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Fixed point on complex partial b-metric spaces with application to a system of Urysohn type integral equations

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Abstract. Sufficient conditions for existence of common fixed point on complex partial b-metric spaces are obtained. Our results generalize and extend several well-known results. In the end we explore applications of our key results to solve a system of Urysohn type integral equations.

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1. Introduction

Backhtin [3] and Czerwik [5], presented the concept of b-metric spaces and provided a framework to extend the results already known in the classical setting of metric spaces. Azam, Fisher, and Khan [2] gave the notion of complex valued metric spaces and proved some common fixed point theorems under the contraction condition. Rao, Swamy, and Prasad [10] introduced the definition of complex valued b-metric space, and a scheme to extend the results in this setting, as well as proving the common fixed point theorem under contraction conditions. Dhivya and Marudai [7] introduced the concept of complex partial metric space and suggested a plan to extend the results to this setting, as well as obtained common fixed point theorems under the rational contraction condition. Afterward several other researchers have introduced and studied intriguing concepts in metric spaces and their applications [[1], [4], [6], [8],[11]]. Recently Gunaseelan [9] further extended and introduced the concept of complex partial b-metric space and proved the existence of fixed point of contractive mappings. In this paper, we prove some common fixed point theorems on complex partial b-metric spaces under rational type weakly increasing mappings with application to solve a system of Urysohn type integral equations.

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2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $\omega_1, \omega_2, \omega_3 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:
$\omega_1 \preceq \omega_2$ if and only if $\mathcal{R}(\omega_1) \leq \mathcal{R}(\omega_2)$, $\mathcal{I}(\omega_1) \leq \mathcal{I}(\omega_2)$.
Consequently, one can infer that $\omega_1 \preceq \omega_2$ if one of the following conditions is satisfied:
(i) $\mathcal{R}(\omega_1) = \mathcal{R}(\omega_2)$, $\mathcal{I}(\omega_1) < \mathcal{I}(\omega_2)$,
(ii)$\mathcal{R}(\omega_1) < \mathcal{R}(\omega_2)$, $\mathcal{I}(\omega_1) = \mathcal{I}(\omega_2)$,
(iii)$\mathcal{R}(\omega_1) < \mathcal{R}(\omega_2)$, $\mathcal{I}(\omega_1) < \mathcal{I}(\omega_2)$,
(iv)$\mathcal{R}(\omega_1) = \mathcal{R}(\omega_2)$, $\mathcal{I}(\omega_1) = \mathcal{I}(\omega_2)$.
In particular, we write $\omega_1 \precnsim \omega_2$ if $\omega_1 \neq \omega_2$ and one of $(i), (ii)$ and (iii) is satisfied and we write $\omega_1 \prec \omega_2$ if (iii) is satisfied. Notice that
(a) If $0 \preceq \omega_1 \precnsim \omega_2$, then $|\omega_1| < |\omega_2|$,
(b) If $\omega_1 \preceq \omega_2$ and $\omega_2 \prec \omega_3$ then $\omega_1 \prec \omega_3$,
(c) If $\eta, \gamma \in \mathbb{R}$ and $\eta \leq \gamma$ then $\eta\omega_1 \preceq \gamma\omega_1$ for all $0 \preceq \omega_1 \in \mathbb{C}$.

Definition 2.1. [10] Let Υ be a non-void set and let $s \geq 1$ be a given real number. A function $\delta \colon \Upsilon \times \Upsilon \to \mathbb{C}$ is called a complex valued b-metric on Υ if for all $\mathcal{Z}, \sigma, \beta \in \Upsilon$ the following conditions are satisfied:
(i) $0 \preceq \delta(\mathcal{Z}, \sigma)$ and $\delta(\mathcal{Z}, \sigma) = 0$ if and only if $\mathcal{Z} = \sigma$;
(ii) $\delta(\mathcal{Z}, \sigma) = \delta(\sigma, \mathcal{Z})$;
(iii) $\delta(\mathcal{Z}, \sigma) \preceq s[\delta(\mathcal{Z}, \beta) + \delta(\beta, \sigma)]$.

The pair (Υ, δ) is called a complex valued b-metric space.
Here $\mathbb{C}^+(= \{(\beta, \sigma) | \beta, \sigma \in \mathbb{R}^+\})$ and $\mathbb{R}^+(= \{\beta \in \mathbb{R} | \beta \geq 0\})$ denote the set of non negative complex numbers, and the set of non negative real numbers, respectively.

Definition 2.2. [7] A complex partial metric on a non-void set Υ is a function $\wp_{cb} : \Upsilon \times \Upsilon \to \mathbb{C}^+$ such that for all $\mathcal{Z}, \sigma, \beta \in \Upsilon$:
(i) $0 \preceq \wp_{cb}(\mathcal{Z}, \mathcal{Z}) \preceq \wp_{cb}(\mathcal{Z}, \sigma)$(small self-distances)
(ii) $\wp_{cb}(\mathcal{Z}, \sigma) = \wp_{cb}(\sigma, \mathcal{Z})(symmetry)$
(iii) $\wp_{cb}(\mathcal{Z}, \mathcal{Z}) = \wp_{cb}(\mathcal{Z}, \sigma) = \wp_{cb}(\sigma, \sigma)$ if and only if $\mathcal{Z} = \sigma(equality)$
(iv) $\wp_{cb}(\mathcal{Z}, \sigma) \preceq \wp_{cb}(\mathcal{Z}, \beta) + \wp_{cb}(\beta, \sigma) - \wp_{cb}(\beta, \beta)(triangularity)$.
A complex partial metric space is a pair $(\Upsilon, \wp_{cb})$ such that Υ is a non-void set and $\wp_{cb}$ is the complex partial metric on Υ.

Definition 2.3. [9] A complex partial b-metric on a non-void set Υ is a function $\delta_{cb} : \Upsilon \times \Upsilon \to \mathbb{C}^+$ such that for all $\mathcal{Z}, \sigma, \beta \in \Upsilon$:
(i) $0 \preceq \delta_{cb}(\mathcal{Z}, \mathcal{Z}) \preceq \delta_{cb}(\mathcal{Z}, \sigma)$(small self-distances)
(ii) $\delta_{cb}(\mathcal{Z}, \sigma) = \delta_{cb}(\sigma, \mathcal{Z})(symmetry)$
(iii) $\delta_{cb}(\mathcal{Z}, \mathcal{Z}) = \delta_{cb}(\mathcal{Z}, \sigma) = \delta_{cb}(\sigma, \sigma)$ iff $\mathcal{Z} = \sigma(equality)$
(iv) $\exists$ a real number $s \geq 1$ and s is independent of $\mathcal{Z}, \sigma, \beta$ such that $\delta_{cb}(\mathcal{Z}, \sigma) \preceq s[\delta_{cb}(\mathcal{Z}, \beta) + \delta_{cb}(\beta, \sigma) - \delta_{cb}(\beta, \beta)(triangularity)$.
A complex partial b-metric space is a pair (Υ, δ_{cb}) such that Υ is a non-void set and δ_{cb} is the complex partial b-metric on Υ. The number s is called the coefficient of (Υ, δ_{cb}).

Example 2.4. [9] Let $\Upsilon = \mathbb{R}^+$ and $\delta_{cb} : \Upsilon \times \Upsilon \to \mathbb{C}^+$ be defined by $\delta_{cb}(\mathcal{Z}, \sigma) = [max\{\mathcal{Z}, \sigma\}]^3 + |\mathcal{Z} - \sigma|^3 + i\{[max\{\mathcal{Z}, \sigma\}]^3 + |\mathcal{Z} - \sigma|^3\}$ for all $\mathcal{Z}, \sigma \in \Upsilon$. Then (Υ, δ_{cb}) is a complex partial b-metric space with coefficient $s = 2^3$, but it is neither a complex valued b-metric nor a complex partial metric.

Now, we define Cauchy sequence and convergent sequence in complex partial b-metric spaces.

Definition 2.5. [9] Let (Υ, δ_{cb}) be a complex partial b-metric space with coefficient s. Then:

(i) The sequence $\{\mathcal{Z}_n\}$ in Υ converges to $\mathcal{Z} \in \Upsilon$, if $\lim_{n\to+\infty} \delta_{cb}(\mathcal{Z}_n, \mathcal{Z}) = \delta_{cb}(\mathcal{Z}, \mathcal{Z})$.

(ii) The sequence $\{\mathcal{Z}_n\}$ is said to be Cauchy sequence in (Υ, δ_{cb}) if $\lim_{n,m\to+\infty} \delta_{cb}(\mathcal{Z}_n, \mathcal{Z}_m)$ exists and is finite.

(iii) The space (Υ, δ_{cb}) is said to be a complete complex partial b-metric space if for every Cauchy sequence $\{\mathcal{Z}_n\}$ in Υ there exists $\mathcal{Z} \in \Upsilon$ such that $\lim_{n,m\to+\infty} \delta_{cb}(\mathcal{Z}_n, \mathcal{Z}_m) = \lim_{n\to+\infty} \delta_{cb}(\mathcal{Z}_n, \mathcal{Z}) = \delta_{cb}(\mathcal{Z}, \mathcal{Z})$.

(iv) A mapping $R : \Upsilon \to \Upsilon$ is said to be continuous at $\mathcal{Z}_0 \in \Upsilon$ if for every $\epsilon > 0$, there exists $t > 0$ such that $R(B_{\delta_{cb}}(\mathcal{Z}_0, t)) \subset B_{\delta_{cb}}(R(\mathcal{Z}_0, \epsilon))$.

Definition 2.6. Let (Υ, δ_{cb}) be a complex partial b-metric space with coefficient $s \geq 1$. Let $(\Upsilon, \preceq)$ be a partially ordered set and for all elements comparable to each other. A pair $(\sqcup, \sqcap)$ of self-maps of Υ is said to be weakly increasing if $\sqcup\mathcal{Z} \preceq \sqcap \sqcup \mathcal{Z}$ and $\sqcap\mathcal{Z} \preceq \sqcup \sqcap \mathcal{Z}$ for all $\mathcal{Z} \in \Upsilon$. If $\sqcup = \sqcap$, then we have $\sqcup\mathcal{Z} \preceq \sqcup^2\mathcal{Z}$ for all $\mathcal{Z} \in \Upsilon$ and in this case, we say that $\sqcup$ is weakly increasing mapping.

Definition 2.7. Let (Υ, δ_{cb}) be a complex partial b-metric space with coefficient $s \geq 1$. A point $\mathcal{Z} \in \Upsilon$ is said to be common fixed point for the pair of self mappings $(\sqcup, \sqcap)$ on Υ is such that $\mathcal{Z} = \sqcup\mathcal{Z} = \sqcap\mathcal{Z}$.

Theorem 2.8. *[9] Let (Υ, δ_{cb}) be a complete complex partial b-metric space with coefficient $s \geq 1$ and $\sqcup : \Upsilon \to \Upsilon$ be a mapping satisfying:*

$$\delta_{cb}(\sqcup\mathcal{Z}, \sqcup\sigma) \preceq \curlywedge[\delta_{cb}(\mathcal{Z}, \sqcup\mathcal{Z}) + \delta_{cb}(\sigma, \sqcup\sigma)]$$

for all $\mathcal{Z}, \sigma \in \Upsilon$, where $\curlywedge \in [0, \frac{1}{s}]$. Then $\sqcup$ has a unique fixed point $\mathcal{Z}^* \in \Upsilon$ and $\delta_{cb}(\mathcal{Z}^*, \mathcal{Z}^*) = 0$.

3. Main Results

In this section we prove some common fixed point theorems on complex partial b-metric space for rational type weakly increasing mappings.

Theorem 3.1. *Let (Υ, δ_{cb}) be a complete complex partial b-metric space with the coefficient $s \geq 1$ and $\sqcap, \sqcup \colon \Upsilon \to \Upsilon$ be two weakly increasing mappings such that*

$$\delta_{cb}(\sqcup \mathcal{Z}, \sqcap \sigma) \preceq \frac{\curlyvee \delta_{cb}(\mathcal{Z}, \sqcup \mathcal{Z}) \delta_{cb}(\sigma, \sqcap \sigma)}{\delta_{cb}(\mathcal{Z}, \sigma)} + \curlywedge \delta_{cb}(\mathcal{Z}, \sigma)$$

for all $\mathcal{Z}, \sigma \in \Upsilon$, $\delta_{cb}(\mathcal{Z}, \sigma) \neq 0$ with $\curlyvee \geq 0, \curlyvee + \curlywedge < 1$ or $\delta_{cb}(\sqcup \mathcal{Z}, \sqcap \sigma) = 0$ if $\delta_{cb}(\mathcal{Z}, \sigma) = 0$. If $\sqcup$ or $\sqcap$ is continuous then the pair $(\sqcup, \sqcap)$ has a common fixed point $\beta \in \Upsilon$ and $\delta_{cb}(\beta, \beta) = 0$.

Proof. Let $\mathcal{Z}_0$ be an arbitrary point in Υ and define a sequence as follows:

$$\begin{aligned} \mathcal{Z}_{2k+1} &= \sqcup \mathcal{Z}_{2k} \\ \mathcal{Z}_{2k+2} &= \sqcap \mathcal{Z}_{2k+1}, k = 0, 1, 2, \ldots. \end{aligned}$$

Since $\sqcup$ and $\sqcap$ are weakly increasing,

$$\begin{aligned} \mathcal{Z}_1 &= \sqcup \mathcal{Z}_0 \preceq \sqcap \sqcup \mathcal{Z}_0 = \sqcap \mathcal{Z}_1 = \mathcal{Z}_2 \\ \mathcal{Z}_2 &= \sqcap \mathcal{Z}_1 \preceq \sqcup \sqcap \mathcal{Z}_1 = \sqcup \mathcal{Z}_2 = \mathcal{Z}_3. \end{aligned}$$

Continuing this way, we have $\mathcal{Z}_1 \preceq \mathcal{Z}_2 \preceq \ldots \preceq \mathcal{Z}_n \preceq \mathcal{Z}_{n+1} \ldots$.
Assume that $\delta_{cb}(\mathcal{Z}_{2k}, \mathcal{Z}_{2k+1}) > 0$ for all $k \in \mathbb{N}$. If not, then $\mathcal{Z}_{2k} = \mathcal{Z}_{2k+1}$ for some k. Then for all those k, $\mathcal{Z}_{2k} = \mathcal{Z}_{2k+1} = \sqcup \mathcal{Z}_{2k}$ and the proof is completed. Assume that $\delta_{cb}(\mathcal{Z}_{2k}, \mathcal{Z}_{2k+1}) > 0$ for $k = 0, 1, 2, \ldots$. As $\mathcal{Z}_{2k}$ and $\mathcal{Z}_{2k+1}$ are comparable, so we have

$$\begin{aligned} \delta_{cb}(\mathcal{Z}_{2k+1}, \mathcal{Z}_{2k+2}) &= \delta_{cb}(\sqcup \mathcal{Z}_{2k}, \sqcap \mathcal{Z}_{2k+1}) \\ &\preceq \curlyvee \frac{\delta_{cb}(\mathcal{Z}_{2k}, \sqcup \mathcal{Z}_{2k}) \delta_{cb}(\mathcal{Z}_{2k+1}, \sqcap \mathcal{Z}_{2k+1})}{\delta_{cb}(\mathcal{Z}_{2k}, \mathcal{Z}_{2k+1})} + \curlywedge \delta_{cb}(\mathcal{Z}_{2k}, \mathcal{Z}_{2k+1}) \\ &\preceq \curlyvee \delta_{cb}(\mathcal{Z}_{2k+1}, \sqcap \mathcal{Z}_{2k+2}) + \curlywedge \delta_{cb}(\mathcal{Z}_{2k}, \mathcal{Z}_{2k+1}) \\ \delta_{cb}(\mathcal{Z}_{2k+1}, \mathcal{Z}_{2k+2}) &\preceq \frac{\curlywedge}{1 - \curlyvee} \delta_{cb}(\mathcal{Z}_{2k}, \mathcal{Z}_{2k+1}). \end{aligned}$$

Now with $\eta = \dfrac{\curlywedge}{1 - \curlyvee}$, we have

$$\delta_{cb}(\mathcal{Z}_{2k+1}, \mathcal{Z}_{2k+2}) \preceq h \delta_{cb}(\mathcal{Z}_{2k}, \mathcal{Z}_{2k+1}) \preceq \ldots \preceq h^{2k+1} \delta_{cb}(\mathcal{Z}_0, \mathcal{Z}_1).$$

For $n > m$, we get

$$\begin{aligned} \delta_{cb}(\mathcal{Z}_m, \mathcal{Z}_n) &\preceq s \delta_{cb}(\mathcal{Z}_m, \mathcal{Z}_{m+1}) + s^2 \delta_{cb}(\mathcal{Z}_{m+1}, \mathcal{Z}_{m+2}) + \cdots + s^n \delta_{cb}(\mathcal{Z}_{n-1}, \mathcal{Z}_n) \\ &\quad - \delta_{cb}(\mathcal{Z}_{m+1}, \mathcal{Z}_{m+1}) - \delta_{cb}(\mathcal{Z}_{m+2}, \mathcal{Z}_{m+2}) - \delta_{cb}(\mathcal{Z}_{m+3}, \mathcal{Z}_{m+3}) \\ &\quad - \cdots - \delta_{cb}(\mathcal{Z}_{n-1}, \mathcal{Z}_{n-1}) \\ &\preceq (s\eta^m + s^2 \eta^{m+1} + \cdots + s^n \eta^{n-1}) \delta_{cb}(\mathcal{Z}_0, \mathcal{Z}_1) \\ &= s\eta^m (1 + s\eta + \cdots + s^{n-1} \eta^{n-m-1}) \delta_{cb}(\mathcal{Z}_0, \mathcal{Z}_1) \\ &\preceq \frac{s\eta^m}{1 - s\eta} \delta_{cb}(\mathcal{Z}_0, \mathcal{Z}_1). \end{aligned}$$

Consequently,

$$|\delta_{cb}(\mathcal{Z}_m, \mathcal{Z}_n)| \leq \frac{(s\eta)^m}{1-\eta} |\delta_{cb}(\mathcal{Z}_1, \mathcal{Z}_0)| \to 0$$

as $m, n \to +\infty$ which implies that $\lim_{m,n\to+\infty} \delta_{cb}(\mathcal{Z}_m, \mathcal{Z}_n) = 0$ such that $\mathcal{Z}_n$ is a Cauchy sequence in Υ. Since (Υ, δ_{cb}) is complete, there exists $\beta \in \Upsilon$ such that $\mathcal{Z}_n \to \beta$ and

$$\delta_{cb}(\beta, \beta) = \lim_{m,n\to+\infty} \delta_{cb}(\beta, \mathcal{Z}_n) = \lim_{m,n\to+\infty} \delta_{cb}(\mathcal{Z}_n, \mathcal{Z}_n) = 0.$$

Without loss of generality, suppose that $\sqcap$ is continuous in (Υ, δ_{cb}). Therefore $\sqcap\mathcal{Z}_{2n+1} \to \sqcap\beta$ in (Υ, δ_{cb}). That is

$$\delta_{cb}(\sqcap\beta, \sqcap\beta) = \lim_{n\to+\infty} \delta_{cb}(\sqcap\beta, \sqcap\mathcal{Z}_{2n+1}) = \lim_{n\to+\infty} \delta_{cb}(\sqcap\mathcal{Z}_{2n+1}, \sqcap\mathcal{Z}_{2n+1}).$$

But

$$\delta_{cb}(\sqcap\beta, \sqcap\beta) = \lim_{n\to+\infty} \delta_{cb}(\sqcap\mathcal{Z}_{2n+1}, \sqcap\mathcal{Z}_{2n+1}) = \lim_{n\to+\infty} \delta_{cb}(\mathcal{Z}_{2n+2}, \mathcal{Z}_{2n+2}) = 0.$$

Next we will prove β is a fixed point of $\sqcap$.

$$\delta_{cb}(\sqcap\beta, \beta) \preceq s\{\delta_{cb}(\sqcap\beta, \sqcap\mathcal{Z}_{2n+1}) + \delta_{cb}(\sqcap\mathcal{Z}_{2n+1}, \beta)\} - \delta_{cb}(\sqcap\mathcal{Z}_{2n+1}, \sqcap\mathcal{Z}_{2n+1}).$$

As $n \to +\infty$, we obtain $|\delta_{cb}(\sqcap\beta, \beta)| \leq 0$. Thus, $\delta_{cb}(\sqcap\beta, \beta) = 0$. Hence $\delta_{cb}(\beta, \beta) = \delta_{cb}(\beta, \sqcap\beta) = \delta_{cb}(\sqcap\beta, \sqcap\beta) = 0$ and so $\sqcap\beta = \beta$. Therefore $\sqcup\beta = \sqcap\beta = \beta$ and $\delta_{cb}(\beta, \beta) = 0$. $\square$

In the absence of the continuity condition for the mapping $\sqcap$, we get the following theorem.

Theorem 3.2. *Let (Υ, δ_{cb}) be a complete complex partial b-metric space with the coefficient $s \geq 1$ and $\sqcap, \sqcup \colon \Upsilon \to \Upsilon$ be two weakly increasing mappings such that*

$$\delta_{cb}(\sqcup\mathcal{Z}, \sqcap\sigma) \preceq \frac{\curlyvee \delta_{cb}(\mathcal{Z}, \sqcup\mathcal{Z})\delta_{cb}(\sigma, \sqcap\sigma)}{\delta_{cb}(\mathcal{Z}, \sigma)} + \curlywedge \delta_{cb}(\mathcal{Z}, \sigma)$$

for all $\mathcal{Z}, \sigma \in \Upsilon$, $\delta_{cb}(\mathcal{Z}, \sigma) \neq 0$ with $\curlyvee \geq 0, \curlywedge \geq 0, \curlyvee + \curlywedge < 1$ or $\delta_{cb}(\sqcup\mathcal{Z}, \sqcap\sigma) = 0$ if $\delta_{cb}(\mathcal{Z}, \sigma) = 0$. Suppose Υ satisfies the condition that, for every increasing sequence $\{\mathcal{Z}_n\}$ with $\mathcal{Z}_n \to \beta$ in Υ, we necessarily have $\beta = \sup \mathcal{Z}_n$, then the pair $(\sqcup, \sqcap)$ has a common fixed point $\beta \in \Upsilon$ and $\delta_{cb}(\beta, \beta) = 0$.

Proof. We know that $\mathcal{Z}_n \preceq \beta$ for all $n \in \mathbb{N}$. Following the proof of Theorem 3.1, it is enough to prove that β is a fixed point of $\sqcup$. Suppose β is not a fixed point, then we have $\delta_{cb}(\beta, \sqcup\beta) = \omega > 0$ for some $\omega \in \mathbb{C}$, we obtain

$$\begin{aligned}
\omega &\preceq s[\delta_{cb}(\beta, \mathcal{Z}_{2n+2}) + \delta_{cb}(\mathcal{Z}_{2n+2}, \sqcup\beta)] - \delta_{cb}(\mathcal{Z}_{2n+2}, \mathcal{Z}_{2n+2}) \\
&= s[\delta_{cb}(\beta, \mathcal{Z}_{2n+2}) + \delta_{cb}(\sqcap\mathcal{Z}_{2n+1}, \sqcup\beta)] - \delta_{cb}(\mathcal{Z}_{2n+2}, \mathcal{Z}_{2n+2}) \\
&\preceq s[\delta_{cb}(\beta, \mathcal{Z}_{2n+2}) + \curlyvee \frac{\delta_{cb}(\mathcal{Z}_{2n+1}, \sqcap\mathcal{Z}_{2n+1})\delta_{cb}(\beta, \sqcup\beta)}{\delta_{cb}(\mathcal{Z}_{2n+1}, \beta)} \\
&+ \curlywedge \delta_{cb}(\mathcal{Z}_{2n+1}, \beta)] - \delta_{cb}(\mathcal{Z}_{2n+2}, \mathcal{Z}_{2n+2}).
\end{aligned}$$

Suppose $\delta_{cb}(\beta,\beta) = 0$, taking limit as $n \to +\infty$, we have $\omega \preceq 0$, which is a contradiction. Therefore β is a fixed point of $\sqcup$.
For $\delta_{cb}(\beta,\beta) \neq 0$, taking limit as $n \to +\infty$, we have $\omega \preceq \curlyvee\delta_{cb}(\beta,\sqcup\beta) + \curlywedge\delta_{cb}(\beta,\beta)$ and so, $|\omega| \leq (\curlyvee + \curlywedge)|\omega|$, since $\curlyvee + \curlywedge < 1$, we get a contradiction, which implies that $\beta = \sqcup\beta$. Therefore, by Theorem 3.1, we get $\sqcup\beta = \sqcap\beta = \beta$ and $\delta_{cb}(\beta,\beta) = 0$. □

Theorem 3.3. *In addition to Theorem 3.2, suppose that the set of common fixed points of $\sqcup$ and $\sqcap$ is totally ordered if and only if $\sqcup$ and $\sqcap$ have a unique common fixed point.*

Proof. Suppose now that the common fixed points of $\sqcup$ and $\sqcap$ are totally ordered. We have to prove that common fixed points of $\sqcup$ and $\sqcap$ are unique. Assume that on the contrary β and q are distinct common fixed points of $\sqcup$ and $\sqcap$. By supposition, we replace μ by β and σ by q in Theorem 3.1, we obtain for $\delta_{cb}(\beta,q) \neq 0$,

$$\begin{aligned}\delta_{cb}(\beta,q) &= \delta_{cb}(\sqcup\beta,\sqcap q)\\ &\preceq \curlyvee\frac{\delta_{cb}(\beta,\sqcup\beta)\delta_{cb}(q,\sqcap q)}{\delta_{cb}(\beta,q)} + \curlywedge\delta_{cb}(\beta,q)\\ &\preceq \curlywedge\delta_{cb}(\beta,q),\end{aligned}$$

which is a contradiction. Hence $\beta = q$. Conversely, if $\sqcup$ and $\sqcap$ have only one common fixed point then the set of common fixed point of $\sqcup$ and $\sqcap$ being singleton is totally ordered. □

Corollary 3.4. *Let (Υ,δ_{cb}) be a complete complex partial b-metric space with the coefficient $s \geq 1$ and $\sqcap\colon \Upsilon \to \Upsilon$ be a weakly increasing mapping such that*

$$\delta_{cb}(\sqcap\mathcal{Z},\sqcap\sigma) \preceq \frac{\curlyvee\delta_{cb}(\mathcal{Z},\sqcap\mathcal{Z})\delta_{cb}(\sigma,\sqcap\sigma)}{\delta_{cb}(\mathcal{Z},\sigma)} + \curlywedge\delta_{cb}(\mathcal{Z},\sigma)$$

for all $\mathcal{Z},\sigma \in \Upsilon$, $\delta_{cb}(\mathcal{Z},\sigma) \neq 0$ with $\curlyvee \geq 0, \curlywedge \geq 0, \curlyvee + \curlywedge < 1$ or $\delta_{cb}(\sqcup\mathcal{Z},\sqcap\sigma) = 0$ if $\delta_{cb}(\mathcal{Z},\sigma) = 0$. Suppose $\sqcap$ is continuous or for every increasing sequence $\{\mathcal{Z}_n\}$ with $\mathcal{Z}_n \to \beta$ in Υ, we necessarily have $\beta = \sup\mathcal{Z}_n$, then $\sqcap$ has a fixed point $\beta \in \Upsilon$ and $\delta_{cb}(\beta,\beta) = 0$. Moreover, the set of fixed points of $\sqcap$ is totally ordered if and only if $\sqcap$ has a unique fixed point.

Corollary 3.5. *Let (Υ,δ_{cb}) be a complete complex partial b-metric space with the coefficient $s \geq 1$ and $\sqcap\colon \Upsilon \to \Upsilon$ be a weakly increasing mapping such that*

$$\delta_{cb}(\sqcap^n\mathcal{Z},\sqcap^n\sigma) \preceq \frac{\curlyvee\delta_{cb}(\mathcal{Z},\sqcap^n\mathcal{Z})\delta_{cb}(\sigma,\sqcap^n\sigma)}{\delta_{cb}(\mathcal{Z},\sigma)} + \curlywedge\delta_{cb}(\mathcal{Z},\sigma)$$

for all $\mathcal{Z},\sigma \in \Upsilon$, $\delta_{cb}(\mathcal{Z},\sigma) \neq 0$ with $\curlyvee \geq 0, \curlywedge \geq 0, \curlyvee + \curlywedge < 1$ or $\delta_{cb}(\sqcup\mathcal{Z},\sqcap\sigma) = 0$ if $\delta_{cb}(\mathcal{Z},\sigma) = 0$. Then $\sqcap$ has a unique fixed point.

Proof. By Corollary 3.4, we obtain $\beta \in \Upsilon$ such that $\sqcap^n \beta = \beta$. Suppose $\delta_{cb}(\sqcap\beta, \beta) = 0$, the proof is finished. If $\delta_{cb}(\sqcap\beta, \beta) \neq 0$, we have

$$\begin{aligned}
\delta_{cb}(\sqcap\beta, \beta) &= \delta_{cb}(\sqcap \sqcap^n \beta, \sqcap^n \beta) = \delta_{cb}(\sqcap^n \sqcap \beta, \sqcap^n \beta) \\
&\preceq \curlyvee \frac{\delta_{cb}(\sqcap\beta, \sqcap^n\beta)\delta_{cb}(\beta, \sqcap^n\beta)}{\delta_{cb}(\sqcap\beta, \beta)} + \curlywedge \delta_{cb}(\beta, \sqcap\beta) \\
&\preceq (\curlyvee + \curlywedge)\delta_{cb}(\beta, \sqcap\beta),
\end{aligned}$$

which is a contradiction. Therefore $\sqcap\beta = \beta$. □

Example 3.6. Let $\Upsilon = \{1, 2, 3, 4\}$ be endowed with the order $\mathcal{Z} \preceq \sigma$ if and only if $\sigma \leq \mathcal{Z}$. Then $\preceq$ is a partial order in Υ. Define the complex partial b-metric space $\delta_{cb} : \Upsilon \times \Upsilon \to \mathbb{C}^+$ as follows:

$(\mathcal{Z}, \sigma)$	$\delta_{cb}(\mathcal{Z}, \sigma)$
(1,1), (2,2)	0
(1,2),(2,1),(1,3),(3,1),(2,3),(3,2),(3,3)	e^{2iy}
(1,4),(4,1),(2,4),(4,2),(3,4),(4,3),(4,4)	$9e^{2iy}$

It is easy to verify that (Υ, δ_{cb}) is a complete complex partial b-metric space with the coefficient $s \geq 1$ for $y \in [0, \frac{\pi}{2}]$. Define $\sqcup, \sqcap : \Upsilon \to \Upsilon$ by $\sqcup\mathcal{Z} = 1$,

$$\sqcap(\mathcal{Z}) = \begin{cases} 1 & \text{if } \mathcal{Z} \in \{1, 2, 3\} \\ 2 & \text{if } \mathcal{Z} = 4. \end{cases}$$

Then, $\sqcup$ and $\sqcap$ are weakly increasing with respect to $\preceq$ and continuous. Now for $\curlyvee = \curlywedge = \frac{1}{9}$, we consider the following cases:

(a) If $\mathcal{Z} = 1$ and $\sigma \in \Upsilon - \{4\}$, then $\sqcup(\mathcal{Z}) = \sqcap(\sigma) = 1$ and $\delta_{cb}(\sqcup(\mathcal{Z}), \sqcap(\sigma)) = 0$ and the conditions of Theorem 3.1 are satisfied.

(b) If $\mathcal{Z} = 1$, $\sigma = 4$, then $\sqcup\mathcal{Z} = 1$, $\sqcap\sigma = 2$,

$$\begin{aligned}
\delta_{cb}(\sqcup\mathcal{Z}, \sqcap\sigma) = e^{2iy} &\preceq 9 \curlywedge e^{i2y} \\
&= \curlyvee \frac{(0)9e^{2iy}}{9e^{2iy}} + \curlywedge 9e^{2iy} \\
&= \curlyvee \frac{\delta_{cb}(\mathcal{Z}, \sqcup\mathcal{Z})\delta_{cb}(\sigma, \sqcap\sigma)}{\delta_{cb}(\mathcal{Z}, \sigma)} + \curlywedge \delta_{cb}(\mathcal{Z}, \sigma).
\end{aligned}$$

(c) If $\mathcal{Z} = 2$, $\sigma = 4$, then $\sqcup\mathcal{Z} = 1$, $\sqcap\sigma = 2$,

$$\begin{aligned}
\delta_{cb}(\sqcup\mathcal{Z}, \sqcap\sigma) = e^{2iy} &\preceq (\curlyvee + 9\curlywedge)e^{i2y} \\
&= \curlyvee \frac{(e^{2iy})9e^{2iy}}{9e^{2iy}} + \curlywedge 9e^{2iy} \\
&= \curlyvee \frac{\delta_{cb}(\mathcal{Z}, \sqcup\mathcal{Z})\delta_{cb}(\sigma, \sqcap\sigma)}{\delta_{cb}(\mathcal{Z}, \sigma)} + \curlywedge \delta_{cb}(\mathcal{Z}, \sigma).
\end{aligned}$$

(d) If $\mathcal{Z} = 3$, $\sigma = 4$, then $\sqcup\mathcal{Z} = 1$, $\sqcap\sigma = 2$,

$$\begin{aligned}\delta_{cb}(\sqcup\mathcal{Z}, \sqcap\sigma) = e^{2iy} &\preceq (\curlyvee + 9\curlywedge)e^{i2y} \\ &= \curlyvee\frac{(e^{2iy})9e^{2iy}}{9e^{2iy}} + \curlywedge 9e^{2iy} \\ &= \curlyvee\frac{\delta_{cb}(\mathcal{Z}, \sqcup\mathcal{Z})\delta_{cb}(\sigma, \sqcap\sigma)}{\delta_{cb}(\mathcal{Z}, \sigma)} + \curlywedge\delta_{cb}(\mathcal{Z}, \sigma).\end{aligned}$$

(e) If $\mathcal{Z} = 4$, $\sigma = 4$, then $\sqcup\mathcal{Z} = 1$, $\sqcap\sigma = 2$,

$$\begin{aligned}\delta_{cb}(\sqcup\mathcal{Z}, \sqcap\sigma) = e^{2iy} &\preceq 9(\curlyvee + \curlywedge)e^{i2y} \\ &= \curlyvee\frac{(9e^{2iy})9e^{2iy}}{9e^{2iy}} + \curlywedge 9e^{2iy} \\ &= \curlyvee\frac{\delta_{cb}(\mathcal{Z}, \sqcup\mathcal{Z})\delta_{cb}(\sigma, \sqcap\sigma)}{\delta_{cb}(\mathcal{Z}, \sigma)} + \curlywedge\delta_{cb}(\mathcal{Z}, \sigma).\end{aligned}$$

Moreover, for $\curlyvee = \curlywedge = \frac{1}{9}$, with $\curlywedge + \curlywedge = \frac{2}{9} < 1$, the conditions of Theorem 3.1 are satisfied. Therefore, 1 is the unique common fixed point of $\sqcup$ and $\sqcap$.

4. Application

Now we prove an existence theorem for the common solution of two Urysohn type integral equations. Consider the following system of Urysohn type integral equations.

$$\begin{cases}\mathcal{Z}(q) = b(q) + \int_x^y G_1(q, s, \mathcal{Z}(s))ds \\ \mathcal{Z}(q) = b(q) + \int_x^y G_2(q, s, \mathcal{Z}(s))ds,\end{cases} \tag{4.1}$$

where

(R0) $\mathcal{Z}(q)$ is an unknown variable for each $q \in [x, y]$, $x > 0$,

(R1) $b(q)$ is the deterministic free term defined for $q \in [x, y]$,

(R2) $G_1(q, s)$ and $G_2(q, s)$ are deterministic kernels defined for $q, s \in [x, y]$.

Let $\Upsilon = (C[x, y], \mathbb{R}^n)$, $q > 0$ and $\delta_{cb} : \Upsilon \times \Upsilon \to \mathbb{R}^n$ defined by

$$\delta_{cb}(\mathcal{Z}, \sigma) = |\mathcal{Z} - \sigma|^2 + 2 + i(|\mathcal{Z} - \sigma|^2 + 2),$$

for all $\mathcal{Z}, \sigma \in \Upsilon$.
Obviously $(C[x, y], \mathbb{R}^n, \delta_{cb})$ is a complete complex partial b-metric space with the constant $s \geq 1$. Further let us consider a Urysohn type integral system as (4.1) under the following conditions:

(1) $b(q) \in \Upsilon$;

(2) for all $q, s \in [x, y]$, we have

$$G_1(q, s, i(s)) \preceq G_2(s, k, b(s) + \int_x^y G_1(s, k, i(k))dk)$$

and

$$G_2(q, s, i(s)) \preceq G_1(q, s, b(q) + \int_x^y G_2(q, s, i(s))ds);$$

(3) $G_1, G_2 : [x, y] \times [x, y] \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions satisfying

$$|G_1(q, s, i(s)) - G_2(q, s, j(s))| \preceq \sqrt{\frac{|i - j|^2}{2(y - x)} - \frac{2}{y - x}}.$$

We apply Theorem 3.1 to prove the existence of a unique solution of the system (4.1).

Theorem 4.1. *Let* $(C[x, y], \mathbb{R}^n, \wp_{cb})$ *be a complete complex partial b-metric space, then the system* (4.1) *under the conditions (1)-(3) has a unique common solution.*

Proof. For $\mathcal{Z} \in \Upsilon$ and $q \in [x, y]$, we define the continuous mappings $\sqcup, \sqcap : \Upsilon \to \Upsilon$ by

$$\sqcup\mathcal{Z}(q) = b(q) + \int_x^y G_1(q, s, \mathcal{Z}(s))ds,$$

and

$$\sqcap\mathcal{Z}(s) = b(q) + \int_x^y G_2(q, s, \mathcal{Z}(s))ds.$$

From the condition (2), the mappings $\sqcup$ and $\sqcap$ are weakly increasing with respect to $\preceq$. Indeed, for all $q \in [x, y]$, we have

$$\begin{aligned}
\sqcup\mathcal{Z}(q) &= b(q) + \int_x^y G_1(q, s, \mathcal{Z}(s))ds \\
&\preceq b(q) + \int_x^y G_2(s, k, b(s) + \int_x^y G_1(s, k, i(k))dk) \\
&= b(q) + \int_x^y G_2(q, s, \sqcup\mathcal{Z}(s))ds \\
&= \sqcap(\sqcup\mathcal{Z}(q)).
\end{aligned}$$

Therefore $\sqcup\mathcal{Z}(q) \preceq \sqcap(\sqcup\mathcal{Z}(q))$. Similarly, one can easily see that $\sqcap\mathcal{Z}(s) \preceq$

$\sqcup(\sqcap\mathcal{Z}(s))$. Next we have

$$\begin{aligned}
\delta_{cb}(\sqcup\mathcal{Z}(q),\sqcap\sigma(q)) &= |\sqcup\mathcal{Z}(q)-\sqcap\sigma(q)|^2+2+i(|\sqcup\mathcal{Z}(q)-\sqcap\sigma(q)|^2+2)\\
&= \int_x^y |G_1(q,s,\mathcal{Z}(s))-G_2(q,s,\sigma(s))|^2 dp+2\\
&+ i\bigg(\int_x^y |G_1(q,s,\mathcal{Z}(s))-G_2(q,s,\sigma(s))|^2 dp+2\bigg)\\
&\preceq \int_x^y \left(\frac{|i-j|^2}{2(y-x)}-\frac{2}{y-x}\right)dp+2\\
&+ i\bigg(\int_x^y \left(\frac{|i-j|^2}{2(y-x)}-\frac{2}{y-x}\right)dp+2\bigg)\\
&= \frac{|i-j|^2}{2}+i\bigg(\frac{|i-j|^2}{2}\bigg)\\
&\preceq \frac{|i-j|^2}{2}+1+i\bigg(\frac{|i-j|^2}{2}+1\bigg)\\
&= \curlywedge(|i-j|^2+2+i(|i-j|^2+2))\\
&= \curlywedge\delta_{cb}(i,j),
\end{aligned}$$

for all $\mathcal{Z},\sigma\in\Upsilon$. Hence, all the conditions of Theorem 3.1 are satisfied for $\curlyvee+\curlywedge(=\frac{1}{2})<1$ with $\curlyvee=0$. Therefore, the system of integral equations (4.1) has a unique common solution. $\square$

5. Conclusion

In this paper, we proved some common fixed point theorems on complex partial b-metric spaces for a pair of weakly increasing mapping satisfying rational type contraction conditoion. An illustrative example and application to Urysohn type integral equations on complex partial b-metric space is given.

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7. Conflict of interest

The authors declare that they have no conflict of interest.

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