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A note on $(k, \mu)'$-almost Kenmotsu manifolds

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Abstract. In this paper, we classify $(k, \mu)'$-almost Kenmotsu manifolds admits some special vector fields such as concircular and torse-forming. Furthermore, we characterize $(k, \mu)'$-almost Kenmotsu manifolds with an η-Ricci soliton whose potential vector field is projective and affine conformal. Beside these we study gradient η-Ricci soliton on $(k, \mu)'$-almost Kenmotsu manifolds. Finally, the existence of an η-Ricci soliton on a 3-dimensional $(k, \mu)'$-almost Kenmotsu manifold is ensured by a proper example.

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1. Introduction

On a Riemannian manifold (N, g) a Ricci soliton is given by

$$\pounds_X g + 2Ric + 2\lambda g = 0, \tag{1.1}$$

$\pounds$ being the Lie-derivative, λ a constant, Ric denotes the Ricci tensor and the vector field X is called the potential vector field. Ricci solitons are the special solutions of the Ricci flow equation

$$\frac{\partial}{\partial t} g = -2Ric, \tag{1.2}$$

which was introduced by Hamilton [16].

To extend the notion of Ricci soliton Cho and Kimura [8] initiated to introduce η-Ricci solitons. η-Ricci solitons on a Riemannian manifold (N, g) satisfies

$$\pounds_X g + 2Ric + 2\lambda g + 2\psi\eta \otimes \eta = 0, \tag{1.3}$$

where ψ is a constant. $\psi = 0$ implies the η-Ricci soliton becomes a Ricci soliton and for $\psi \neq 0$ the η-Ricci soliton is said to be proper. η-Ricci solitons have been studied by several authors such as Blaga ([1], [2], [4]), De and De [9],

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De and Haseeb [10], Haseeb and De [17], De and Sardar [24], De et. al. [19], Sarkar and Sardar ([26], [25]), Caliskan and Saglamer [6] and many others.

If the vector field X is a gradient of a smooth function $f : N \to \mathbb{R}$ (called the potential function), then the soliton is said to be a gradient η-Ricci soliton and the above equation (1.3) becomes

$$Hess f + Ric + \lambda g + \psi\eta \otimes \eta = 0, \tag{1.4}$$

where Hess f is the Hessian of f. For $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$ indicate respectively the soliton is shrinking, steady or expanding. Gradient η-Ricci solitons have been investigated by several authors such as ([3], [29], [30]) and many others.

In a Riemannian manifold (N, g) a vector field X is called torse-forming [32] if

$$\nabla_U X = fU + \omega(U)X, \tag{1.5}$$

where f is a smooth function, ω is a 1-form and ∇ is the Levi-Civita connection of g. X is called concircular [15] if $\omega = 0$ and X is called recurrent [27] if $f = 0$.

Also a vector field X is said to be an affine conformal [14] if

$$(\pounds_X \nabla)(U, V) = (U\rho)V + (V\rho)U - g(U, V)D\rho, \tag{1.6}$$

or projective [33] if

$$(\pounds_X \nabla)(U, V) = p(U)V - p(V)U, \tag{1.7}$$

$\pounds$ being the Lie-derivative, p an exact 1-form and ρ being a smooth function on N. The vector field X is called affine if ρ is constant in (1.6) and $p = 0$ in (1.7).

If X is projective, then from (1.7) we get

$$\nabla_U \nabla_V X - \nabla_{\nabla_U V} X = R(U, X)V + p(U)V + p(V)U. \tag{1.8}$$

Torse-forming vector field, concircular vector field, affine conformal vector field and projective vector field have been studied by several authors such as ([7], [11], [18], [29], [28], [32], [34]) and many others.

Inspired by the foregoing studies we are interested to characterize the above vector fields on $(k, \mu)'$-akm.

We organize the paper as follows:
After preliminaries in Section 2, we consider some vector fileds on $(k, \mu)'$-almost Kenmotsu manifolds. In the next section we study η-Ricci solitons on $(k, \mu)'$-almost Kenmotsu manifolds whose potential vector field is projective and affine conformal. Section 5 deals with the study of gradient η-Ricci solitons on $(k, \mu)'$-almost Kenmotsu manifolds. Finally, we construct an example of a 3-dimensional $(k, \mu)'$-almost Kenmotsu manifold admitting η-Ricci soliton.

2. Preliminaries

An $(2m+1)$-dimensional Riemannian manifold (N,g) is said to be an almost contact metric manifold [5] if it admits a $(1,1)$-type tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 U = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V) \tag{2.2}$$

for any vector fields U, V. The vector field ξ is called the Reeb or characteristic vector field.

Let us consider the Riemannian product $N^{2m+1} \times \mathbb{R}$ of an almost contact manifold and $\mathbb{R}$. Then we define on the product an almost complex structure J by

$$J(U, \sigma\frac{d}{dt}) = (\phi U - \sigma\xi, \eta(U)\frac{d}{dt}),$$

where U denotes a vector field tangent to N^{2m+1}, t is the coordinate of $\mathbb{R}$ and σ is a C^∞-function on $N^{2m+1} \times \mathbb{R}$. From Blair [5], the normality of an almost contact structure is expressed by the vanishing of the tensor $N_\phi = [\phi,\phi] + 2d\eta \otimes \xi$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ. An almost Kenmotsu manifold (in short, akm) is an almost contact metric manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, $\Phi(U,V) = g(U, \phi V)$. A Kenmotsu manifold is a normal akm. In an akm the relation

$$(\nabla_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U \tag{2.3}$$

holds. Also the following formulas hold in akm ([13], [12]):

$$h\xi = h'\xi = 0, \quad tr(h) = tr(h') = 0, \quad h\phi + \phi h = 0, \tag{2.4}$$

$$\nabla_U \xi = U - \eta(U)\xi + h'U, \tag{2.5}$$

where $h = \frac{1}{2}\pounds_\xi \phi$ and $h' = h \circ \phi$.

In an akm if the characteristic vector field ξ belongs to $(k,\mu)'$-nullity distribution, that is,

$$R(U,V)\xi = k(\eta(V)U - \eta(U)V) + \mu(\eta(V)h'U - \eta(U)h'V), \tag{2.6}$$

k and μ are constants, suck akm is called a $(k,\mu)'$-akm [13]. In a $(k,\mu)'$-akm (N,g) we have [13]

$$h'^2 U = -(k+1)U + (k+1)\eta(U)\xi \tag{2.7}$$

and $\mu = -2$. Equation (2.7) reflects that $h' = 0$ if and only if $k = -1$ and $h' \neq 0$ everywhere if and only if $k < -1$. It follows from (2.6) that

$$R(\xi, U)V = k(g(U,V)\xi - \eta(V)U) - 2(g(h'U,V)\xi - \eta(V)h'U) \tag{2.8}$$

for any $U, V \in \chi(N)$.

Proposition 2.1. *([31]) In a $(k,\mu)'$-akm, the Ricci operator Q is given by*

$$QU = -2mU + 2m(k+1)\eta(U)\xi - 2mh'U \tag{2.9}$$

for $k < -1$, where Q is defined by $Ric(U,V) = g(QU,V)$ and the scalar curvature $r = 2m(k-2m)$.

Proposition 2.2. *([23]) In a $(k,\mu)'$-akm, the relation*

$$(\nabla_U h')V = -g(h'U + h'^2U, V)\xi - \eta(V)(h'U + h'^2U) - (\mu+2)\eta(U)h'V \tag{2.10}$$

holds.

Proposition 2.3. *([12]) An almost Kenmotsu manifold $(\mathbf{N}^{2m+1}, \phi, \xi, \eta, g)$ is a Kenmotsu manifold if and only if $h = 0$.*

Proposition 2.4. *([12]) Let $(N^{2m+1}, \phi, \xi, \eta, g)$ be an akm and assume that $h = 0$. Then, N^{2m+1} is locally a warped product $N' \times_f M^{2m}$, where M^{2m} is an almost Kähler manifold, N' is an open interval with coordinate t, and $f = ce^t$ for some positive constant c.*

Definition 2.5. An almost Kenmotsu manifold N^{2m+1} is said to be an η-Einstein manifold if its Ricci tensor Ric is of the form

$$Ric(U,V) = a_1 g(U,V) + b_1 \eta(U)\eta(V),$$

where a_1, b_1 are scalars of which $b_1 \neq 0$.

In [20], the author proved that an η-Einstein Kenmotsu manifold is an Einstein manifold, provided b_1 = constant (or, a_1 = constant). Also, Pastore and Saltarelli [23] proved that η-Einstein (k,μ)-akm is an Einstein manifold if any one of the associated scalars is constant. Again, Mandal and De [21] studied the above result in $(k,\mu)'$-akm. They explain the following:

Proposition 2.6. *An η-Einstein $(k,\mu)'$-almost Kenmotsu manifold becomes an Einstein manifold, provided one of the associated scalars a_1 or b_1, is constant.*

3. Some special vector fields on $(k,\mu)'$-almost Kenmotsu manifolds

We assume that the vector field X is concircular in $(k,\mu)'$-akm. Then equation (1.5) infers

$$\nabla_U X = fU \tag{3.1}$$

for any $U \in \chi(N)$.

The above equation implies

$$R(U,V)X = (Uf)V - (Vf)U. \tag{3.2}$$

Taking inner product of (3.2) with the vector filed W, we get

$$g(R(U,V)X,W) = (Uf)g(V,W) - (Vf)g(U,W). \tag{3.3}$$

Contracting U and W in (3.3), we obtain

$$Ric(V,X) = -2m(Vf). \tag{3.4}$$

Putting $U = W = \xi$ in (3.3), we acquire

$$k[g(V,X) - \eta(V)\eta(X)] + \mu g(h'V,X) = (\xi f)\eta(V) - (Vf). \tag{3.5}$$

Proposition 2.1 readily gives

$$Ric(U,\xi) = 2mk\eta(U). \tag{3.6}$$

Substituting V by ξ in (3.4) and using (3.6) entails that

$$\xi f = -k\eta(X). \tag{3.7}$$

Using (3.7) in (3.5), we infer

$$(Vf) + kg(V,X) + \mu g(h'V,X) = 0, \tag{3.8}$$

which implies

$$Df + kX + \mu h'X = 0. \tag{3.9}$$

Using (3.1) from the above equation we acquire

$$\nabla_U Df = -kfU - \mu((\nabla_U h')X). \tag{3.10}$$

It is well known that

$$g(\nabla_U Df, V) = g(U, \nabla_V Df). \tag{3.11}$$

In view of (3.10) and (3.11), we get

$$\mu[g((\nabla_U h')X,V) - g((\nabla_V h')X,U)] = 0 \tag{3.12}$$

Utilizing (2.10) in (3.12) gives

$$\begin{aligned}\mu[&g(h'U,X)\eta(V) + g(h'^2U,X)\eta(V) - g(h'V,X)\eta(U)\\ &-g(h'^2V,X)\eta(U) + (\mu+2)\{g(h'X,V)\eta(U) - g(h'X,U)\eta(V)\}] = 0.\end{aligned} \tag{3.13}$$

Putting $V = \xi$ in the foregoing equation entails that

$$\mu[g(h'U,V) + g(h'^2U,X) - (\mu+2)g(h'X,U)] = 0. \tag{3.14}$$

Replacing U by ϕU in (3.14) and using (2.1), we provide

$$\mu[(\mu+1)g(hX,U) - (k+1)g(X,\phi U)] = 0. \tag{3.15}$$

Interchanging X and U in (3.15) and using (3.15), we infer

$$\mu(k+1)g(X,\phi U)=0, \tag{3.16}$$

which implies $k=-1$, since in a $(k,\mu)'$-akm, $\mu=-2$ [13]. Hence from (2.7), we get $h=0$. Therefore, from Proposition 2.3 and 2.4, we conclude the following:
Theorem 3.1. *If a $(k,\mu)'$-almost Kenmotsu manifold is endowed with a concircular vector field, then N^{2m+1} becomes a Kenmotsu manifold and N^{2m+1} is locally a warped product $N'\times_f M^{2m}$, where M^{2m} is an almost Kähler manifold, N' is an open interval with coordinate t, and $f=ce^t$ for some positive constant c.*

Theorem 3.2. *A non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold does not admit any concircular vector field.*

Let us assume that the vector field X is torse-forming in a $(k,\mu)'$-akm and $\omega=\eta$ in (1.5). Then we have

$$\nabla_U X=fU+\eta(U)X. \tag{3.17}$$

From the above equation (3.17), we get

$$\nabla_V\nabla_U X=(Vf)U+f\nabla_V U+\eta(U)\nabla_V X+(\nabla_V\eta(U))X. \tag{3.18}$$

Interchanging U and V in (3.18) gives

$$\nabla_U\nabla_V X=(Uf)V+f\nabla_U V+\eta(V)\nabla_U X+(\nabla_U\eta(V))X. \tag{3.19}$$

From (3.17), (3.18) and (3.19), we obtain

$$R(U,V)X=(Uf)V-(Vf)U+\eta(V)fU-\eta(U)fV. \tag{3.20}$$

Contracting U in the foregoing equation, we get

$$Ric(V,X)=-2m(Vf)+2mf\eta(V). \tag{3.21}$$

Taking inner product on (3.20) with ξ and using (2.6) yields

$$\begin{aligned}&k[g(V,X)\eta(U)-g(U,X)\eta(V)]\\&+\mu[g(h'V,X)\eta(U)-g(h'U,X)\eta(V)]\\&=(Uf)\eta(V)-(Vf)\eta(U).\end{aligned} \tag{3.22}$$

Putting $V=\xi$ in (3.22) gives

$$k[\eta(U)\eta(X)-g(U,X)]-\mu g(h'U,X)=(Uf)-(\xi f)\eta(U). \tag{3.23}$$

Substituting V by ξ in (3.21), we get

$$\xi f=-k\eta(X)+f. \tag{3.24}$$

Using (3.24) in (3.23), we infer

$$(Uf) + kg(U,X) - f\eta(U) + \mu g(h'X,U) = 0, \tag{3.25}$$

which implies

$$Df + kX - f\xi + \mu h'X = 0. \tag{3.26}$$

The above equation implies

$$\nabla_U Df = -k\nabla_U X + (Uf)\xi + f\nabla_U\xi - \mu((\nabla_U h')X). \tag{3.27}$$

Using (2.5), (3.17) and (3.25) in the above equation entails that

$$\begin{aligned}\nabla_U Df &= -k[fU + \eta(U)X] - [kg(U,X) - f\eta(U) + \mu g(h'X,U)]\xi \\ &\quad + f[U - \eta(U)\xi + h'U] - \mu((\nabla_U h')X).\end{aligned} \tag{3.28}$$

In view of (3.11) and (3.28), we provide

$$\begin{aligned}&\mu[g((\nabla_U h')X,V) - g((\nabla_V h')X,U) \\ &+ g(h'X,U)\eta(V) - g(h'X,V)\eta(U)] = 0.\end{aligned} \tag{3.29}$$

Again, utilizing (2.10) in (3.29) gives

$$\begin{aligned}&\mu[(\mu+1)\{g(h'U,X)\eta(V) - g(h'V,X)\eta(U)\} \\ &+(k+1)\{g(U,X)\eta(V) - g(V,X)\eta(U)\}] = 0.\end{aligned} \tag{3.30}$$

Setting $V = \xi$ in (3.30), we get

$$\mu[(\mu+1)g(h'U,X) + (k+1)\{g(U,X) - \eta(U)\eta(X)\}] = 0. \tag{3.31}$$

Replacing U by ϕU in (3.31) entails that

$$\mu[(\mu+1)\{g(hU,X) - \eta(X)\eta(U)\} - (k+1)g(\phi U,X)] = 0. \tag{3.32}$$

Interchanging U and X in (3.32) and using (3.32), we acquire

$$\mu(k+1)g(\phi U,X) = 0, \tag{3.33}$$

which implies $k = -1$, since in a $(k,\mu)'$-akm, $\mu = -2$ [13]. Hence from (2.7), we get $h = 0$. Therefore, from Proposition 2.3 and 2.4 we have:

Theorem 3.3. *If a $(k,\mu)'$-almost Kenmotsu manifold admitting a torse-forming vector field, then N^{2m+1} becomes a Kenmotsu manifold and N^{2m+1} is locally a warped product $N' \times_f M^{2m}$, where M^{2m} is an almost Kähler manifold, N' is an open interval with coordinate t, and $f = ce^t$ for some positive constant c, provided $\omega = \eta$.*

Theorem 3.4. *In a non-Kenmotsu $(k,\mu)'$-almost Kenmotsu manifold, the torse-forming vector field does not exist, provided $\omega = \eta$.*

4. η-Ricci solitons on $(k,\mu)'$-almost Kenmotsu manifolds

Let us assume that a $(k,\mu)'$-akm admit an η-Ricci soliton (g,ξ,λ,ψ). Then from (1.3), we get

$$(\pounds_\xi g)(U,V)+2Ric(U,V)+2\lambda g(U,V)+2\psi\eta(U)\eta(V)=0. \tag{4.1}$$

From (2.5), we infer

$$(\pounds_\xi g)(U,V)=2[g(U,V)-\eta(U)\eta(V)+g(h'U,V)]. \tag{4.2}$$

Using (4.2) in (4.1), we obtain

$$Ric(U,V)=-(\lambda+1)g(U,V)-(\psi-1)\eta(U)\eta(V)-g(h'U,V). \tag{4.3}$$

Putting $U=V=\xi$ and using (2.9) gives

$$\lambda+\psi=-2mk. \tag{4.4}$$

Thus we have the following theorem:

Theorem 4.1. *If a $(k,\mu)'$-almost Kenmotsu manifold admits an η-Ricci soliton, then the Ricci tensor is of the form (4.3) and the constants λ and ψ are related by $\lambda+\psi=-2mk$.*

Now, let (g,X,λ,ψ) be an η-Ricci soliton in a $(k,\mu)'$-akm such that the potential vector field X is pointwise collinear with ξ, that is, $X=b\xi$, where b is a function.

Then from (1.3), we get

$$\begin{aligned}&bg(\nabla_U\xi,V)+bg(U,\nabla_V\xi)+(Ub)\eta(V)\\&+(Vb)\eta(U)+2Ric(U,V)+2\lambda g(U,V)+2\psi\eta(U)\eta(V)=0.\end{aligned} \tag{4.5}$$

Using (2.5) in the above equation yields

$$\begin{aligned}&2b[g(U,V)-\eta(U)\eta(V)+g(h'U,V)]+(Ub)\eta(V)+(Vb)\eta(U)\\&+2Ric(U,V)+2\lambda g(U,V)+2\psi\eta(U)\eta(V)=0.\end{aligned} \tag{4.6}$$

Putting $V=\xi$ in (4.6) entails

$$(Ub)+(\xi b)\eta(U)+4nk\eta(U)+2(\lambda+\psi)\eta(U)=0. \tag{4.7}$$

Again, putting $U=\xi$ in (2.8), we get

$$\xi b=-(2mk+\lambda+\psi). \tag{4.8}$$

Using (4.8) in (4.7), we obtain

$$Ub=(2mk+\lambda+\psi)\eta(U), \tag{4.9}$$

which implies that

$$db=(2mk+\lambda+\psi)\eta. \tag{4.10}$$

Now, applying d on the foregoing equation, we infer

$$(2mk+\lambda+\psi)d\eta = 0. \tag{4.11}$$

Since $d\eta \neq 0$, hence we get

$$2mk+\lambda+\psi = 0. \tag{4.12}$$

Using (4.12) in (4.10), we obtain

$$db = 0,$$

which implies b is a constant.

Hence we conclude the following:

Theorem 4.2. *If a $(k,\mu)'$-almost Kenmotsu manifold admits an η-Ricci soliton (g, X, λ, ψ) such that the potential vector field X is pointwise collinear with ξ, then X is a constant multiple of ξ and the constants λ and ψ are related by $2mk+\lambda+\psi = 0$.*

Let us assume that the potential vector field X is an affine conformal vector field. Then from (1.6), we get

$$(\pounds_X \nabla)(U,V) = (U\rho)V + (V\rho)U - g(U,V)D\rho. \tag{4.13}$$

Equation (1.3) implies

$$(\pounds_X g)(V,W) = -2Ric(V,W) - 2\lambda g(V,W) - 2\psi\eta(V)\eta(W) \tag{4.14}$$

for any V, W. It is well known that the following formula is satisfied [33]:

$$\begin{aligned}(\pounds_X \nabla_U g - \nabla_U \pounds_X g - \nabla_{[X,U]})(V,W) &= -g((\pounds_X \nabla)(U,V),W) \\ &\quad -g((\pounds_X \nabla)(U,W),V).\end{aligned}$$

In view of the parallelism of the Riemannian metric g, the above formula becomes

$$(\nabla_U \pounds_X g)(V,W) = g((\pounds_X \nabla)(U,V),W) + g((\pounds_X \nabla)(U,W),V). \tag{4.15}$$

From (4.14), we obtain

$$\begin{aligned}(\nabla_U \pounds_X g)(V,W) &= -2(\nabla_U Ric)(V,W) \\ &\quad -2\psi[\{g(U,V) - \eta(U)\eta(V) + g(h'U,V)\}\eta(W) \\ &\quad +\{g(U,W) - \eta(U)\eta(W) + g(h'U,W)\}\eta(V)].\end{aligned} \tag{4.16}$$

Using (4.13) and (4.16) in (4.15), we get

$$\begin{aligned}(U\rho)g(V,W) &= -(\nabla_U Ric)(V,W) \\ &\quad -\psi[\{g(U,V) - \eta(U)\eta(V) + g(h'U,V)\}\eta(W) \\ &\quad +\{g(U,W) - \eta(U)\eta(W) + g(h'U,W)\}\eta(V)]\end{aligned} \tag{4.17}$$

Putting $V = W = \xi$ in the foregoing equation, we infer

$$U\rho = 0, \tag{4.18}$$

which implies ρ is a constant.

Hence we can state the following:

Theorem 4.3. *If a $(k,\mu)'$-almost Kenmotsu manifold admits an η-Ricci soliton such that the potential vector field X is an affine conformal vector field, then the potential vector field reduces to an affine vector field.*

By applying the same process as given in the proof of Theorem 4.3 we also state the following:

Theorem 4.4. *If a $(k,\mu)'$-almost Kenmotsu manifold admits an η-Ricci soliton such that the potential vector field X is a projective vector field, then the vector field reduces to an affine vector field.*

5. Gradient η-Ricci solitons on $(k,\mu)'$-almost Kenmotsu manifolds

This section is devoted to studing gradient η-Ricci solitons on $(k,\mu)'$-akm. Now equation (1.4) implies

$$\nabla_U Df = -QU - \lambda U - \psi\eta(U)\xi. \tag{5.1}$$

Using (5.1) and after some calculations, we obtain

$$\begin{aligned} R(U,V)Df &= -[(\nabla_U Q)V - (\nabla_V Q)U] \\ &\quad -\psi[\eta(V)U - \eta(U)V + \eta(V)h'U - \eta(U)h'V]. \end{aligned} \tag{5.2}$$

Now from (2.9), we have

$$\begin{aligned} (\nabla_U Q)V - (\nabla_V Q)U &= 2m(k+1)[\eta(V)U - \eta(U)V + \eta(V)h'U \\ &\quad -\eta(U)h'V] - 2m[(\nabla_U h')V - (\nabla_V h')U]. \end{aligned} \tag{5.3}$$

Using Proposition 2.2 in the above equation entails that

$$(\nabla_U Q)V - (\nabla_V Q)U = 2m(k+1)[\eta(V)U - \eta(U)V + \eta(V)h'U - \eta(U)h'V]. \tag{5.4}$$

Using (5.4) in (5.2), we infer

$$R(U,V)Df = -[2m(k+1) + \psi][\eta(V)U - \eta(U)V + \eta(V)h'U - \eta(U)h'V]. \tag{5.5}$$

Contracting (5.5), we get

$$Ric(V, Df) = -2m[2m(k+1) + \psi]. \tag{5.6}$$

Equation (2.9) can be written as

$$Ric(U,V) = -2mg(U,V) + 2m(k+1)\eta(U)\eta(V) - 2mg(h'U,V). \tag{5.7}$$

Replacing U by Df in (5.7), we get

$$Vf = [2m(k+1)+\psi]\eta(V) + (k+1)(\xi f)\eta(V) - (h'Vf). \tag{5.8}$$

Putting $V=\xi$ in the above equation gives

$$-k(\xi f) = 2m(k+1)+\psi. \tag{5.9}$$

Taking inner product of (5.5) with ξ, we obtain

$$k[\eta(V)(Uf) - \eta(U)(Vf)] + \mu[\eta(V)(h'Uf) - \eta(U)(h'Vf)] = 0. \tag{5.10}$$

Putting $U=\xi$ in the foregoing equation, we get

$$k\eta(V)(\xi f) - k(Vf) - \mu(h'Vf) = 0. \tag{5.11}$$

In view of (5.8) and (5.11) and using (5.9), we infer

$$(k-\mu)[k(Vf) + \{2m(k+1)+\psi\}\eta(V)] = 0. \tag{5.12}$$

It follows that either $k=\mu$ or,

$$k(Vf) + \{2m(k+1)+\psi\}\eta(V) = 0. \tag{5.13}$$

If we take $2m(k+1)+\psi = 0$, then from (5.12) we get $Vf=0$, since $k \neq \mu$ in general. Hence f is constant and so from (5.1) we get it is an η-Einstein manifold. Hence from Proposition 2.5, we can state the following:

Theorem 5.1. *If the metric of a $(k,\mu)'$-almost Kenmotsu manifold is a gradient η-Ricci soliton, then it is an Einstein manifold, provided $2m(k+1)+\psi=0$.*

6. Example

We consider the 3-dimensional manifold $N^3 = \{(x,y,z) \in \mathbb{R}^3\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let ξ, e_2, e_3 be three vector fields in $\mathbb{R}^3$ which satisfies [13]

$$[\xi, e_2] = -e_2 - e_3, \quad [\xi, e_3] = -e_2 - e_3, \quad [e_2, e_3] = 0.$$

Let g be the Riemannian metric defined by

$$g(\xi,\xi) = g(e_2,e_2) = g(e_3,e_3) = 1 \quad \textit{and} \quad g(\xi,e_2) = g(\xi,e_3) = g(e_2,e_3) = 0.$$

Let η be the 1-form defined by $\eta(W) = g(W,\xi)$, for any $W \in \chi(N^3)$.

Let ϕ be the (1,1)-tensor field defined by

$$\phi\xi = 0, \quad \phi e_2 = e_3, \quad \phi e_3 = -e_2.$$

Then using the linearity of ϕ and g, we have

$$\eta(\xi) = 1,$$

$$\phi^2 U = -U + \eta(U)\xi,$$

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)$$

for any $U, V \in \chi(N^3)$. Thus the structure (ϕ, ξ, η, g) is an almost contact structure.

Moreover, $h'\xi = 0, \quad h'e_2 = e_3 \quad and \quad h'e_3 = e_2$.

In [22] the authors obtained the expression of the curvature tensor and the Ricci tensor as follows:

$$R(\xi, e_2)\xi = 2(e_2 + e_3), \quad R(\xi, e_2)e_2 = -2\xi, \quad R(\xi, e_2)e_3 = -2\xi,$$

$$R(e_2, e_3)\xi = R(e_2, e_3)e_2 = R(e_2, e_3)e_3 = 0,$$

$$R(\xi, e_3)\xi = 2(e_2 + e_3), \quad R(\xi, e_3)e_2 = -2\xi, \quad R(\xi, e_3)e_3 = -2\xi.$$

With help of the expressions of the curvature tensor, we conclude that the characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution with $k = -2$ and $\mu = -2$.

Using the expression of the curvature tensor, we find the values of the Ricci tensor as follows:

$$Ric(\xi, \xi) = -4, \quad Ric(e_2, e_2) = Ric(e_3, e_3) = -2.$$

From (4.3) we obtain $Ric(\xi, \xi) = -(\lambda + \psi), \quad Ric(e_2, e_2) = -(\lambda + 1) \quad and \quad Ric(e_3, e_3) = -(\lambda + 1)$.

Therefore $\psi = 3$ and $\lambda = 1$. The data (g, ξ, λ, ψ) defines an η-Ricci soliton on $(k, \mu)'$-akm.

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