Novi Sad J. Math. Vol. 54, No. 1, 2024, 55-65
https://doi.org/10.30755/NSJOM.12981

On almost pseudo M-Projectively symmetric Riemannian manifold

Mohabbat Ali[1,2], Quddus Khan[3] and Mohd Vasiulla[4]

Abstract. In this paper, we have studied an almost pseudo M-projectively symmetric Riemannian manifold and obtained some interesting and fruitful results on it.

AMS Mathematics Subject Classification (2010): 53C15; 53C25

Key words and phrases: M-projective curvature tensor; almost pseudo symmetric manifold; Einstein manifold; quasi-Einstein manifold; Codazzi type of Ricci tensor; Ricci symmetric manifold

1. Introduction

Let (M^n, g) be a Riemannian manifold of dimension n with the Riemannian metric g and ∇ be the Levi-Civita connection with respect to the metric g. In 1971, Pokhariyal and Mishra [8] introduced and studied a new curvature tensor of type (1,3) in an n-dimensional Riemannian manifold known as M-projective curvature tensor (M) and defined by

$$\begin{aligned} M(X,Y,Z) = K(X,Y,Z) - \frac{1}{2(n-1)}&[\mathrm{Ric}(Y,Z)X - \mathrm{Ric}(X,Z)Y \\ &+ g(Y,Z)R(X) - g(X,Z)R(Y)], \end{aligned} \tag{1.1}$$

where K denotes the Riemannian curvature tensor of type (1,3), Ric denotes Ricci tensor of type (0,2) and R denotes Ricci tensor of type (1,1), and defined by

$$\mathrm{Ric}(X,Y) = g(R(X),Y). \tag{1.2}$$

Consequently (1.1) gives

$$\begin{aligned} \tilde{M}(X,Y,Z,V) = \tilde{K}(X,Y,Z,V) - \frac{1}{2(n-1)}&[\mathrm{Ric}(Y,Z)g(X,V) \\ &- \mathrm{Ric}(X,Z)g(Y,V) + \mathrm{Ric}(X,V)g(Y,Z) \\ &- \mathrm{Ric}(Y,V)g(X,Z)], \end{aligned} \tag{1.3}$$

[1]Department of Applied Sciences & Humanities, Jamia Millia Islamia, New Delhi, e-mail: ali.math509@gmail.com , ORCID iD: orcid.org/0000-0001-8518-4942
[2]Corresponding author
[3]Department of Applied Sciences & Humanities, Jamia Millia Islamia, New Delhi, e-mail: qkhan@jmi.ac.in, ORCID iD: orcid.org/0000-0002-7731-4732
[4]Department of Applied Sciences & Humanities, Jamia Millia Islamia, New Delhi, e-mail: vsmlk45@gmail.com, ORCID iD: orcid.org/0000-0002-4214-8148

where $\tilde{K}$ and $\tilde{M}$ denote the Riemannian curvature tensor of type (0,4) and M-projective curvature tensor of type (0,4), respectively, and are defined by

$$\tilde{K}(X,Y,Z,V) = g(K(X,Y,Z),V), \tag{1.4}$$

and

$$\tilde{M}(X,Y,Z,V) = g(M(X,Y,Z),V). \tag{1.5}$$

From (1.3), we have

$$\begin{cases} \tilde{M}(X,Y,Z,V) = -\tilde{M}(Y,X,Z,V), \\ \tilde{M}(X,Y,Z,V) = -\tilde{M}(X,Y,V,Z), \\ \tilde{M}(X,Y,Z,V) = \tilde{M}(Z,V,X,Y) \end{cases} \tag{1.6}$$

and

$$\tilde{M}(X,Y,Z,V) + \tilde{M}(Y,Z,X,V) + \tilde{M}(Z,X,Y,V) = 0. \tag{1.7}$$

If $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and i running from 1 to n, then, from (1.3), we have

$$\sum_{i=1}^{n} \tilde{M}(X,Y,e_i,e_i) = 0 = \sum_{i=1}^{n} \tilde{M}(e_i,e_i,Z,V) \tag{1.8}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \tilde{M}(e_i,Y,Z,e_i) &= \sum_{i=1}^{n} \tilde{M}(Y,e_i,e_i,Z) \\ &= \frac{n}{2(n-1)}[\mathrm{Ric}(Y,Z) - \frac{r}{n}g(Y,Z)], \end{aligned} \tag{1.9}$$

where $r = \sum_{i=1}^{n} \mathrm{Ric}(e_i,e_i)$ is the scalar curvature.

It is known that in Riemannian manifold the curvature tensor K satisfies the following

$$(\mathrm{div}K)(X,Y,Z) = (\nabla_X \mathrm{Ric})(Y,Z) - (\nabla_Y \mathrm{Ric})(X,Z), \tag{1.10}$$

where $(\mathrm{div}K)(X,Y,Z) = \sum_{i=1}^{n} g((\nabla_{e_i} K)(X,Y,Z),e_i)$ and 'div' denotes the divergence.

A non-flat Riemannian manifold (M^n,g), $(n>2)$, is called a pseudo symmetric manifold [2] if its curvature tensor $\tilde{K}$ satisfies the following condition:

$$\begin{aligned} (\nabla_U \tilde{K})(X,Y,Z,V) &= 2\alpha(U)\tilde{K}(X,Y,Z,V) + \alpha(X)\tilde{K}(U,Y,Z,V) \\ &\quad + \alpha(Y)\tilde{K}(X,U,Z,V) + \alpha(Z)\tilde{K}(X,Y,U,V) \\ &\quad + \alpha(V)\tilde{K}(X,Y,Z,U), \end{aligned} \tag{1.11}$$

for all differentiable vector fields X, Y, Z, U, V and α is called the associated 1-form, and is defined by

$$g(X,P) = \alpha(X). \tag{1.12}$$

Such a manifold is denoted by $(PS)_n$. If $\alpha = 0$ in (1.11), then the pseudo symmetric manifold becomes a locally symmetric manifold.

A non-flat Riemannian manifold (M^n, g), $(n > 2)$, whose M-projective curvature tensor $\tilde{M}$ satisfies the following condition:

$$\begin{aligned}(\nabla_U \tilde{M})(X,Y,Z,V) &= 2\alpha(U)\tilde{M}(X,Y,Z,V) + \alpha(X)\tilde{M}(U,Y,Z,V)\\ &\quad + \alpha(Y)\tilde{M}(X,U,Z,V) + \alpha(Z)\tilde{M}(X,Y,U,V)\\ &\quad + \alpha(V)\tilde{M}(X,Y,Z,U).\end{aligned} \tag{1.13}$$

is called a pseudo M-projectively symmetric manifold and is denoted by $(PMPS)_n$.

De and Ghazi [3] introduced the notion of almost pseudo symmetric manifolds. A Riemannian manifold (M^n, g), $(n > 2)$, is said to be an almost pseudo symmetric manifold if its curvature tensor $\tilde{K}$ of type (0,4) satisfies the following condition:

$$\begin{aligned}(\nabla_U \tilde{K})(X,Y,Z,V) &= [\alpha(U) + \beta(U)]\tilde{K}(X,Y,Z,V) + \alpha(X)\tilde{K}(U,Y,Z,V)\\ &\quad + \alpha(Y)\tilde{K}(X,U,Z,V) + \alpha(Z)\tilde{K}(X,Y,U,V)\\ &\quad + \alpha(V)\tilde{K}(X,Y,Z,U),\end{aligned} \tag{1.14}$$

where α and β are non-zero 1-forms defined by $g(X,P) = \alpha(X)$, $g(X,Q) = \beta(X)$. Such a manifold is denoted by $(APS)_n$. Here the vector fields P and Q are called the basic vector fields of the manifold corresponding to the associated 1-forms α and β, respectively. If the basic vector fields P and Q are orthonormal, then

$$\alpha(P) = 1, \quad \beta(Q) = 1 \quad \textit{and} \quad g(P,Q) = 0. \tag{1.15}$$

If $\alpha = \beta$ in (1.14) , then the $(APS)_n$ reduces to a $(PS)_n$.

A Riemannian manifold is said to be almost pseudo M-projectively symmetric manifold (M^n, g), if the following condition is satisfied

$$\begin{aligned}(\nabla_U \tilde{M})(X,Y,Z,V) &= [\alpha(U) + \beta(U)]\tilde{M}(X,Y,Z,V) + \alpha(X)\tilde{M}(U,Y,Z,V)\\ &\quad + \alpha(Y)\tilde{M}(X,U,Z,V) + \alpha(Z)\tilde{M}(X,Y,U,V)\\ &\quad + \alpha(V)\tilde{M}(X,Y,Z,U),\end{aligned} \tag{1.16}$$

where α and β are as stated earlier. Such a manifold is denoted by $(APMPS)_n$.

A Riemannian manifold is said to be an Einstein manifold if [7]

$$\mathrm{Ric}(X,Y) = \frac{r}{n} g(X,Y). \tag{1.17}$$

A non-flat Riemannian manifold is said to be a quasi-Einstein manifold if [4]

$$\mathrm{Ric}(X,Y) = a g(X,Y) + b\alpha(X)\alpha(Y), \tag{1.18}$$

where a and b are scalar functions.

The Ricci tensor of type (0,2) is said to be of Codazzi type of Ricci tensor if it satisfies the condition ([1], p.- 355)

$$(\nabla_X \mathrm{Ric})(Y,Z) = (\nabla_Z \mathrm{Ric})(Y,X). \tag{1.19}$$

From (1.19), it follows that

$$\mathrm{dr}(X) = 0. \tag{1.20}$$

The above results will be used in next sections.

2. $(APMPS)_n$, $(n > 2)$ with constant scalar curvature

Taking covariant derivative of (1.3) with respect to U and using (1.16), we have

$$\begin{aligned}(\nabla_U \tilde{K})(X,Y,Z,V) = {} & \frac{1}{2(n-1)}[(\nabla_U \mathrm{Ric})(Y,Z)g(X,V) - (\nabla_U \mathrm{Ric})(X,Z)g(Y,V) \\ & + (\nabla_U \mathrm{Ric})(X,V)g(Y,Z) - (\nabla_U \mathrm{Ric})(Y,V)g(X,Z)] \\ & + [\alpha(U)+\beta(U)]\tilde{M}(X,Y,Z,V) + \alpha(X)\tilde{M}(U,Y,Z,V) \\ & + \alpha(Y)\tilde{M}(X,U,Z,V) + \alpha(Z)\tilde{M}(X,Y,U,V) \\ & + \alpha(V)\tilde{M}(X,Y,Z,U).\end{aligned} \tag{2.1}$$

Contracting (2.1) over U and V and then using (1.8) and (1.9), we get

$$\begin{aligned}(\mathrm{div}K)(X,Y,Z) = {} & \frac{1}{2(n-1)}[(\nabla_X \mathrm{Ric})(Y,Z) - (\nabla_Y \mathrm{Ric})(X,Z) \\ & + \frac{1}{2}g(Y,Z)\mathrm{dr}(X) - \frac{1}{2}g(X,Z)\mathrm{dr}(Y)] \\ & + 2\alpha(M(X,Y,Z)) + \beta(M(X,Y,Z)) \\ & + \frac{n}{2(n-1)}\alpha(X)[\mathrm{Ric}(Y,Z) - \frac{r}{n}g(Y,Z)] \\ & - \frac{n}{2(n-1)}\alpha(Y)[\mathrm{Ric}(X,Z) - \frac{r}{n}g(X,Z)].\end{aligned} \tag{2.2}$$

In view of (1.10) the relation (2.2) gives

$$\begin{aligned}& \frac{(2n-3)}{2(n-1)}[(\nabla_X \mathrm{Ric})(Y,Z) - (\nabla_Y \mathrm{Ric})(X,Z)] \\ & = \frac{1}{4(n-1)}[g(Y,Z)\mathrm{dr}(X) - g(X,Z)\mathrm{dr}(Y)] \\ & + 2\alpha(M(X,Y,Z)) + \beta(M,X,Y,Z) \\ & + \frac{n}{2(n-1)}\alpha(X)[Ric(Y,Z) - \frac{r}{n}g(Y,Z)] \\ & - \frac{n}{2(n-1)}\alpha(Y)\{\mathrm{Ric}(X,Z) - \frac{r}{n}g(X,Z)\}].\end{aligned} \tag{2.3}$$

Taking $Y = Z = e_i$ in (2.3), we get

$$\frac{(2n-3)}{2(n-1)}\Big(\frac{\mathrm{dr}(X)}{2}\Big) = \frac{(n-\frac{1}{2})}{4(n-1)}\mathrm{dr}(X) - \frac{n}{2(n-1)}[\alpha(R(X)) - \frac{r}{n}\alpha(X)]. \tag{2.4}$$

As the scalar curvature r is constant, then the relation (2.4) reduces to

$$\alpha(R(X)) = \frac{r}{n}\alpha(X).$$

Consequently in view of (1.12), the above relation gives

$$\mathrm{Ric}(X,\rho) = \frac{r}{n}g(X,\rho).$$

This leads to the following:

Theorem 2.1. *In an almost pseudo M-projectively symmetric Riemannian manifold of constant scalar curvature,* $\frac{r}{n}$ *is an eigenvalue of Ricci tensor* Ric *corresponding to the eigenvector* ρ.

3. $(APMPS)_n$, $(n > 2)$ with Codazzi type of Ricci tensor

We suppose that $2\alpha(M(X,Y,Z)) + \beta(M(X,Y,Z)) = 0$, then by virtue of (1.19) and (1.20) the relation (2.3) reduces to

$$\begin{aligned}&\frac{n}{2(n-1)}\alpha(X)[\mathrm{Ric}(Y,Z) - \frac{r}{n}g(Y,Z)]\\ &\qquad - \frac{n}{2(n-1)}\alpha(Y)[\mathrm{Ric}(X,Z) - \frac{r}{n}g(X,Z)] = 0.\end{aligned} \tag{3.1}$$

Putting $X = P$ in (3.1), we get

$$t[\mathrm{Ric}(Y,Z) - \frac{r}{n}g(Y,Z)] = \mathrm{Ric}(Y,Z) - \frac{r}{n}\alpha(Y)\alpha(Z), \tag{3.2}$$

where t=$\alpha(P)$ is a non-zero scalar.
From (3.2) we have

$$\mathrm{Ric}(Y,Z) = \frac{rt}{n(t-1)}g(Y,Z) - \frac{r}{n(t-1)}\alpha(Y)\alpha(Z). \tag{3.3}$$

This can be written as

$$\mathrm{Ric}(Y,Z) = ag(Y,Z) + b\alpha(Y)\alpha(Z);\ a = \frac{rt}{n(t-1)}\ \textit{and}\ b = -\frac{r}{n(t-1)} \tag{3.4}$$

which is a quasi-Einstein manifold. Thus, we have the following result:

Theorem 3.1. *Consider an almost pseudo M-projectively symmetric manifold whose Ricci tensor is of Codazzi type. The manifold is a quasi-Einstein manifold, provided* $2\alpha(M(X,Y,Z)) + \beta(M(X,Y,Z)) = 0$.

Let us suppose that the M-projective curvature tensor $M(X,Y,Z,V)$ satisfies Bianchi's second identity, that is,

$$(\nabla_U \tilde{M})(X,Y,Z,V) + (\nabla_X \tilde{M})(U,Y,Z,V) + (\nabla_Y \tilde{M})(X,U,Z,V) = 0. \tag{3.5}$$

Taking cyclic sum of (1.16) over U, X, Y and then using (3.5), we get

$$\begin{aligned}
&[\alpha(U)+\beta(U)]\tilde{M}(X,Y,Z,V) + \alpha(X)\tilde{M}(U,Y,Z,V)\\
&+\alpha(Y)\tilde{M}(X,U,Z,V) + \alpha(Z)\tilde{M}(X,Y,U,V) + \alpha(V)\tilde{M}(X,Y,Z,U)\\
&+[\alpha(X)+\beta(X)]\tilde{M}(Y,U,Z,V) + \alpha(U)\tilde{M}(Y,X,Z,V)\\
&+\alpha(X)\tilde{M}(U,Y,Z,V) + \alpha(Z)\tilde{M}(Y,U,X,V) + \alpha(V)\tilde{M}(Y,U,Z,X)\\
&+[\alpha(Y)+\beta(Y)]\tilde{M}(U,X,Z,V) + \alpha(Y)\tilde{M}(X,U,Z,V)\\
&+\alpha(U)\tilde{M}(Y,X,Z,V) + \alpha(Z)\tilde{M}(U,X,Y,V) + \alpha(V)\tilde{M}(U,X,Z,Y) = 0.
\end{aligned} \tag{3.6}$$

In view of (1.6) and (1.7) the relation (3.6) gives

$$\begin{aligned}
&[\beta(U)-\alpha(U)]\tilde{M}(X,Y,Z,V) + [\beta(X)-\alpha(X)]\tilde{M}(Y,U,Z,V)\\
&\qquad + [\beta(Y)-\alpha(Y)]\tilde{M}(U,X,Z,V) = 0
\end{aligned} \tag{3.7}$$

which implies

$$\begin{aligned}
&\eta(U)\tilde{M}(X,Y,Z,V) + \eta(X)\tilde{M}(Y,U,Z,V)\\
&\qquad + \eta(Y)\tilde{M}(U,X,Z,V) = 0,
\end{aligned} \tag{3.8}$$

where $\eta(U) = \beta(U) - \alpha(U) = g(U,\rho)$.
Contracting (3.8) over U and V and using (1.9), we get

$$\begin{aligned}
&\eta(M(X,Y,Z)) - \frac{n}{2(n-1)}\eta(X)\{\mathrm{Ric}(Y,Z) - \frac{r}{n}g(Y,Z)\}\\
&\qquad + \frac{n}{2(n-1)}\eta(Y)\{\mathrm{Ric}(X,Z) - \frac{r}{n}g(X,Z)\} = 0.
\end{aligned} \tag{3.9}$$

Using (1.17) in (3.9), we have

$$\eta(M(X,Y,Z)) = 0. \tag{3.10}$$

Now putting $U = \rho$ in (3.8) and using (3.10), we get

$$\eta(\rho)M(X,Y,Z) = 0. \tag{3.11}$$

Hence, either the manifold is M-projectively flat, or $\eta(\rho) = 0$. But an M-projectively flat manifold is of constant curvature. Then ρ is a null vector field or the manifold is of constant curvature.

Thus we can state the following theorem:

Theorem 3.2. *In an Einstein $(APMPS)_n$, $(n > 2)$, if the M-projective curvature tensor satisfies Bianchi's second identity, then the manifold is either a manifold of constant curvature, or the vector field ρ defined by $g(U,\rho) = \beta(U) - \alpha(U)$ is a null vector field.*

4. Ricci symmetric $(APMPS)_n$, $(n > 2)$

Contracting (2.1) over X and V and using (1.9), we get

$$\begin{aligned}(\nabla_U \mathrm{Ric})(Y,Z) \\ = {} & \frac{1}{2(n-1)}[(n-1)(\nabla_U \mathrm{Ric})(Y,Z) + g(Y,Z)\mathrm{dr}(U) \\ & -(\nabla_U \mathrm{Ric})(Y,Z)] + \frac{n}{2(n-1)}[\alpha(U)+\beta(U)][\mathrm{Ric}(Y,Z) - \frac{r}{n}g(Y,Z)] \\ & +\alpha(M(U,Y,Z)) + \frac{n}{2(n-1)}\alpha(Y)[\mathrm{Ric}(U,Z) - \frac{r}{n}g(U,Z)] \\ & +\frac{n}{2(n-1)}\alpha(Z)[\mathrm{Ric}(Y,U) - \frac{r}{n}g(Y,U)] + \alpha(M(U,Z,Y)).\end{aligned} \tag{4.1}$$

If the manifold is Ricci symmetric then

$$(\nabla_U \mathrm{Ric})(Y,Z) = 0 \quad \forall \quad U,Y,Z \tag{4.2}$$

which on contraction over Y and Z gives

$$\mathrm{dr}(U) = 0. \tag{4.3}$$

Using (4.2) and (4.3) in (4.1), we have

$$\begin{aligned} & \frac{n}{2(n-1)}[\alpha(U)+\beta(U)][\mathrm{Ric}(Y,Z) - \frac{r}{n}g(Y,Z)] \\ & + \alpha(M(U,Y,Z)) + \frac{n}{2(n-1)}\alpha(Y)[\mathrm{Ric}(U,Z) - \frac{r}{n}g(U,Z)] \\ & + \frac{n}{2(n-1)}\alpha(Z)[\mathrm{Ric}(Y,U) - \frac{r}{n}g(Y,U)] + \alpha(M(U,Z,Y)) = 0.\end{aligned} \tag{4.4}$$

Taking $Y = Z = e_i$ in (4.4), we get

$$\alpha(R(U)) = \frac{r}{n}\alpha(U). \tag{4.5}$$

Putting $Z = P$ in (4.4) and using (1.15), (4.5) we get

$$\mathrm{Ric}(Y,U) = \frac{r}{n}g(Y,U).$$

Hence the manifold is an Einstein manifold.

This leads to the following theorem:

Theorem 4.1. *A Ricci symmetric $(APMPS)_n$, $(n > 2)$, is an Einstein manifold provided the basic vector fields are orthonormal vector fields.*

Taking covariant derivative of (1.3) with respect to U and using the condition of Ricci symmetric manifold, we get

$$(\nabla_U \tilde{M})(X,Y,Z,V) = (\nabla_U \tilde{K})(X,Y,Z,V).$$

This leads to the following:

Theorem 4.2. *Every Ricci symmetric $(APMPS)_n$, $(n > 2)$, is an almost pseudo symmetric manifold.*

Taking covariant derivative of (1.3) with respect to U and then taking cyclic sum with respect to U, X and Y, we get

$$\begin{aligned}
&(\nabla_U \tilde{M})(X,Y,Z,V) + (\nabla_X \tilde{M})(Y,U,Z,V) + (\nabla_Y \tilde{M})(U,X.Z,V)\\
&\quad =(\nabla_U \tilde{K})(X,Y,Z,V) + (\nabla_X \tilde{K})(Y,U,Z,V) + (\nabla_Y \tilde{K})(U,X.Z,V)\\
&\qquad - \frac{1}{2(n-1)}[(\nabla_U \mathrm{Ric})(Y,Z)g(X,V) + (\nabla_Y \mathrm{Ric})(X,Z)g(U,V)\\
&\qquad + (\nabla_X \mathrm{Ric})(U,Z)g(Y,V) - (\nabla_U \mathrm{Ric})(X,Z)g(Y,V)\\
&\qquad - (\nabla_X \mathrm{Ric})(Y,Z)g(U,V) - (\nabla_Y \mathrm{Ric})(U,Z)g(X,V)\\
&\qquad + (\nabla_U \mathrm{Ric})(X,V)g(Y,Z) + (\nabla_X \mathrm{Ric})(Y,V)g(U,Z)\\
&\qquad + (\nabla_Y Ric)(U,V)g(X,Z) - (\nabla_U Ric)(Y,V)g(X,Z)\\
&\qquad - (\nabla_Y \mathrm{Ric})(X,V)g(U,Z) - (\nabla_X \mathrm{Ric})(U,V)g(Y,Z)]
\end{aligned} \tag{4.6}$$

which in view of Bianchi's second identity, the above relation gives

$$\begin{aligned}
&(\nabla_U \tilde{M})(X,Y,Z,V) + (\nabla_X \tilde{M})(Y,U,Z,V) + (\nabla_Y \tilde{M})(U,X.Z,V)\\
&\qquad - \frac{1}{2(n-1)}[(\nabla_U \mathrm{Ric})(Y,Z)g(X,V) + (\nabla_Y \mathrm{Ric})(X,Z)g(U,V)\\
&\qquad + (\nabla_X \mathrm{Ric})(U,Z)g(Y,V) - (\nabla_U \mathrm{Ric})(X,Z)g(Y,V)\\
&\qquad - (\nabla_X \mathrm{Ric})(Y,Z)g(U,V) - (\nabla_Y \mathrm{Ric})(U,Z)g(X,V)\\
&\qquad + (\nabla_U \mathrm{Ric})(X,V)g(Y,Z) + (\nabla_X \mathrm{Ric})(Y,V)g(U,Z)\\
&\qquad + (\nabla_Y Ric)(U,V)g(X,Z) - (\nabla_U Ric)(Y,V)g(X,Z)\\
&\qquad - (\nabla_Y \mathrm{Ric})(X,V)g(U,Z) - (\nabla_X \mathrm{Ric})(U,V)g(Y,Z)].
\end{aligned} \tag{4.7}$$

If the manifold is a Ricci symmetric manifold [6], then the relation (4.7) reduces to

$$(\nabla_U \tilde{M})(X,Y,Z,V) + (\nabla_X \tilde{M})(Y,U,Z,V) + (\nabla_Y \tilde{M})(U,X.Z,V) = 0.$$

Hence we can state the following theorem:

Theorem 4.3. *In a Ricci symmetric $(APMPS)_n$, $(n > 2)$, the M-projective curvature tensor satisfies Bianchi's second identity.*

5. Existence of an $(APMPS)_n$

We define a Riemannian metric g on the 4-dimensional real number space $\mathbb{R}^4$ by the relation

$$ds^2 = g_{ij}dx^i dx^j = x^1[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] - (dx^4)^2, \tag{5.1}$$

where $i, j = 1, 2, 3, 4$. Then the non-vanishing components of covariant and contravariant metric tensor are

$$g_{11} = g_{22} = g_{33} = x^1,\ g_{44} = -1 \tag{5.2}$$

and

$$g^{11} = g^{22} = g^{33} = \frac{1}{x^1},\ g^{44} = -1 \tag{5.3}$$

In the metric considered the only non-vanishing components of the Christoffel symbols are (see [5])

$$\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{1}{2x^1}, \tag{5.4}$$

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = -\frac{1}{2x^1}. \tag{5.5}$$

The non-zero derivatives of equations (5.4) and (5.5) are as follows:

$$\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = -\frac{1}{2(x^1)^2}, \tag{5.6}$$

$$\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{1}{2(x^1)^2}. \tag{5.7}$$

The Riemannian curvature tensor is as follows:

$$R^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=II} \tag{5.8}$$

The non-zero components of (I) in (5.8) are as follows:

$$K^1_{221} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = -\frac{1}{2(x^1)^2},$$
$$K^1_{331} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = -\frac{1}{2(x^1)^2}$$

and the non-zero components of (II) in (5.8) are:

$$K^1_{221} = \begin{Bmatrix} m \\ 21 \end{Bmatrix}\begin{Bmatrix} 1 \\ m2 \end{Bmatrix} - \begin{Bmatrix} m \\ 22 \end{Bmatrix}\begin{Bmatrix} 1 \\ m1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 21 \end{Bmatrix}\begin{Bmatrix} 1 \\ 12 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 22 \end{Bmatrix}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{1}{4(x^1)^2},$$

$$K^1_{331} = \begin{Bmatrix} m \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 1 \\ m1 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ 13 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 33 \end{Bmatrix}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{1}{4(x^1)^2},$$

$$K^2_{332} = \begin{Bmatrix} m \\ 32 \end{Bmatrix}\begin{Bmatrix} 2 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 32 \end{Bmatrix}\begin{Bmatrix} 2 \\ 13 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{1}{4(x^1)^2}.$$

Now using these components in (5.8), we get

$$K^1_{221} = K^1_{331} = -\frac{1}{4(x^1)^2} \qquad \textit{and} \quad K^2_{332} = \frac{1}{4(x^1)^2}.$$

Thus the non-vanishing components of the Riemannian curvature tensor of type (0,4), up to symmetry, are

$$\tilde{K}_{1221} = \tilde{K}_{1331} = -\frac{1}{2x^1}, \quad \tilde{K}_{2332} = \frac{1}{4x^1},$$

and the Ricci tensor of type (0,2)

$$\mathrm{Ric}_{11} = g^{ij}K_{1ij1} = -\frac{1}{(x^1)^2},$$

$$\mathrm{Ric}_{22} = g^{ij}K_{2ij2} = -\frac{1}{4(x^1)^2},$$

$$\mathrm{Ric}_{33} = g^{ij}K_{3ij3} = -\frac{1}{4(x^1)^2},$$

$$\mathrm{Ric}_{44} = g^{ij}K_{4ij4} = 0.$$

The scalar curvature r is

$$r = g^{11}\mathrm{Ric}_{11} + g^{22}\mathrm{Ric}_{22} + g^{33}\mathrm{Ric}_{33} + g^{44}\mathrm{Ric}_{44} = -\frac{3}{2(x^1)^3}.$$

Now, the non-vanishing components of the M-projective curvature tensor are as follows:

$$\tilde{M}_{1221} = \tilde{M}_{1331} = -\frac{7}{24x^1}, \quad \tilde{M}_{2332} = \frac{1}{3x^1}, \tag{5.9}$$

and their covariant derivatives

$$\tilde{M}_{1221,1} = \tilde{M}_{1331,1} = \frac{7}{24(x^1)^2}, \quad \tilde{M}_{2332,1} = -\frac{1}{3(x^1)^2}, \tag{5.10}$$

where ‘,’ denotes the covariant derivative with respect to the metric tensor.

Note that the associated 1-forms are as follows:

$$\alpha_i(x) = \begin{cases} 0, & \text{if i=1} \\ x^1, & \text{otherwise} \end{cases}, \quad \beta_i(x) = \begin{cases} -\frac{1}{x^1}, & \text{if i=1} \\ -x^1, & \text{otherwise,} \end{cases} \tag{5.11}$$

at any point $x \in \mathbb{R}^4$. To verify the relation (1.16), it is sufficient to check the following equations:

$$\tilde{M}_{1221,1} = [\alpha_1 + \beta_1]\tilde{M}_{1221} + \alpha_1\tilde{M}_{1221} + \alpha_2\tilde{M}_{1121} + \alpha_2\tilde{M}_{1211} + \alpha_1\tilde{M}_{1221}, \tag{5.12}$$

$$\tilde{M}_{1331,1} = [\alpha_1 + \beta_1]\tilde{M}_{1331} + \alpha_1\tilde{M}_{1331} + \alpha_3\tilde{M}_{1131} + \alpha_3\tilde{M}_{1311} + \alpha_1\tilde{M}_{1331}, \tag{5.13}$$

and

$$\tilde{M}_{2332,1} = [\alpha_1 + \beta_1]\tilde{M}_{2332} + \alpha_2\tilde{M}_{1332} + \alpha_3\tilde{M}_{2132} + \alpha_3\tilde{M}_{2312} + \alpha_2\tilde{M}_{2331}. \tag{5.14}$$

Since for the other cases relation (1.16) holds trivially. By (5.9), (5.10) and (5.11), we get

$$\begin{aligned}
R.H.S. \quad of \quad (5.12) &= [\alpha_1 + \beta_1]\tilde{M}_{1221} + \alpha_1 \tilde{M}_{1221} + \alpha_1 \tilde{M}_{1221} \\
&= [3\alpha_1 + \beta_1]\tilde{M}_{1221} \\
&= 3(0)\Big(-\frac{7}{24x^1}\Big) + \Big(-\frac{1}{x^1}\Big) \times \Big(-\frac{7}{24x^1}\Big) \\
&= \frac{7}{24(x^1)^2} \\
&= \tilde{M}_{1221,1} \\
&= L.H.S. \quad of \quad (5.12)
\end{aligned}$$

By a similar argument it can be shown that (5.13) and (5.14) are also true. So the manifold ($\mathbb{R}^4$, g) is an $(APMPS)_4$.

In consequence of the above, one can say that

Theorem 5.1. *There exist a manifold* $(\mathbb{R}^4, g)$ *which is an almost pseudo M-projectively symmetric Riemannian manifold with the above mentioned choice of the 1-forms.*

Acknowledgement

The authors wish to express their sincere thanks and gratitude to the referee for his/her valuable suggestions towards the improvement of the paper.

References

[1] Ali, M., Khan, Q., and Vasiulla, M. On generalized pseudo-symmetric manifold. *Bull. Cal. Math. Soc. 113*, 4 (2021), 353–362.

[2] Chaki, M. C. On pseudo symmetric manifolds. *An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat. 33*, 1 (1987), 53–58.

[3] De, U. C., and Gazi, A. K. On almost pseudo symmetric manifolds. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 51* (2008), 53–68 (2009).

[4] De, U. C., and Pal, P. On generalized M-projectively recurrent manifolds. *Ann. Univ. Paedagog. Crac. Stud. Math. 13* (2014), 77–101.

[5] De, U. C., Shaikh, A. A., and Sengupta, J. *Tensor Calculus, Second Edition.* Narosa Publ. Pvt. Ltd, New Delhi, 2007.

[6] Desai, P., and Amur, K. On symmetric spaces. *Tensor (N.S.) 29*, 2 (1975), 119–124.

[7] Khan, Q. *Differential geometry of manifolds.* PHI, New Delhi, 2012.

[8] Pokhariyal, G. P., and Mishra, R. S. Curvature tensors and their relativistic significance. II. *Yokohama Math. J. 19*, 2 (1971), 97–103.

Received by the editors May 7, 2021
First published online March 15, 2022