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A note on Ricci and Yamabe solitons on almost Kenmotsu manifolds

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Abstract. The object of the present paper is to study almost Kenmotsu manifolds admitting Ricci and Yamabe solitons with conformal Reeb foliation. We found that there exist no non-zero parallel 2-form in such a manifold with conformal Reeb foliation. We also study the torque and concurrent vector fields on almost Kenmotsu manifolds. Next we prove certain condition for a vector field to be Killing. Finally, we construct an example to verify some results.

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1. Introduction

The notion of Yamabe flow was introduced by Hamilton at the same time as the Ricci flow, as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on $(M^n, g)(n \geq 3)$ [13]. A time-dependent metric $g(\cdot, t)$ on a Riemannian or pseudo Riemannian manifold M is said to evolve by the Yamabe flow if the metric g satisfies

$$\frac{\partial g(t)}{\partial t} = -\breve{\kappa} g(t), \quad g(0) = g_0, \tag{1.1}$$

on M, where $\breve{\kappa}$ is the scalar curvature corresponds to g.

A Yamabe soliton is a special soliton of the Yamabe flow that moves by one parameter family of diffeomorphisms ϕ_t generated by a fixed vector field V on M [8]. Ye has found that a point-wise elliptic gradient estimate for the Yamabe flow on a locally conformally flat compact Riemannian manifold [29].

The significance of Yamabe flow lies in the fact that it is a natural geometric deformation to metric of constant scalar curvature. One notes that Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation) in mathematical physics. In dimension n=2, the Yamabe flow is equivalent to the Ricci flow (defined by $\frac{\partial}{\partial t} g(t)$=-2$\breve{\alpha}(t)$, where $\breve{\alpha}$ stands for the Ricci tensor). Just as Ricci soliton is a special solution of the Ricci flow,

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a Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphism ϕ_t generated by a fixed vector field V on M , and homotheries, i.e., $g(;t) = \varsigma(t)\phi_*(t)g_0$.

A Riemannian manifold (M, g) is said to be an almost Yamabe soliton $(M, g, V, \check{\lambda})$ if there exists a vector field V on M which satisfies [1]:

$$\frac{1}{2}(\mathfrak{L}_V g) = (\breve{\kappa} - \check{\lambda})g, \tag{1.2}$$

where $\mathfrak{L}_V$ denotes the Lie-derivative of the metric g along the vector field V, $\breve{\kappa}$ stands for the scalar curvature, while $\check{\lambda}$ is a smooth function. Moreover, we say that an almost Yamabe soliton is expanding, steady, or shrinking, if $\check{\lambda} < 0, \check{\lambda} = 0$ or $\check{\lambda} > 0$, respectively. An almost Yamabe soliton is said to be the Yamabe soliton if $\check{\lambda}$ is a constant in Definition (1.2). It is obvious that Einstein manifolds are almost Yamabe solitons. Given a Yamabe soliton, if $V{=}Df$ holds for a smooth function f on M, the equation (1.2) becomes $Hess\ f{=}(\check{r} - \check{\lambda})g$, where $Hess\ f$ denotes the $Hessian$ of f and D denotes the gradient operator of g on M^n. In this case f is called the potential function of the Yamabe soliton and f is said to be a gradient Yamabe soliton.

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called a Ricci soliton if it moves only by a one parameter group of diffeomorphisms and scaling. To be precise, a Ricci soliton on a Riemannian manifold (M, g) is a triple $(g, V, \breve{\beta})$ satisfying [13]

$$(\mathfrak{L}_V g)(X, Y) + 2\breve{Ric}(X, Y) + 2\breve{\beta}g(X, Y) = 0, \tag{1.3}$$

where $\breve{Ric}$ is a Ricci tensor and $\mathfrak{L}_V$ is the Lie-derivative along the vector field V on M and $\breve{\beta} \in \Re$. The Ricci soliton is said to be shrinking, steady and expanding when $\breve{\beta}$ is negative, zero and positive, respectively. Yamabe solitons coincide with Ricci solitons (defined by (1.3)) in dimension $n{=}2$. But for higher dimensions, Ricci solions and Yamabe solitons have different nature.

Moreover, from (1.2) it is clear that for a Yamabe soliton the vector field V is a conformal vector field, that is,

$$\mathfrak{L}_V g = 2\breve{\omega} g, \tag{1.4}$$

where $\breve{\omega}$ is called the conformal coefficient, that is, $\breve{\omega}{=}(\breve{\kappa} - \check{\lambda})$. In particular, if $\breve{\omega}{=}0$, is equivalent to V being Killing.

Before going to our main work, we recall definition which will be used later on.

Definition 1.1. [20] A vector field X on an almost contact Riemannian manifold M is said to be an infinitesimal transformation if there exists a smooth function $\breve{\upsilon}$ on M such that

$$(\mathfrak{L}_X \eta)(Y) = \breve{\upsilon}\eta(Y), \tag{1.5}$$

for every smooth vector field X and Y. If $\breve{\upsilon}{=}0$, then X is called a strict infinitesimal transformation.

Definition 1.2. [22] On a Riemannian, or pseudo-Riemannian manifold (M, g), a nowhere zero vector field $\breve{\tau}$ is called a torqued vector field if it satisfies

$$\nabla_X \breve{\tau} = \varphi X + \breve{\alpha}(X)\breve{\tau}, \quad \breve{\alpha}(\breve{\tau}) = 0, \tag{1.6}$$

where the function φ *is called the torqued function and* 1*-form* $\breve{\alpha}$ *is called the torqued form of* $\breve{\tau}$.

Definition 1.3. [27] A vector field $\breve{\rho}$ on a Riemannian, or pseudo-Riemannian manifold (M, g), is called concurrent, if it satisfies

$$\nabla_X \breve{\rho} = \phi X + \psi(X)\breve{\rho}, \tag{1.7}$$

the 1-form ψ vanishes identically and ϕ=1.

Definition 1.4. [28] A pseudo-Riemannian manifold (M, g), is called an almost quasi-Einstein manifold if

$$Ric = pg + q(\vartheta \otimes \nu + \nu \otimes \vartheta), \tag{1.8}$$

where p, q are smooth functions and ϑ, ν are 1 -forms.

During the last two decades, the geometry of Yamabe flow has been the focus of attention of many mathematicians. In particular, Barbosa et al. [1], Brendle [4], Cho et al. [7], Chow [8], Yang and Zhang [26]. According to Hsu [14], the metric of any compact gradient Yamabe soliton is a metric of constant curvature. Yamabe solitons on a three-dimensional Sasakian manifold were studied by Sharma [22]. A complete classification of Yamabe solitons of non-reductive homogeneous 4-spaces was given by Calvaruso et al.[5]. Wang [24] proved that a three-dimensional Kenmotsu manifolds with a Yamabe soliton is of constant sectional curvature -1 and the soliton is expanding with $\breve{\lambda}$=-6. Yamabe solitons on tree-dimensional $N(k)$-paracontact metric manifold were studied by and Young and Mandal [23] and many more.

The above works motivate us to study almost Kenmotsu manifolds admitting Ricci and Yamabe solitons with conformal Reeb foliation. The outline of the article is as follows: Section 2 is devoted to the basic concept of almost Kenmotsu manifolds with conformal Reeb foliation. In Section 3, we investigate the second order parallel tensor field on such a manifold. Next, we consider the infinitesimal contact transformation in Section 4. In Section 5, we investigate the application of torqued and concurrent vector field on such a manifold. In Section 6, we deduce the certain condition for a vector field to be Killing in such a manifold. Finally, a non-trivial example is given to validate our some results.

2. Almost Kenmotsu manifolds

Geometry of Kenmotsu manifolds was publicized by Kenmotsu [16], such manifolds are known not only as a special case of almost contact metric manifolds [2], but also as analogues of Hermitian manifolds. They were investigated by many authors in the last four decades. Recently, G. Pitis published a book in which many interesting results on such manifolds were collected [20]. Later on they were generalized to almost Kenmotsu manifolds by Janssens and Vanhecke [15]. Since then some authors started to study almost Kenmotsu manifolds under various conditions and many fundamental formulas were obtained ([10],[11],[17],[18]).

On a $(2n+1)$-dimensional smooth differentiable manifold M^{2n+1}, if there exist a triplet (ϕ, ξ, η) satisfying

$$\phi^2 = -id + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{2.1}$$

where id denotes the identity mapping, ϕ a $(1,1)$-type tensor field, ξ a global vector field and η a 1-form, then the triplet is called an almost contact structure and M^{2n+1} is called an almost contact manifold. If in addition there exists a Riemannian metric g on an almost contact manifold $M^{2n+1}(\phi, \xi, \eta, g)$ which is compatible with the almost contact structure, i.e.,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any vector fields X and $Y \in \chi(M)$, then M^{2n+1} is called an almost contact metric manifold, where $\chi(M)$, denotes the Lie algebra of all differentiable vector fields on M^{2n+1}.

Let us consider the Riemannian product $M^{2n+1} \times \Re$ of an almost contact manifold and $\Re$. We define on the product an almost complex structure J by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt}),$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\Re$ and f is a C^∞-function on $M^{2n+1} \times \Re$. If the almost complex structure J is integrable, i.e., the Nijenhuis tensor of J vanishes, then the almost contact structure is said to be normal. By Blair [2], the normality of an almost contact structure is equivalent to $[\phi, \phi]$=-$2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ. The fundamental 2-form Ω of an almost contact metric M^{2n+1} is defined by $\Omega(X, Y)$=$g(X, \phi Y)$ for any vector fields $X, Y \in \chi(M)$. An almost Kenmotsu manifold is defined as an almost contact metric manifold such that η is closed and $d\Omega$=$2\eta \wedge \Omega$.
On an almost Kenmotsu manifold M^{2n+1}, we consider a $(1,1)$-type tensor field h=$\frac{1}{2}\mathfrak{L}_\xi \phi$, $h^{'}$=$h \circ \phi$ and l=$R(\cdot, \xi)\xi$, where R is the curvature tensor of g and l is the Lie derivative operator. Thus, $h, h^{'}$ and l are symmetric and satisfy the following relations ([10],[11]):

$$h\xi = l\xi = 0, \quad tr(h) = tr(h') = 0, \quad h\phi + \phi h = 0, \tag{2.3}$$

$$\nabla \xi = h' + id - \eta \otimes \xi, \phi\, l\phi - l = 2(h^2 - \phi^2), \tag{2.4}$$

$$\nabla_\xi h = -\phi - 2h - \phi h^2 - \phi l, \tag{2.5}$$

$$tr(l) = S(\xi,\xi) = g(Q\xi,\xi) = -2n - trh^2, \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)(Y + h'Y) - \eta(Y)(X + h'X)) + (\nabla_X h')Y - (\nabla_Y h')X, \tag{2.7}$$

for any $X, Y \in \chi(M)$, where ∇ denotes the Levi-Civita connection of g, S the Ricci tensor, Q the Ricci operator with respect to g and tr the trace operator. The $(1,1)$-type symmetric tensor field h'=$h \circ \phi$ is anticommuting with ϕ and $h'\xi$=0. Also it is clear that ([3], [19], [25]):

$$h = 0 \Leftrightarrow h' = 0, \quad h'^2 = (k+1)\phi^2 \Leftrightarrow h^2 = (k+1)\phi^2. \tag{2.8}$$

In the recent years the authors Pastore and Saltarelli [19], Gosh and Majhi [12] studied almost Kenmotsu manifolds with conformal Reeb foliation. In [19], they proved that an almost Kenmotsu manifold satisfying $R(X,\xi)\cdot R = 0$, for any vector filed X, is a Kenmotsu manifold of constant curvature -1. It is well known in the contact case the vanishing of the tensor h=$\frac{1}{2}\mathfrak{L}_\xi \phi$ means that the Reeb vector field is Killing. According to Pastore and Saltarelli [19] for an almost Kenmotsu manifolds h=0 means the Reeb foliation is conformal (in fact homothetic).

We recall the following propositions and Lemma which will be used later on

Proposition 2.1. *[19] Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with conformal Reeb foliation. Then for any vector fields X and Y, one has*

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.9}$$

$$R(X,\xi)\xi = \phi^2 X, \tag{2.10}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{2.11}$$

$$Ric(X,\xi) = -2n\eta(X). \tag{2.12}$$

Proposition 2.2. *[27] In an n-dimensional Riemannian, or pseudo Riemannian, manifold (M^n, g) endowed with a conformal vector field V, we have*

$$(\mathfrak{L}_V Ric)(X,Y) = -(n-1)(\nabla_X D\breve{\omega}, Y) + (\Delta\breve{\omega})g(X,Y),$$

$$(\mathfrak{L}_V \breve{\kappa}) = -2\breve{\omega}\breve{\kappa} + 2(n-1)\Delta\breve{\omega},$$

for any vector fields X and Y, where D denotes the gradient operator and Δ =-$divD$ denotes the Laplacan operator of g.

Lemma 2.3. *[6] If $(g, V, \breve{\beta})$ is a Ricci soliton of a Riemannian manifold then we have*

$$\frac{1}{2}\|\mathfrak{L}_V g\|^2 = V(\breve{scal}) + 2div\,(\breve{\beta}V - QV)$$

where $\breve{scal}$ is the scalar curvature.

3. Second order parallel tensor fields

We consider a parallel symmetric $(0,2)$-tensor field $\breve{T}$ on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$ with conformal Reeb foliation. Then from $\nabla\breve{T}$=0, we get

$$\breve{T}(R(X,Y)Z,V)+\breve{T}(Z,R(X,Y)V)=0, \tag{3.1}$$

where X, Y, Z and V are arbitrary vector fields on M^{2n+1}.
Since $\breve{T}$ is symmetric, when we fix X=Z=V=ξ in (3.1), we have

$$\breve{T}(\xi,R(\xi,Y)\xi)=0. \tag{3.2}$$

Take a non-empty connected open subset U of M^{2n+1}. Using (2.11) in (3.2), we obtain

$$\breve{T}(\xi,Y)-\eta(Y)\breve{T}(\xi,\xi)=0. \tag{3.3}$$

Differentiating (3.3) covariantly along U and using (2.4), it yields

$$\breve{T}(U,Y)=\breve{T}(\xi,\xi)g(U,Y). \tag{3.4}$$

This implies that $\breve{T}$ and g are parallel tensor fields, with $\breve{\varepsilon}$=$\breve{T}(\xi,\xi)$ is a constant on U. With the help of parallelity property of $\breve{T}$ and g, it is obvious that $\breve{T}$=$\varepsilon\, g$ on M^{2n+1}.
This leads to the following result:

Theorem 3.1. *A parallel symmetric $(0,2)$-tensor field $\breve{T}$ in an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$ with conformal Reeb foliation is a constant multiple of the associated metric tensor.*

Using Theorem 3.1, we can state the following:

Corollary 3.2. *If the Ricci tensor field of an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$ with conformal Reeb foliation is parallel, then it is an Einstein manifold.*

Corollary 3.3. *Let $(g,V,\breve{\beta})$ be a Ricci soliton on almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$ with conformal Reeb foliation be parallel. Then V is solenoidal if and only if it is expanding or shrinking as $\breve{\beta}<0$ or $\breve{\beta}>0$, respectively.*

We suppose that $\breve{T}$ is a parallel 2-form on M^{2n+1}, that is, $\breve{T}(X,Y)$=-$\breve{T}(Y,X)$ and $\nabla\breve{T}$=0. Then it follows that

$$\mathrm{T}(\xi,\xi)=0. \tag{3.5}$$

Taking covariant derivative of (3.5) and using (2.4), we obtain $\breve{T}(X,Y)$=0. This implies that $\breve{T}$=0 on U, where U is a non-empty open subset of M^{2n+1}. On the other hand $\breve{T}$ is parallel on U. Thus we can state the following:

Theorem 3.4. *In an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$ with conformal Reeb foliation, there exists no non-zero parallel 2-form.*

4. Infinitesimal contact transformation

In this section we study transformations which transform an almost Kenmotsu structure (ϕ, ξ, η, g) with conformal Reeb foliation into another almost Kenmotsu structure $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$. We denote this transformation by placing 'bar' on the geometric object which are transformed by the transformation ψ [21].

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with conformal Reeb foliation, the infinitesimal contact transformation leaves the $(0,2)$-tensor field $\breve{T}$ invariant. Then

$$(\mathfrak{L}_V\breve{T})(X,Y) = 0, \tag{4.1}$$

Putting $Y{=}\xi$ in (4.1), we get

$$(\mathfrak{L}_V\breve{T})(X,\xi) = 0. \tag{4.2}$$

On the other hand, we have

$$(\mathfrak{L}_V\breve{T})(X,\xi) = \mathfrak{L}_V(\breve{T}(X,\xi)) - \breve{T}(\mathfrak{L}_V X,\xi) - \breve{T}(X,\mathfrak{L}_V\xi)$$

Using (1.5),(3.3) and (4.2), the above equation reduces to

$$\breve{T}(X,\mathfrak{L}_V\xi) = -\breve{\upsilon}\breve{T}(\xi,\xi)\,\eta(X), \tag{4.3}$$

Replacing $X{=}\xi$ in (4.3), we have

$$\eta(\mathfrak{L}_V\xi) = \breve{\upsilon}. \tag{4.4}$$

Again from (1.5), we obtain

$$(\mathfrak{L}_V\eta)\xi = \breve{\upsilon}, \tag{4.5}$$

which implies that

$$\mathfrak{L}_V(\eta(\xi)) - \eta(\mathfrak{L}_V\xi) = \breve{\upsilon}. \tag{4.6}$$

In view of (4.5) and (4.6), we get $\breve{\upsilon}{=}0$. Hence we can state the following:

Theorem 4.1. *In an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation, the infinitesimal contact transformation which leaves the $(0,2)$-tensor field $\breve{T}$ invariant is an infinitesimal strict contact transformation.*

Corollary 4.2. *In an almost Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ with conformal Reeb foliation bearing Ricci soliton, the infinitesimal contact transformation which leaves the Ricci tensor field invariant is an infinitesimal strict contact transformation, or the soliton is always shrinking.*

Using Proposition 2.2 and the definition of the infinitesimal contact transformation, we have the following corollary.

Corollary 4.3. *In an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation, the infinitesimal contact transformation which leaves the Ricci tensor field is invariant then the conformal vector field V satisfies*

$$2n(\nabla_X D\breve{\kappa}, Y) - (\Delta\breve{\kappa})g(X,Y) = 0$$

$$\breve{\omega}\breve{\kappa} = 2n\Delta\breve{\kappa},$$

for any vector fields X and Y, where D denotes the gradient operator and Δ=-$divD$ denotes the Laplace operator of g.

5. Application of torqued and concurrent vector fields

In this section we discuss the application of torque and concurrent vector fields, that is, the potential vector field V is a torqued vector field $\breve{\tau}$. Then

$$\begin{aligned}(\mathfrak{L}_{\breve{\tau}}g)(X,Y) &= g(\nabla_X\breve{\tau}, Y) + g(X, \nabla_Y\breve{\tau}),\\ &= \breve{\alpha}(X)g(\breve{\tau}, Y) + \breve{\alpha}(Y)g(X, \breve{\tau}).\end{aligned} \tag{5.1}$$

for any vector fields X, Y tangent to M.

In view of (1.3) and (5.1), we get

$$\breve{Ric} = -\breve{\beta}\, g - \frac{1}{2}\{\, \breve{\alpha}\otimes\gamma + \gamma\otimes\breve{\alpha}\,\}, \tag{5.2}$$

where the dual of the 1-form $\breve{\tau}$ is denoted by γ. Thus the manifold is almost quasi-Einstein manifold. So we state the following result.

Theorem 5.1. *If the potential vector field V of a Ricci soliton $(g, V, \breve{\beta})$ is a torque vector field $\breve{\tau}$ in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation then the manifold is an almost quasi-Einstein manifold.*

Again let $(g, V, \breve{\beta})$ be a Ricci soliton on $(M^{2n+1}, \phi, \xi, \eta, g)$, with torque vector field $\breve{\tau}$. Let $\breve{A}$ be any vertical vector field. Then from (5.2), we obtain

$$\breve{Ric}(\breve{\tau}, \breve{A}) = -\frac{1}{2}\left\{\, \breve{\alpha}(\breve{\tau})g(\breve{\tau}, \breve{A}) + \breve{\alpha}(\breve{A})g(\breve{\tau}, \breve{\tau})\,\right\}. \tag{5.3}$$

Since $\breve{\alpha}(\breve{\tau}) = 0$, we have

$$\breve{Ric}(\breve{\tau}, \breve{A}) = -\frac{1}{2}\breve{\alpha}(\breve{A})g(\breve{\tau}, \breve{\tau}). \tag{5.4}$$

Also from (2.12), we obtain

$$\breve{Ric}(\breve{\tau}, \breve{A}) = -2ng(\breve{\tau}, \breve{A}). \tag{5.5}$$

In view of (5.4) and (5.5), $\breve{\tau}$ is nowhere zero and therefore $g(\breve{\tau}, \breve{\tau}) \neq 0$, which gives $\breve{\alpha}(\breve{A})$=0. Any vector field $\breve{A}$ orthogonal to $\breve{\tau}$ means $\breve{\alpha}$=0. Hence, the potential field $\breve{\tau}$ is a concircular vector field. Conversely, let $(g, V, \breve{\beta})$ be a Ricci soliton with concircular vector field $\breve{\tau}$. Then one can easily prove that the manifold is Einstein. Thus we can state the following result.

Theorem 5.2. *An almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation bearing Ricci soliton $(g, \xi, \breve{\beta})$ with torqued potential vector field $\breve{\tau}$ is an Einstein manifold if and only if $\breve{\tau}$ is a concircular vector field.*

It is well known that ∇g=0 and $\breve{\beta}$ given by (1.3) is constant, so $\nabla(\breve{\beta})g$=0. Which means, $\mathfrak{L}_\xi g+2\breve{Ric}$ is parallel. Next, $\mathfrak{L}_\xi g+2\breve{Ric}$=$\Phi$ is a constant multiple of the metric tensor g, that is,

$$(\mathfrak{L}_\xi g + 2\breve{Ric})(X, Y) = \Phi(X, Y) = \Phi(\xi, \xi)g(X, Y),$$

where $\Phi(\xi, \xi)$ is given by

$$(\mathfrak{L}_\xi g + 2\breve{Ric})(\xi, \xi) = \Phi(\xi, \xi) = -4n. \tag{5.6}$$

In view of (5.6), equation (1.3) takes the form

$$(\mathfrak{L}_\xi g) + 2\breve{Ric} + 2\breve{\beta}g = (-4n + 2\breve{\beta})g. \tag{5.7}$$

This implies that $\breve{\beta}$=$2n$. Then we have the following:

Theorem 5.3. *If the symmetric tensor $\mathfrak{L}_\xi g + 2\breve{Ric}$=$\Phi$ is parallel in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation, then the soliton $(g, V, \breve{\beta})$ is always expanding*

Also from (1.2) and (5.1), we have

$$2(\breve{\kappa} - \breve{\lambda})g(X, Y) = \{\breve{\alpha}(X)g(\breve{\tau}, Y) + \breve{\alpha}(Y)g(X, \breve{\tau})\}. \tag{5.8}$$

Since $\breve{\alpha}(\breve{\tau})$=0 and $\breve{\tau}$ is a torqued vector field, then $g(\breve{\tau}, \breve{\tau}) \neq 0$. This implies that $\breve{\lambda}$=$\breve{\kappa}$. Using (1.2) we get $\mathfrak{L}_V g$=0, thus V is a Killing vector field. Since $\breve{\lambda}$ is constant, so $\breve{\kappa}$ is also constant. Thus we can state the following:

Theorem 5.4. *If the potential vector field V of a Yamabe soliton $(g, V, \breve{\lambda})$ is a torque vector field $\breve{\tau}$ in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation then the manifold is a space of constant curvature, or the flow vector field V is Killing.*

Corollary 5.5. *If the metric of an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation is a Yamabe soliton, then it is of constant curvature, or the flow vector field ξ is Killing.*

Corollary 5.6. *Let $(g, V, \breve{\lambda})$ be a Yamabe soliton in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation, then V is solenoidal if and only if the manifold is a space of constant curvature.*

Also, from (1.2), we get $\mathfrak{L}_V g$=0, thus V is Killing. Differentiating covariantly along an arbitrary vector field X, we have $\nabla_X \mathfrak{L}_V g$=0. Now, we consider the identity [27]:

$$(\nabla_X \mathfrak{L}_V g)(U, W) = g((\mathfrak{L}_V \nabla)(X, Y), W) + g((\mathfrak{L}_V \nabla)(X, W), Y), \tag{5.9}$$

which is equivalent to

$$(\mathfrak{L}_V \nabla_X g - \nabla_X \mathfrak{L}_V g - \nabla_{[V,X]} g)(U,W) = -g((\mathfrak{L}_V \nabla)(X,Y),W) - g(((\mathfrak{L}_V \nabla)(X,W),Y).$$

This implies that

$$g((\mathfrak{L}_V \nabla)(W,X),U) + g((\mathfrak{L}_V \nabla)(W,U),X) = 0. \tag{5.10}$$

With the help of (5.9), (5.10) and the skew-symmetric property of ϕ, we get $(\mathfrak{L}_V \nabla)(U,W)$=0, which implies that $(\mathfrak{L}_V \nabla)(\xi,\xi)$=0. Also, by the geodesic properties of ξ, we have

$$(\mathfrak{L}_V \nabla)(X,U) = -\nabla_X \nabla_U V - \nabla_{\nabla_X U} V + R(V,X)U,$$

which yields $\nabla_\xi \nabla_\xi V + R(V,\xi)\xi = 0$. Therefore V is Jacobi along the direction of ξ. Thus, it leads to the following:

Theorem 5.7. *If the potential vector field V of a Yamabe soliton $(g, V, \breve{\lambda})$ is a torque vector field $\breve{\tau}$ in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation then the flow vector field is Jacobi along the direction of ξ.*

Let V be a pointwise collinear vector field with the structure vector field ξ, that is, V=$c\xi$, where c is a smooth function on $(M^{2n+1}, \phi, \xi, \eta, g)$. Then from (1.2), we obtain

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0, \tag{5.11}$$

for any vector fields $X, Y \in \chi(M)$.

Putting V=$c\xi$ in (5.11) and using (2.4), we have

$$X(c)\eta(Y) + 2cg(X,Y) - 2c\eta(X)\eta(Y) + Y(c)\eta(X) = 0, \tag{5.12}$$

For fix Y=ξ in (5.12), we get

$$X(c) + \xi(c)\eta(X) = 0, \tag{5.13}$$

Again taking X=ξ in (5.13), it yields $\xi(c)$=0, using this fact in (5.13) we obtain $X(c)$=0, this means $d(c)$=0, that is, c is constant.
Therefore we have the following:

Theorem 5.8. *If an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation admits a Yamabe soliton and V is a pointwise collinear vector field with the structure vector field ξ, then V is a constant multiple of ξ.*

With the help of (5.12), we have f=0, for X and Y belonging to the contact distribution. Thus we have following result.

Corollary 5.9. *If an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation admits a Yamabe soliton for a vector field V and a constant $\breve{\lambda}$, then V does not have to be pointwise collinear with ξ.*

In particular, if we consider the potential vector field V which is a concurrent vector field, then we have

$$(\mathfrak{L}_V g)(X,Y) = 2g(X,Y), \tag{5.14}$$

for any vector fields X, Y tangent to M.

In view of (1.2) and (5.14), we get $\breve{\lambda}$=$\breve{\kappa}$-1 which implies that Yamabe soliton is expanding, steady, or shrinking, respectively, if $\breve{\kappa} \leq 0$, $\breve{\kappa}$=1 or $\breve{\kappa} > 1$. Thus we state the following:

Theorem 5.10. *If the potential vector field V of a Yamabe soliton $(g, V, \breve{\lambda})$ is a concurrent vector field in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation then the Yamabe soliton will be expanding, steady, or shrinking, according to $\breve{\kappa} \leq 0$, $\breve{\kappa} = 1$ or $\breve{\kappa} > 1$.*

Corollary 5.11. *If the potential vector field V is a concurrent vector field in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation then it admits a Yamabe soliton of type $\left(g, V, \breve{\lambda} = \frac{1}{2} - \breve{\kappa}\right)$.*

Also from (1.3) and (5.14), we get $\breve{Ric}(X,Y)$=-$(1+\breve{\beta})g(X,Y)$, that is, the manifold is Einstein. With the help of (2.11), we obtain $\breve{\beta}$=$(2n-1)$. Thus we have the result.

Theorem 5.12. *If the potential vector field V of a Ricci soliton $(g, V, \breve{\beta})$ is a concurrent vector field in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation then the manifold is Einstein and it is always expanding.*

Corollary 5.13. *If the potential vector field V of a Ricci soliton $(g, V, \breve{\beta})$ is a concurrent vector field in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation then it admits a Yamabe soliton which will be expanding, steady, or shrinking, according to $\breve{\kappa} \leq 0$, $\breve{\kappa}$=1 or $\breve{\kappa} > 1$.*

Also from (2.5), we have

$$(\mathfrak{L}_\xi g)(X,Y) = 2g(\phi X, \phi Y). \tag{5.15}$$

Using (1.3) with V=ξ we get

$$\breve{Ric}(X,Y) = -(1+\breve{\beta})g(X,Y) + \eta(X)\eta(Y). \tag{5.16}$$

$$\breve{Q}X = -(1+\breve{\beta})X + \eta(X)\xi, \tag{5.17}$$

$$\breve{Q}\xi = -\breve{\beta}\xi, \tag{5.18}$$

$$\breve{scal} = -2n\,(1+\breve{\beta}). \tag{5.19}$$

Using (5.18) and (5.19) in Lemma 2.3, we find

$$\frac{1}{2}\left\|\mathfrak{L}_\xi g\right\|^2 = -2n\xi(\breve{\beta}) + 4div\,(\breve{\beta}\xi). \tag{5.20}$$

Since $(\mathfrak{L}_\xi g)$=$2g(\phi X, \phi Y)$ and $\breve{\beta}$=$2n$ is constant followed by comparing (2.12) with (5.18). Hence from (5.20) it is clear that ξ is Killing if the vector field $\breve{\beta}\xi$ will be solenoidal. Moreover, $\breve{Ric}(X,Y) \neq 0$. Thus we have the following result.

Theorem 5.14. *If an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation admits contact Ricci soliton, M^{2n+1} is neither locally isometric to the product of a line, nor a Calabi-Yau manifold.*

6. The sufficient condition for a vector field to be Killing

In this section we deduce the condition, for a vector field is to be Killing in an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation under Yamabe soliton. Then from (1.4), we have

$$(\mathfrak{L}_V g)(X,Y) = 2\breve{\omega} g(X,Y) \tag{6.1}$$

In view of (2.4) we get $\nabla_\xi \xi$=0. So the integral curves are geodesic.

Now, taking X=Y=ξ, (6.1), it follows that

$$(\mathfrak{L}_V g)(\xi,\xi) = 2\breve{\omega}.$$

On the other hand, we have

$$(\mathfrak{L}_V g)(\xi,\xi) = 2g(\nabla_\xi V, \xi), \quad \text{and} \quad 2\nabla_\xi(g(V,\xi)) = 2g(\nabla_\xi V, \xi),$$

Then from above we get

$$2\breve{\omega} = (\mathfrak{L}_V g)(\xi,\xi) = 2g(\nabla_\xi V,\xi) = 2g(\nabla_\xi V,\xi). \tag{6.2}$$

If V is orthogonal to ξ, then $\breve{\omega}$=0. Therefore $\breve{\lambda}$=$\breve{\kappa}$ and hence $(\mathfrak{L}_V g)(X,Y)$=0; that is, V is a Killing vector field. This leads to the following:

Theorem 6.1. *If the potential vector field V of a Yamabe soliton $(g, V, \breve{\lambda})$ on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ with conformal Reeb foliation is orthogonal to ξ the manifold is a space of constant curvature, or the flow vector field V is Killing.*

In addition, it is well known that in a Riemannian manifold

$$g(R(X,Y)Z,U) + g(R(X,Y)U,Z) = 0, \tag{6.3}$$

for all vector fields X, Y, Z, U.

Let the vector field V on M^{2n+1} with conformal Reeb foliation such that $\mathfrak{L}_V R$=0, then from (6.3), we obtain

$$(\mathfrak{L}_V g)(R(X,Y)Z,U) + (\mathfrak{L}_V g)(R(X,Y)U,Z) = 0, \tag{6.4}$$

Putting X=Z=U=ξ, in (6.4), using (2.10), we get

$$(\mathfrak{L}_V g)(X,\xi) = \eta(X)(\mathfrak{L}_V g)(\xi,\xi). \tag{6.5}$$

Again taking X=Z=ξ, in (6.4), we have

$$(\mathfrak{L}_V g) = (\mathfrak{L}_V g)(\xi,\xi)g. \tag{6.6}$$

From (2.12), we have $Ric(\xi,\xi)$=-$2n$, then $\mathfrak{L}_V R$=0. It implies $\mathfrak{L}_V Ric$=0, thus we get $Ric(\mathfrak{L}_V\xi,\xi)$=0, but $Ric(\xi,\xi)$=-$2n$, then $\mathfrak{L}_V\xi$=0. Since $g(\mathfrak{L}_V\xi,\xi)$=0. Therefore $(\mathfrak{L}_V g)(\xi,\xi)$=0. So, in view of (6.2) we get $\breve{\omega}$=0, that is, the vector field V is Killing vector field. In this way it leads the following result.

Theorem 6.2. *If a vector field V on an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$ with conformal Reeb foliation is conformal to a Yamabe soliton, leaves the curvature tensor R invariant, the manifold is a space of constant curvature, or the flow vector field V is Killing.*

7. An Example

We recall some basic theorems as follows. After that, we verify our results by taking an example.

Theorem 7.1. *[11] Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold. Suppose that the characteristic vector field ξ belongs to the (k,μ)-nullity distribution. Then k=-1;h=0, and M^{2n+1} is locally a warped product of an open interval and an almost Kähler manifold.*

Keeping in mind Theorem 7.1, Deshmukh et al. [9], prove that

Theorem 7.2. *[9] Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with ξ belonging to the $((k,\mu)$-nullity distribution. Then M^{2n+1} is Ricci semisymmetric if and only if the manifold is an Einstein manifold.*

Also, Ghos et al. [12] mentioned that

Theorem 7.3. *[12] Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an almost Kenmotsu manifold with conformal Reeb foliation then the manifold is Ricci semisymmetric if and only if the manifold is an Einstein one*

We consider the three-dimensional manifold $M = \{(x,y,z) \in \Re^3 : (x,y,z) \neq 0\}$, where (x,y,z) are standard coordinates of $\Re^3$. The vector fields

$$e_1 = e^{-z}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \quad e_2 = e^{-z}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right), \quad e_3 = \frac{\partial}{\partial z},$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form such that $\eta(X) = g(X, e_3)$, for any $X \in \Gamma(TM)$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_3) = 1, \phi^2(X) = -X + \eta(X)e_3, g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(TM)$.

The ∇ be the Levi-Civita connection with respect to metric tensor g. Then we get

$$[e_1, e_2] = 0, [e_2, e_3] = e_2, [e_1, e_3] = e_1.$$

Also,

$$(\mathfrak{L}_\xi \phi)(e_1) = 0.$$

Thus $he_1 = \frac{1}{2}(\mathfrak{L}_\xi \phi)(e_1) = 0$. Similarly, we have $he_2 = he_3 = 0$, which implies that $h'e_1 = h'e_2 = h'e_3 = 0$.

Using Koszul's formula for the metric tensor g, we get the following

$$\begin{cases} \nabla_{e_2} e_3 = e_2, & \nabla_{e_3} e_3 = 0, \quad \nabla_{e_2} e_2 = -e_3, \\ \nabla_{e_1} e_1 = -e_3, & \nabla_{e_3} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1, \\ \nabla_{e_1} e_2 = 0, & \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_1 = 0. \end{cases}$$

From the above relations, we have

$$\nabla_X \xi = -\phi^2 X + h'X,$$

for any $X \in \Gamma(TM)$. Therefore, the structure (ϕ, η, ξ, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Omega = 2\eta \wedge \Omega$, so that M is an almost Kenmotsu manifold with conformal Reeb foliation.

The components of the curvature tensor R as follows:

$$\begin{cases} R(e_1, e_2) e_3 = 0, & R(e_1, e_3) e_2 = 0, \quad R(e_1, e_3) e_3 = -e_1, \\ R(e_2, e_3) e_1 = 0, & R(e_1, e_2) e_1 = e_2, \quad R(e_1, e_2) e_2 = -e_1, \\ R(e_3, e_1) e_1 = -e_3, & R(e_2, e_3) e_3 = -e_2, \quad R(e_3, e_2) e_2 = -e_3. \end{cases}$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belonging to the (-1)-nullity distribution, Thus Theorem 7.1 is verified.

For the Ricci tensor Ric, we have

$$Ric(e_1, e_1) = -2 \quad Ric(e_2, e_2) = -2 \quad Ric(e_3, e_3) = -2.$$

Also,

$$scal = \sum_{i=1}^{3} g(e_i e_i) S(e_i, e_i) = -6$$

This implies that the manifold is Ricci semisymmetric.

Let $\{e_1, e_2, e_3\}$ be a basis of the tangent space at any point. For any vector $X, Y \in \chi(M^{2n+1})$, we have

$$X = a_1 e_1 + b_1 e_2 + c_1 e_3, \;\; Y = a_2 e_1 + b_2 e_2 + c_2 e_3,$$

where $a_i, b_i, c_i \in \Re \backslash \{0\}$ such that $a_1 a_2 + b_1 b_2 = 0, \quad c_1 c_2 \neq 0$ for all $i = 1, 2, 3$. Thus

$g(X,Y)$=$\mathrm{a_1 a_2 + b_1 b_2 + c_1 c_2}$, and $Ric(X,Y)$=-2$\{\mathrm{a_1 a_2 + b_1 b_2 + c_1 c_2}\}$.

Then we obtain $Ric(X,Y)$=$-2\mathrm{g(X,Y)}$, therefore we notice that M is an Einstein manifold. Hence Theorem 7.2 and Theorem 7.3 are also hold.

Now, we consider $\mathfrak{L}_\xi g + 2\breve{Ric}$ is parallel, so we have

$$(\mathfrak{L}_\xi g + 2\breve{Ric})(e_i, e_i) = \Phi(e_i, e_i) = \Phi(e_3, e_3) g(X, Y).$$

The value of $\Phi(e_3, e_3)$ is given by

$$\Phi(e_3, e_3) = (\mathfrak{L}_\xi g + 2\breve{Ric})(e_3, e_3) = -4.$$

From (1.3), we obtain

$$(\mathfrak{L}_\xi g) + 2\breve{Ric} + 2\breve{\beta} g = (-4 + 2\breve{\beta}) g.$$

It implies that for $\breve{\beta}$=2, the Ricci soliton is always expanding. Therefore Theorem 5.3 holds.

Again, we also have

$$\begin{aligned} (\mathfrak{L}_\xi g)(X,Y) &= 2g(\phi X, \phi Y) \\ &= 2(a_1 a_2 + b_1 b_2). \end{aligned}$$

In view of (1.2), we yield

$$\begin{aligned} (\mathfrak{L}_V g)(X,Y) &= 2(\breve{\kappa} - \breve{\lambda}) g(X,Y) \\ &= 2(\breve{\kappa} - \breve{\lambda})(a_1 a_2 + b_1 b_2 + c_1 c_2). \end{aligned}$$

If we fix V=ξ, equation (1.2) holds only if $a_1 a_2 + b_1 b_2 = 0, \quad c_1 c_2 \neq 0$. Then we get $\breve{\lambda}$=$\breve{\kappa}$, this implies that the manifold is a space of constant curvature and e_3=ξ, is Killing, which satisfies Theorem 5.4

Finally, we suppose that the potential vector field V is a concurrent for Yamabe soliton in such a manifold. Then we have

$$\begin{aligned} (\mathfrak{L}_V g)(X,Y) &= 2g(X,Y) \\ &= 2(a_1 a_2 + b_1 b_2 + c_1 c_2). \end{aligned}$$

Again, from (1.2), it follows that

$$\begin{aligned}(\mathfrak{L}_V g)(X,Y) &= 2(\breve{\kappa}-\breve{\lambda})g(X,Y)\\ &= 2(\breve{\kappa}-\breve{\lambda})(a_1a_2+b_1b_2+c_1c_2).\end{aligned}$$

According to above equation, we obtain $\breve{\lambda} = \breve{\kappa} - 1$. Thus Yamabe soliton is expanding, steady, or shrinking, according to $\breve{\kappa} \leq 0$, $\breve{\kappa} = 1$ or $\breve{\kappa} > 1$, therefore Theorem 5.10 is satisfied.

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