W$  on  $M_2$ . On the other hand, from (3.21), we have

$$(3.31) \quad \begin{aligned} (\nabla_U S)(V, W) &= (\nabla_U^2 S^2)(V, W) + (\nabla_U^2 h^k)(V, W) \\ &\quad + (1 - m_2)(\nabla_U^2 h_2^k)(V, W) - m_1(\nabla_U^2 h_2^l)(V, W) \\ &\quad - 2U(f_1)f_1g_2(V, W)k^\diamond - g(V, W)U(k^\diamond) \\ &\quad + m_2\{h_2^k(U, V)W(k) + h_2^k(U, W)V(k)\} \\ &\quad - m_1\{h_2^l(U, V)W(l) + h_2^l(U, W)V(l) - h_2^l(U, V)W(k)\} \\ &\quad - m_1\{-h_2^k(U, W)V(l) - h_2^k(U, V)W(l) - h_2^l(U, W)V(k)\} \\ &\quad - 2U(k)S(V, W) - V(k)S(U, W) - W(k)S(V, U) \\ &\quad + g(U, V)S(\nabla k, W) + g(U, W)S(V, \nabla k) \end{aligned}$$

for  $U, V, W \in \mathcal{L}(M_2)$ . Thus, using (3.25) and (3.30) in (3.31), we obtain (3.27).  $\square$

For a singly warped product, one can obtain the following result.

**Corollary 3.6.** *Let  $M$  be an Einstein-like singly warped product of class  $\mathcal{P}$ . Then*

**a)**  $(M_1, g_1)$  is an Einstein-like manifold of class  $\mathcal{P}$  if and only if

$$(3.32) \quad (\nabla_X^1 h^l)(Y, Z) = 0,$$

**b)**  $(M_2, g_2)$  is an Einstein-like manifold of class  $\mathcal{P}$  if and only if

$$(3.33) \quad (\nabla_U^2 h_2^l)(V, W) = -h_2^l(U, V)W(l) - h_2^l(U, W)V(l)$$

for  $X, Y, Z \in \mathcal{L}(M_1)$  and  $U, V, W \in \mathcal{L}(M_2)$ .

**Class  $\mathcal{I} \oplus \mathcal{A}$ .** Let  $(\bar{M}, \bar{g})$  be a Riemannian manifold of dimension  $m$  and a tensor  $\mathcal{T}$  is defined by

$$\mathcal{T} = S - \frac{2\tau}{m+2}\bar{g}.$$

If the tensor  $\mathcal{T}$  is Killing, then  $(\bar{M}, \bar{g})$  are called *Einstein-like manifolds of class  $\mathcal{I} \oplus \mathcal{A}$* . This condition is equivalent to

$$(3.34) \quad 0 = (\bar{\nabla}_{\bar{X}} \mathcal{T})(\bar{X}, \bar{X})$$

for any vector field  $\bar{X}$  on  $\bar{M}$ . The inheritance property of Class  $\mathcal{I} \oplus \mathcal{A}$  is given by the following result.

**Theorem 3.7.** *Let  $M$  be an Einstein-like warped-twisted product of class  $\mathcal{I} \oplus \mathcal{A}$ . Then*

*a)  $(M_1, g_1)$  is an Einstein-like manifold of class  $\mathcal{I} \oplus \mathcal{A}$  if and only if*

$$(3.35) \quad \begin{aligned} & (\nabla_X^1 h^l)(X, X) \\ &= m_2(\nabla_X^1 h_1^k)(X, X) \\ & \quad + 2m_2 X(k)h_1^k(X, X) - 2g(X, X)S(\nabla l, X) \\ & \quad + \frac{2}{m+2} \{ \nabla_X \tau g(X, X) - \frac{m+2}{m_1+2} \nabla_X^1 \tau^1 g_1(X, X) \}, \end{aligned}$$

*b)  $(M_2, g_2)$  is an Einstein-like manifold of class  $\mathcal{I} \oplus \mathcal{A}$  if and only if*

$$(3.36) \quad \begin{aligned} & (\nabla_U^2 h^k)(U, U) + (1 - m_2)(\nabla_U^2 h_2^k)(U, U) \\ &= m_1(\nabla_U^2 h_2^l)(U, U) - 2m_2 h_2^k(U, U)U(k) \\ & \quad + 2U(f_1)f_1 g_2(U, U)k^\diamond + g(U, U)U(k^\diamond) \\ & \quad + m_1 \{ 2h_2^l(U, U)U(l) - 2h_2^l(U, U)U(k) - 2h_2^k(U, U)U(l) \} \\ & \quad + 4U(k)S(U, U) - 2g(U, U)S(\nabla k, U) \\ & \quad + \frac{2}{m+2} \{ (\nabla_U \tau)g(U, U) - \frac{m+2}{m_2+2} \nabla_U^2 \tau^2 g_2(U, U) \} \end{aligned}$$

for  $X \in \mathcal{L}(M_1)$  and  $U \in \mathcal{L}(M_2)$ , where  $k^\diamond = \Delta k + \|\nabla k\|^2$ .

*Proof.* Let  $M$  be an Einstein-like warped-twisted product of class  $\mathcal{I} \oplus \mathcal{A}$ .

**a)**  $(M_1, g_1)$  is an Einstein-like manifold of class  $\mathcal{I} \oplus \mathcal{A}$  if and only a tensor  $\mathcal{T}^1 = S^1 - \frac{2\tau^1}{m_1+2}g_1$  is Killing, i.e.,

$$(3.37) \quad 0 = (\nabla_X^1 \mathcal{T}^1)(X, X)$$

for any vector field  $X$  on  $M_1$ . On the other hand, from (3.34), we have

$$\begin{aligned} 0 &= (\nabla_X \mathcal{T})(X, X) \\ &= (\nabla_X S)(X, X) - \frac{2}{m+2} (\nabla_X (\tau g))(X, X) \\ &= (\nabla_X S)(X, X) - \frac{2}{m+2} (\nabla_X \tau)g(X, X). \end{aligned}$$



for all  $X \in \mathcal{L}(M_1)$ . Then, using (3.5), we find

$$\begin{aligned}
 0 &= (\nabla_X^1 S^1)(X, X) + (\nabla_X^1 h^l)(X, X) \\
 &\quad - m_2(\nabla_X^1 h_1^k)(X, X) - 2m_2 X(k)h_1^k(X, X) \\
 &\quad - (\nabla_X^1 g)(X, X)l^\diamond + 2g(X, X)S(\nabla l, X) \\
 (3.38) \quad &\quad - \frac{2}{m+2}(\nabla_X \tau)g(X, X).
 \end{aligned}$$

If we add and subtract  $\nabla_X^1(\frac{2\tau^1}{m_1+2}g_1)(X, X)$  right side of the equation (3.38), we have

$$\begin{aligned}
 0 &= (\nabla_X^1 S^1)(X, X) - \nabla_X^1(\frac{2\tau^1}{m_1+2}g_1)(X, X) \\
 &\quad + (\nabla_X^1 h^l)(X, X) - m_2(\nabla_X^1 h_1^k)(X, X) - 2m_2 X(k)h_1^k(X, X) \\
 &\quad + 2g(X, X)S(\nabla l, X) - \frac{2}{m+2}(\nabla_X \tau)g(X, X) \\
 (3.39) \quad &\quad + \nabla_X^1(\frac{2\tau^1}{m_1+2}g_1)(X, X).
 \end{aligned}$$

So, using (3.39) and (3.37), we obtain (3.35).

**b)**  $(M_2, g_2)$  is an Einstein-like manifold of class  $\mathcal{I} \oplus \mathcal{A}$  if and only a tensor  $\mathcal{T}^2 = S^2 - \frac{2\tau^2}{m_2+2}g_2$  is Killing, i.e.,

$$(3.40) \quad 0 = (\nabla_U^2 \mathcal{T}^2)(U, U)$$

for any vector field  $U$  on  $M_2$ . On the other hand, we have

$$\begin{aligned}
 0 &= (\nabla_U \mathcal{T})(U, U) \\
 &= (\nabla_U S)(U, U) - \frac{2}{m+2}(\nabla_U(\tau g))(U, U) \\
 &= (\nabla_U S)(U, U) - \frac{2}{m+2}(\nabla_U \tau)g(U, U).
 \end{aligned}$$

for all  $U \in \mathcal{L}(M_2)$ , from (3.34). Then, using (3.9), we find

$$\begin{aligned}
 0 &= (\nabla_U^2 S^2)(U, U) + (\nabla_U^2 h^k)(U, U) + (1 - m_2)(\nabla_U^2 h_2^k)(U, U) \\
 &\quad - m_1(\nabla_U^2 h_2^l)(U, U) \\
 &\quad + 2m_2 h_2^k(U, U)U(k) - (\nabla_U^2 g)(U, U)k^\diamond - g(U, U)U(k^\diamond) \\
 &\quad - m_1\{2h_2^l(U, U)U(l) - 2h_2^l(U, U)U(k) - 2h_2^k(U, U)U(l)\} \\
 (3.41) \quad &\quad - 4U(k)S(U, U) + 2g(U, U)S(\nabla k, U) \\
 &\quad - \frac{2}{m+2}(\nabla_U \tau)g(U, U).
 \end{aligned}$$

If we add and subtract  $\nabla_U^2(\frac{2\tau^2}{m_2+2}g_2)(U, U)$  right side of the equation (3.41), we

have

$$\begin{aligned}
 0 &= (\nabla_U^2 S^2)(U, U) - \nabla_U^2 \left( \frac{2\tau^2}{m_2 + 2} g_2 \right)(U, U) \\
 &\quad + 2m_2 h_2^k(U, U)U(k) - 2U(f_1)f_1 g_2(U, U)k^\diamond - g(U, U)U(k^\diamond) \\
 &\quad - m_1 \{ 2h_2^l(U, U)U(l) - 2h_2^l(U, U)U(k) - 2h_2^k(U, U)U(l) \} \\
 &\quad - 4U(k)S(U, U) + 2g(U, U)S(\nabla k, U) \\
 (3.42) \quad &\quad - \frac{2}{m + 2} (\nabla_U \tau)g(U, U) \\
 &\quad + \nabla_U^2 \left( \frac{2\tau^2}{m_2 + 2} g_2 \right)(U, U).
 \end{aligned}$$

So, using (3.42) and (3.40), we obtain (3.36). □

**Class  $\mathcal{A} \oplus \mathcal{B}$ .** Let  $(\bar{M}, \bar{g})$  be a Riemannian manifold of dimension  $m$ . If  $(\bar{M}, \bar{g})$  has a constant scalar curvature, then  $(\bar{M}, \bar{g})$  are called *Einstein-like manifolds of class  $\mathcal{A} \oplus \mathcal{B}$* . Hence, from (2.10), one can prove the following theorem.

**Theorem 3.8.** *Let  $M$  be an Einstein-like warped-twisted product of class  $\mathcal{A} \oplus \mathcal{B}$ . Then,  $(M_1, g_1)$  and  $(M_2, g_2)$  are Einstein-like manifolds of class  $\mathcal{A} \oplus \mathcal{B}$  if and only if*

$$\begin{aligned}
 c &= \frac{c_1}{f_2^2} + \frac{c_2}{f_1^2} + \tilde{\Delta}_1(l) + \tilde{\Delta}_2(k) - \frac{m_2}{f_2^2} \Delta_1(k) - \frac{m_1}{f_1^2} \Delta_2(l) \\
 &\quad + \frac{(1 - m_2)}{f_1^2} \Delta_2(k) - m_2 g(P_1 \nabla k, P_1 \nabla k) - m_1 \Delta l - 2m_1 g(\nabla l, \nabla l) \\
 &\quad - m_2 \left\{ \Delta k + g(\nabla k, \nabla k) \right\} + m_2 g(P_2 \nabla k, P_2 \nabla k) + 2m_1 g(P_2 \nabla k, \nabla l)
 \end{aligned}$$

where  $c, c_1$  and  $c_2$  are constant scalar curvature of  $(M, g), (M_1, g_1)$  and  $(M_2, g_2)$ , respectively.

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