Novi Sad J. Math. Vol. 53, No. 2, 2023, 197-206
https://doi.org/10.30755/NSJOM.13179

New decomposition forms of bioperation-continuity

Carlos Carpintero[1 2], **Neelamegarajan Rajesh**[3],
Ennis Rosas[4] **and Jorge Vielma**[5]

Abstract. In this paper, we introduce some new types of sets via bioperation and obtain new decomposition forms of bioperation-continuity using these sets and finally using the notions of a bioperation some well known concepts of continuity are generalized.

AMS Mathematics Subject Classification (2010): 54A05; 54A10; 54C05; 54D10

Key words and phrases: topological spaces; (γ, γ')-open set

1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologist worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. By utilizing generalized open sets. Kasahara [5] introduced the concept of an operation on topological spaces and the notion of γ-open sets, the collection of all γ-open sets is denoted by τ_γ. Ogata and Maki [11] introduced the notion of $\tau_{\gamma\vee\gamma'}$ which is the collection of all $\gamma \vee \gamma'$-open sets in a topological space (X, τ) and Umehara et al. [13] introduced the notion of $\tau_{(\gamma,\gamma')}$ which is the collection of all (γ, γ')-open sets in a topological space (X, τ) that generalized the notions of $\gamma \vee \gamma'$-open sets in a topological space (X, τ). In this paper, using the bioperation (γ, γ'), we introduce new types of sets and find the relationships between them and obtain a new forms of decomposition of bioperation-continuity. Finally we can see that these new concepts of continuity using these bioperations, generalizes well-known concepts of continuity.

2. Preliminaries

The closure and the interior of a subset A of (X, τ) are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

[1]Departamento de Ciencias Básicas, Corporación Universitaria del Caribe-CECAR, Sincelejo, Colombia, e-mail: carpintero.carlos@gmail.com

[2]Corresponding author

[3]Department of Mathematics, Rajah Serfoji Govt. College, Thanjavur-613005, Tamilnadu, India, e-mail: nrajesh_topology@yahoo.co.in

[4]Universidad de la Costa, Department of Natural Sciences and Exact, Barranquilla, Colombia, e-mail: ennisrafael@gmail.com

[5]Escuela Superior Politécnica del Litoral,Facultad de Ciencias Naturales y Matemáticas, Campus Gustavo Galindo,Km 30.5, Vía perimetral, P.O Box 09-01-5863, Guayaquil, Ecuador, e-mail: jevielma@espol.edu.ec

Definition 2.1. *[5] Let (X, τ) be a topological space. An operation γ on the topology τ is a function from τ on to power set $\mathcal{P}(X)$ of X such that $V \subset V^{\gamma}$ for each $V \in \tau$, where V^{γ} denotes the value of τ at V. It is denoted by $\gamma : \tau \to \mathcal{P}(X)$.*

Definition 2.2. *[11] A topological space (X, τ) equipped with two operations, say, γ and γ' defined on τ is called a bioperation-topological space, it is denoted by $(X, \tau, \gamma, \gamma')$.*

Definition 2.3. *[11] A subset A of a topological space (X, τ) is said to be $\gamma \vee \gamma'$-open set if for each $x \in A$ there exists an open neighborhood U of x such that $U^{\gamma} \cup U^{\gamma'} \subset A$. The complement of $\gamma \vee \gamma'$-open set is called $\gamma \vee \gamma'$-closed. $\tau_{\gamma \vee \gamma'}$ denotes set of all $\gamma \vee \gamma'$-open sets in (X, τ).*

Definition 2.4. *[13] A subset A of a topological space (X, τ) is said to be (γ, γ')-open set if for each $x \in A$ there exist open neighborhoods U and V of x such that $U^{\gamma} \cup V^{\gamma'} \subset A$. The complement of (γ, γ')-open set is called (γ, γ')-closed. $\tau_{(\gamma, \gamma')}$ denotes set of all (γ, γ')-open sets in (X, τ).*

Remark 2.5. *Observe that from Definitions 2.3 and 2.4, each $\gamma \vee \gamma'$-open set is a (γ, γ')-open set, but the converse is not necessarily true as we can see in the following example.*

Example 2.6. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. We define the operations $\gamma, \gamma' : \tau \to \mathcal{P}(X)$ as follows: $\gamma(\{a\}) = \{a\}$, $\gamma(\{b\}) = \{b, c\}$, $\gamma(\{a, b\}) = \{a, b\}$*
$\gamma'(\{a\}) = \{a, b\}$, $\gamma'(\{b\}) = \{b\}$, $\gamma'(\{a, b\}) = X$
Observe that: $\tau_{\gamma \vee \gamma'} = \{\emptyset, X\}$ and $\tau_{(\gamma, \gamma')} = \{\emptyset, X, \{a, b\}\}$

Example 2.7. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. We define the operators $\gamma, \gamma' : \tau \to \mathcal{P}(X)$ by $\gamma(A) = \mathrm{Cl}(A)$ and $\gamma'(A) = \mathrm{Int}(\mathrm{Cl}(A))$ for all $A \in \tau$. Then $\tau_{(\gamma, \gamma')} = \tau_{\gamma \vee \gamma'} = \{\emptyset, X\}$.*

Definition 2.8. *[13] For a subset A of (X, τ), $\mathrm{Cl}_{(\gamma, \gamma')}(A)$ denotes the intersection of all (γ, γ')-closed sets containing A, that is, $\mathrm{Cl}_{(\gamma, \gamma')}(A) = \cap\{F : A \subset F, X \backslash F \in \tau_{(\gamma, \gamma')}\}$.*

Definition 2.9. *Let A be any subset of X. The $\mathrm{Int}_{(\gamma, \gamma')}(A)$ is defined as $\mathrm{Int}_{(\gamma, \gamma')}(A) = \cup\{U : U$ is a (γ, γ')-open set and $U \subset A\}$.*

Definition 2.10. *Let (X, τ) be a topological space and A be a subset of X and γ and γ' be operations on τ. Then A is said to be*

1. *(γ, γ')-α-open if $A \subset \mathrm{Int}_{(\gamma, \gamma')}(\mathrm{Cl}_{(\gamma, \gamma')}(\mathrm{Int}_{(\gamma, \gamma')}(A)))$*
2. *(γ, γ')-preopen [3] if $A \subset \mathrm{Int}_{(\gamma, \gamma')}(\mathrm{Cl}_{(\gamma, \gamma')}(A))$*
3. *(γ, γ')-semiopen [10] if $A \subset \mathrm{Cl}_{(\gamma, \gamma')}(\mathrm{Int}_{(\gamma, \gamma')}(A))$*
4. *(γ, γ')-semipreopen (or (γ, γ')-β-open) if $A \subset \mathrm{Cl}_{(\gamma, \gamma')}(\mathrm{Int}_{(\gamma, \gamma')}(\mathrm{Cl}_{(\gamma, \gamma')}(A)))$*

5. *(γ,γ')-regular open [9] if* $A = \mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(A))$.

Remark 2.11. *The union of all (γ,γ')-semipreopen sets contained in A is called the (γ,γ')-semipreinterior of A and denoted by $sp\,\mathrm{Int}_{(\gamma,\gamma')}(A)$. The complement of a (γ,γ')-semipreopen set is called a (γ,γ')-semipreclosed set. It is clear that $sp\,\mathrm{Int}_{(\gamma,\gamma')}(A) = A \cap \mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(A)))$.*

Remark 2.12. *Observe that if in Definition 2.10, the operations γ and γ' are the identity operations, we obtain well-known concepts studied in general topology such as: α-open set [7], [12], preopen set [8], semiopen set [2], semipreopen set [2].*

Definition 2.13. *Let (X,τ) and (Y,σ) be two topological spaces and let $\gamma,\gamma' : \tau \to \mathcal{P}(X)$ be operations on τ. A mapping $f : (X,\tau) \to (Y,\sigma)$ is said to be (γ,γ')-continuous (resp. (γ,γ')-α-continuous, (γ,γ')-precontinuous, (γ,γ')-semicontinuous, (γ,γ')-semiprecontinuous) if for each $x \in X$ and each open set V of Y containing $f(x)$ there exists a (γ,γ')-open set U containing x (resp. (γ,γ')-α-open set, (γ,γ')-preopen set, (γ,γ')-semiopen set, (γ,γ')-semipreopen set) such that $f(U) \subset V$.*

3. Some subsets in topological spaces

Through this section, let (X,τ) and (Y,σ) be two topological spaces, and let $\gamma,\gamma' : \tau \to \mathcal{P}(X)$ be operations on τ.

Definition 3.1. *A subset A of a topological space (X,τ) with the operations γ, γ' is called:*

1. *$\alpha^\star_{(\gamma,\gamma')}$-set if* $\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(A))) = \mathrm{Int}_{(\gamma,\gamma')}(A)$,
2. *$t_{(\gamma,\gamma')}$-set if* $\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(A)) = \mathrm{Int}_{(\gamma,\gamma')}(A)$,
3. *$s_{(\gamma,\gamma')}$-set if* $\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(A)) = \mathrm{Int}_{(\gamma,\gamma')}(A)$,
4. *$\beta^\star_{(\gamma,\gamma')}$-set if* $\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(A))) = \mathrm{Int}_{(\gamma,\gamma')}(A)$.

Remark 3.2. *Observe that if in Definition 3.1, the operations γ and γ' are the identity operations, we obtain well-known concepts studied in general topology such as β-set [1], t-set and α^*-set.*

Example 3.3. *Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{a,b\}\}$. We define the operations $\gamma,\gamma' : \tau \to \mathcal{P}(X)$ as follows*

$$A^\gamma = \begin{cases} A & \text{if } A = \{a\} \text{ or } \{c\}, \\ A \cup \{a,c\} & \text{if } A \neq \{a\} \text{ and } \{c\} \end{cases}$$

and $A^{\gamma'} = int(\mathrm{Cl}(A))$.

1. $\tau_{(\gamma,\gamma')} = \{\emptyset, X, \{c\}\}$
2. *$\alpha^\star_{(\gamma,\gamma')}$-set* $= \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$.

3. $t_{(\gamma,\gamma')}$-*set* $= \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$.

4. $s_{(\gamma,\gamma')}$-*set* $= \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$.

5. $\beta^{\star}_{(\gamma,\gamma')}$-*set* $= \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$.

Proposition 3.4. *The following are equivalent for a subset A of a space (X,τ) with the operations γ, γ'*

1. *A is a $\alpha^{\star}_{(\gamma,\gamma')}$-set,*

2. *A is a (γ,γ')-semipreclosed set,*

3. $\mathrm{Int}_{(\gamma,\gamma')}(A)$ *is a (γ,γ')-regular open set.*

Proof. Straightforward. □

Proposition 3.5. *Let A be a subset of a space (X,τ) with the operations γ, γ'*

1. *A (γ,γ')-semiopen set A is a $t_{(\gamma,\gamma')}$-set if and only if it is an $\alpha^{\star}_{(\gamma,\gamma')}$-set.*

2. *A is (γ,γ')-α-open and $\alpha^{\star}_{(\gamma,\gamma')}$-set if and only if it is (γ,γ')-regular open.*

Proof. 1. Let A be a (γ,γ')-semiopen and A an $\alpha^{\star}_{(\gamma,\gamma')}$-set. Since A is (γ,γ')-semiopen,

$$\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(A)) = \mathrm{Cl}_{(\gamma,\gamma')}(A)$$

and

$$\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(A)) = \mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(A))) = \mathrm{Int}_{(\gamma,\gamma')}(A).$$

Therefore, A is a $t_{(\gamma,\gamma')}$-set.
2. Let A be a (γ,γ')-α-open set and an $\alpha^{\star}_{(\gamma,\gamma')}$-set. Then $\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(A)) = A$ and hence $\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(A)) = \mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(A))) = A$. The converse is obvious. □

Definition 3.6. *A subset A of a topological space (X,τ) with the operations γ, γ' is called a*
1. $C_{(\gamma,\gamma')}$-set if $A = U \cap V$, where $U \in \tau_{(\gamma,\gamma')}$ and V is an $\alpha^{\star}_{(\gamma,\gamma')}$-set,
2. $B_{(\gamma,\gamma')}$-set if $A = U \cap V$, where $U \in \tau_{\gamma,\gamma'}$ and V is a $t_{(\gamma,\gamma')}$-set,
3. $S_{(\gamma,\gamma')}$-set if $A = U \cap V$, where $U \in \tau_{(\gamma,\gamma')}$ and V is a $s_{(\gamma,\gamma')}$-set,
4. $\beta_{(\gamma,\gamma')}$-set if $A = U \cap V$, where $U \in \tau_{(\gamma,\gamma')}$ and V is a $\beta^{\star}_{(\gamma,\gamma')}$-set,
5. $\beta^{\star\star}$-open set if $sp\,\mathrm{Int}_{(\gamma,\gamma')}(A) = \mathrm{Int}_{(\gamma,\gamma')}(A)$.

Example 3.7. *Observe that in Example 2.6,*

1. $\tau_{(\gamma,\gamma')} = \{\emptyset, X, \{a,b\}\}$

2. $\alpha^{\star}_{(\gamma,\gamma')}$-*set* $= \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,c\}, \{b,c\}\}$.

3. $t_{(\gamma,\gamma')}$-*set* $= \{\emptyset, X, \{c\}\}$.

4. $s_{(\gamma,\gamma')}$*-set* $= \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,c\}, \{b,c\}\}$.
5. $\beta^{\star}_{(\gamma,\gamma')}$*-set* $= \{\emptyset, X, \{c\}\}$.
6. $C_{(\gamma,\gamma')}$*-set* $= \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$.
7. $B_{(\gamma,\gamma')}$*-set* $= \{\emptyset, X, \{c\}, \{a,b\}\}$.
8. $S_{(\gamma,\gamma')}$*-set* $= \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}\}$.
9. $\beta_{(\gamma,\gamma')}$*-set* $= \{\emptyset, X, \{c\}, \{a,b\}\}$.
10. $\beta^{\star\star}$*-open set* $= \{\emptyset, X, \{a,b\}\}$.

Proposition 3.8. *Let (X,τ) be a topological space with the operations γ, γ' and A a subset of X. Then the following hold:*

1. *If A is a $t_{(\gamma,\gamma')}$-set, then A is an $\alpha^{\star}_{(\gamma,\gamma')}$-set,*
2. *If A is a $s_{(\gamma,\gamma')}$-set, then A is an $\alpha^{\star}_{(\gamma,\gamma')}$-set,*
3. *If A is a $\beta^{\star}_{(\gamma,\gamma')}$-set, then A is both a $t_{(\gamma,\gamma')}$-set and a $s_{(\gamma,\gamma')}$-set.*
4. *$t_{(\gamma,\gamma')}$-set and $s_{(\gamma,\gamma')}$-set are independent notions.*

Proof. (1). Let A be a $t_{(\gamma,\gamma')}$-set. Then $\tau_{(\gamma,\gamma')}$-Int($\tau_{(\gamma,\gamma')}$-Cl(A)) $= \tau_{(\gamma,\gamma')}$-Int($A$) $\supset \tau_{(\gamma,\gamma')}$-Int($\tau_{(\gamma,\gamma')}$-Cl($\tau_{\gamma}$-Int($A$))) $\supset \tau_{(\gamma,\gamma')}$-Int($A$) and hence $\tau_{\gamma\vee\gamma'}$-Int($\tau_{\gamma\vee\gamma'}$-Cl($\tau_{\gamma\vee\gamma'}$-Int(A))) $= \tau_{\gamma\vee\gamma'}$-Int($A$). Therefore, A is an $\alpha^{\star}_{\gamma\vee\gamma'}$-set.
(2) and (3) are proved in a similar form as (1).
(4) The following examples shows that the notions of a $t_{(\gamma,\gamma')}$-set and a $s_{(\gamma,\gamma')}$-set are independent. □

Remark 3.9. *The converse of the statement in Proposition 3.8 are not true as seen in the following examples.*

Example 3.10. *Let $X = \{a,b,c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{a,b\}\}$. We define the operations $\gamma, \gamma' : \tau \to \mathcal{P}(X)$ as follows*

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } A = \{a\} \text{ or } \{c\}, \\ A \cup \{a,c\} & \text{if } A \neq \{a\} \text{ and } \{c\}. \end{cases}$$

Then $\tau_{(\gamma,\gamma')} = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}\}$. If we take $A = \{a\}$, then A is an $\alpha^{\star}_{(\gamma,\gamma')}$-set and a $t_{(\gamma,\gamma')}$-set, but it is neither a $s_{(\gamma,\gamma')}$-set and nor a $\beta^{\star}_{(\gamma,\gamma')}$-set.

Example 3.11. *If in Example 2.6 and Example 3.7, we take $A = \{b\}$, then it is an $\alpha^{\star}_{(\gamma,\gamma')}$-set and a $s_{(\gamma,\gamma')}$-set, but it is neither a $t_{(\gamma,\gamma')}$-set and nor a $\beta^{\star}_{(\gamma,\gamma')}$-set.*

Proposition 3.12. *Let (X,τ) be a topological space with the operations γ, γ' and A a subset of X. Then the following hold:*

1. *If A is an $\alpha^{\star}_{(\gamma,\gamma')}$-set, then it is a $C_{(\gamma,\gamma')}$-set,*

2. *If A is a $t_{(\gamma,\gamma')}$-set, then it is a $B_{(\gamma,\gamma')}$-set,*
3. *If A is a $s_{(\gamma,\gamma')}$-set, then it is a $S_{(\gamma,\gamma')}$-set,*
4. *If A is a $\beta^{\star}_{(\gamma,\gamma')}$-set, then it is a $\beta_{(\gamma,\gamma')}$-set.*

Proof. 1. Let A be an $\alpha^{\star}_{(\gamma,\gamma')}$-set. If we take $U = X \in \tau_{(\gamma,\gamma')}$, then $A = U \cap A$ and hence A is a $C_{(\gamma,\gamma')}$-set.
2. Let A be a $t_{(\gamma,\gamma')}$-set. If we take $U = X \in \tau_{(\gamma,\gamma')}$, then $A = U \cap A$ and hence A is a $B_{(\gamma,\gamma')}$-set.
3. Let A be a $s_{(\gamma,\gamma')}$-set. If we take $U = X \in \tau_{(\gamma,\gamma')}$, then $A = U \cap A$ and hence A is a $S_{(\gamma,\gamma')}$-set.
4. Let A be a $\beta^{\star}_{(\gamma,\gamma')}$-set. If we take $U = X \in \tau_{(\gamma,\gamma')}$, then $A = U \cap A$ and hence A is a $\beta_{(\gamma,\gamma')}$-set. □

Remark 3.13. *The converse of the statements in Proposition 3.12 are not true. In Example 3.7, $\{a,b\}$ is a $C_{(\gamma,\gamma')}$-set (resp. $B_{(\gamma,\gamma')}$-set, $S_{(\gamma,\gamma')}$-set, $\beta_{(\gamma,\gamma')}$-set), but it is not an $\alpha^{\star}_{(\gamma,\gamma')}$-set (resp. $t_{(\gamma,\gamma')}$-set, $s_{(\gamma,\gamma')}$-set, $\beta^{\star}_{(\gamma,\gamma')}$-set).*

Proposition 3.14. *Let (X,τ) be a topological space with the operations γ, γ'.*

1. *Every $B_{(\gamma,\gamma')}$-set is a $C_{(\gamma,\gamma')}$-set,*
2. *Every $S_{(\gamma,\gamma')}$-set is a $C_{(\gamma,\gamma')}$-set,*
3. *Every $\beta_{(\gamma,\gamma')}$-set is both a $B_{(\gamma,\gamma')}$-set and a $S_{(\gamma,\gamma')}$-set.*

Proof. The proof follows from Proposition 3.12 and Definition 3.6. □

Remark 3.15. *The converse of the statements in Proposition 3.14 are not true and $B_{(\gamma,\gamma')}$-set and $S_{(\gamma,\gamma')}$-set are independent notions. In Example 3.10, $\{a,b\}$ is a $B_{(\gamma,\gamma')}$-set but it is not a $S_{(\gamma,\gamma')}$-set and not a $\beta_{(\gamma,\gamma')}$-set. In Example 2.7, $\{b\}$ is a $C_{(\gamma,\gamma')}$-set and a $S_{(\gamma,\gamma')}$-set but it is neither a $B_{(\gamma,\gamma')}$-set nor a $\beta_{(\gamma,\gamma')}$-set.*

Remark 3.16. *Observe that if (X,τ) is a topological space with the operations γ, γ'. Then $\beta^{\star\star}$-open and $\beta^{\star}$-set are independent notions. See Example 3.7.*

Remark 3.17. *We have the following implication diagram.*

$$
\begin{array}{ccccccccc}
\beta_{(\gamma,\gamma')}\text{-set} & & \leftarrow & & \beta^{\star}_{(\gamma,\gamma')}\text{-set} & & \rightarrow & & \beta_{(\gamma,\gamma')}\text{-set} \\
\downarrow & & & \swarrow & & \searrow & & & \downarrow \\
S_{(\gamma,\gamma')}\text{-set} & \leftarrow & s_{(\gamma,\gamma')}\text{-set} & & \downarrow & & t_{(\gamma,\gamma')}\text{-set} & \rightarrow & B_{(\gamma,\gamma')}\text{-set} \\
 & \searrow & & \searrow & & \swarrow & & \swarrow & \\
 & & & & \alpha^{\star}_{(\gamma,\gamma')}\text{-set} & & & & \\
 & & & \searrow & \downarrow & \swarrow & & & \\
 & & & & C_{(\gamma,\gamma')}\text{-set} & & & &
\end{array}
$$

Theorem 3.18. *For a subset A of a space (X, τ) with the operations γ, γ', the following properties are equivalent:*

1. *A is (γ, γ')-open,*
2. *A is a (γ, γ')-α-open set and a $C_{(\gamma,\gamma')}$-set,*
3. *A is a (γ, γ')-preopen set and a $B_{(\gamma,\gamma')}$-set,*
4. *A is a (γ, γ')-semiopen set and a $S_{(\gamma,\gamma')}$-set,*
5. *A is a (γ, γ')-semipreopen set and a $\beta_{(\gamma,\gamma')}$-set.*

Proof. The proofs of (1)→(2), (1)→(3), (1)→ (4), (1)→(5) are obvious.
(5)→(1): Let A be a (γ, γ')-semipreopen set and a $\beta_{(\gamma,\gamma')}$-set. Since A is a $\beta_{(\gamma,\gamma')}$-set, $A = U \cap V$, where U is a (γ, γ')-open set and V is a $\beta^{\star}_{(\gamma,\gamma')}$-set. By the hypothesis, A is also (γ, γ')-semipreopen and we have

$$\begin{aligned}
A &\subset \mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(A))) \\
&= \mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(U \cap V))) \\
&\subset \mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(U) \cap \mathrm{Cl}_{(\gamma,\gamma')}(V))) \\
&= \mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(U)) \cap \mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(V))) \\
&\subset \mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(U))) \cap \mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(V))) \\
&\subset \mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(Cl_{(\gamma,\gamma')}(U))) \cap \mathrm{Int}_{(\gamma,\gamma')}(V).
\end{aligned}$$

Hence

$$\begin{aligned}
A &= U \cap V \\
&= (U \cap V) \cap U \\
&\subset (\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(U))) \cap \mathrm{Int}_{(\gamma,\gamma')}(V)) \cap U \\
&= (\mathrm{Cl}_{(\gamma,\gamma')}(\mathrm{Int}_{(\gamma,\gamma')}(\mathrm{Cl}_{(\gamma,\gamma')}(U))) \cap U) \cap \mathrm{Int}_{(\gamma,\gamma')}(V).
\end{aligned}$$

Notice $A = U \cap V \supset U \cap \mathrm{Int}_{(\gamma,\gamma')}(V)$. Hence $A = U \cap \mathrm{Int}_{(\gamma,\gamma')}(V)$. (2)→(1), (3)→(1), (4)→(1) are shown similarly. □

4. Decompositions of (γ, γ')-continuity

Definition 4.1. *Let (X, τ) and (Y, σ) be two topological spaces, γ, γ' operations on τ. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $C_{(\gamma,\gamma')}$-continuous (resp. $B_{(\gamma,\gamma')}$-continuous, $S_{(\gamma,\gamma')}$-continuous, $\beta_{(\gamma,\gamma')}$-continuous) if for each $V \in \sigma$, $f^{-1}(V)$ is a $C_{(\gamma,\gamma')}$-set (resp. $B_{(\gamma,\gamma')}$-set, $S_{(\gamma,\gamma')}$-set, $\beta_{\gamma\vee\gamma'}$-set).*

Remark 4.2. *It is clear that the definition of $C_{(\gamma,\gamma')}$-continuous function (resp. $B_{(\gamma,\gamma')}$-continuous, $S_{(\gamma,\gamma')}$-continuous, $\beta_{(\gamma,\gamma')}$-continuous) generalize the notions of $C_{\gamma\vee\gamma'}$-continuous function (resp. $B_{\gamma\vee\gamma'}$-continuous, $S_{\gamma\vee\gamma'}$-continuous, $\beta_{\gamma\vee\gamma'}$-continuous) defined in [4] and also in the case that the operations γ and γ' are the identity operations, it is easy to see that Definition 4.1, generalizes the well-known concepts of continuity in general topology such as; α-continuous function [12], semi continuous functions [6], precontinuous function [8], β-continuous function [1].*

Proposition 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a function and γ, γ' operations on τ. Then*

1. *Every $B_{(\gamma,\gamma')}$-continuous function is $C_{(\gamma,\gamma')}$-continuous.*
2. *Every $S_{(\gamma,\gamma')}$-continuous function is $C_{(\gamma,\gamma')}$-continuous.*
3. *Every $\beta_{(\gamma,\gamma')}$-continuous is both $B_{(\gamma,\gamma')}$-continuous and $S_{(\gamma,\gamma')}$-continuous.*

Proof. The proof follows from Proposition 3.14. □

Theorem 4.4. *Let (X, τ) and (Y, σ) be two topological spaces and let γ, γ' operations on τ. For a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

1. *f is (γ, γ')-continuous.*
2. *f is (γ, γ')-α-continuous and $C_{(\gamma,\gamma')}$-continuous.*
3. *f is (γ, γ')-precontinuous and $B_{(\gamma,\gamma')}$-continuous.*
4. *f is (γ, γ')-semicontinuous and $S_{(\gamma,\gamma')}$-continuous.*
5. *f is (γ, γ')-semiprecontinuous and $\beta_{(\gamma,\gamma')}$-continuous.*

Proof. The proof follows from Theorem 3.18. □

Remark 4.5. *The notions of (γ, γ')-α-continuity, $C_{(\gamma,\gamma')}$-continuity, (γ, γ')-continuity, $B_{(\gamma,\gamma')}$-continuity, (γ, γ')-semicontinuity, $S_{(\gamma,\gamma')}$-continuity, (γ, γ')-semiprecontinuity and (γ, γ')-continuity are independent of each other as seen in the following examples.*

Example 4.6. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}\}$. We define the operators $\gamma, \gamma' : \tau \to \mathcal{P}(X)$ by*

$$A^{\gamma} = A^{\gamma'} = \begin{cases} A & \text{if } A = \{a\}, \\ A \cup \{a, c\} & \text{if } A \neq \{a\}. \end{cases}$$

Then $\tau_{(\gamma,\gamma')} = \{\emptyset, X, \{a\}, \{a, c\}\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as $f(a) = f(b) = a$, $f(c) = c$. Then f is $C_{(\gamma,\gamma')}$-continuous (resp. $B_{(\gamma,\gamma')}$-continuous, (γ, γ')-semicontinuous and (γ, γ')-semiprecontinuous), but it is not (γ, γ')-α-continuous (resp. $\gamma \vee \gamma'$-precontinuous, $S_{\gamma\vee\gamma'}$-continuous and $\beta_{(\gamma,\gamma')}$-continuous).

Example 4.7. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. We define the operators $\gamma, \gamma' : \tau \to \mathcal{P}(X)$ by $\gamma(A) = \mathrm{Cl}(A)$ and $\gamma'(A) = \mathrm{Int}(\mathrm{Cl}(A))$ for all $A \in \tau$. Then $\tau_{(\gamma,\gamma')} = \{\emptyset, X\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as $f(a) = f(c) = a$, $f(b) = b$. Then f is both $S_{(\gamma,\gamma')}$-continuous and (γ, γ')-precontinuous, but it is neither (γ, γ')-semicontinuous nor $B_{(\gamma,\gamma')}$-continuous.*

Example 4.8. *Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$ and $\sigma = \{\emptyset, Y, \{a\}, \{b\}, \{a, b\}\}$. We define the operations $\gamma, \gamma' : \tau \to \mathcal{P}(X)$ by*

$$A^{\gamma} = A^{\gamma'} = \begin{cases} \mathrm{Int}(\mathrm{Cl}(A)) & \text{if } A = \{a\}, \\ \mathrm{Cl}(A) & \text{if } A \neq \{a\}. \end{cases}$$

Then $\tau_{(\gamma,\gamma')} = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b, d\}, X\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as $f(a) = f(c) = a$, $f(b) = f(d) = b$. Then f is $\beta_{(\gamma,\gamma')}$-continuous, but it is not (γ, γ')-semiprecontinuous.

Example 4.9. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, c\}\}$ and $\sigma = \{\emptyset, X, \{a\}\}$. We define the operations $\gamma, \gamma' : \tau \to \mathcal{P}(X)$ by*

$$A^{\gamma} = A^{\gamma'} = \begin{cases} \mathrm{Int}(\mathrm{Cl}(A)) & \text{if } A = \{a\}, \\ X & \text{if } A \neq \{a\}. \end{cases}$$

Then $\tau_{(\gamma,\gamma')} = \{\emptyset, \{a\}, X\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ as $f(a) = f(c) = a$, $f(b) = b$. Then f is (γ, γ')-α-continuous but it is not $C_{(\gamma,\gamma')}$-continuous.

Acknowledgement

Thanks to the referees for the valuable observations made that improve the content of the article.

References

[1] Abd El-Monsef, M. E., El-Deeb, S. N., and Mahmoud, R. A. β-open sets and β-continuous mapping. *Bull. Fac. Sci. Assiut Univ. A 12*, 1 (1983), 77–90.

[2] Andrijević, D. Semipreopen sets. *Mat. Vesnik 38*, 1 (1986), 24–32.

[3] Carpintero, C., Rajesh, N., and Rosas, E. On a class of (γ, γ')-preopen sets in a topological space. *Fasc. Math.*, 46 (2011), 25–35.

[4] Carpintero, C., Rajesh, N., and Rosas, E. On decomposition of bioperation-continuity. *C. R. Acad. Bulgare Sci. 73*, 4 (2020), 443–450.

[5] Kasahara, S. Operation-compact spaces. *Math. Japon. 24*, 1 (1979/80), 97–105.

[6] Levine, N. Semi-open sets and semi-continuity in topological spaces. *Amer. Math. Monthly 70* (1963), 36–41.

[7] Mashbour, A., Abd-El-Monsef, M., and El-Deep, S. α-continuous and α-open mapping. 213–218.

[8] Mashhour, A. S., Abd El-Monsef, M. E., and El-Deep, S. N. On precontinuous and weak precontinuous mappings. *Proc. Math. Phys. Soc. Egypt*, 53 (1982), 47–53 (1983).

[9] Nirmala, R., and Rajesh, N. Bioperation-regular open sets. *Aryabhata J. Math. Infor. 8(1)* (2016), 53–62.

[10] Nirmala, R., and Rajesh, N. Generalization of semiopen sets via bioperation.

[11] Ogata, H., and Maki, H. Bioperations on topological spaces. *Math. Japon. 38*, 5 (1993), 981–985.

[12] Reilly, I., and M. Vamanamunty, M. On α-continuity in topological spaces. *J. Indian Acad. Math. 18* (1996), 89–99.

[13] Umehara, J.-i., Maki, H., and Noiri, T. Bioperations on topological spaces and some separation axioms. *Mem. Fac. Sci. Kochi Univ. Ser. A Math. 13* (1992), 45–59.

Received by the editors June 13, 2021
First published online January 27, 2022