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Non-algebraic crossing limit cycle for discontinuous piecewise differential systems formed by a linear system without equilibrium points and quadratic isochronous centers at the origin

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Abstract. The aim of this paper is devoted to study the maximum number of crossing limit cycles for discontinuous piecewise differential systems separated by one straight line $y = 0$ and formed by a linear system without equilibrium points and quadratic isochronous centers. Under some suitable conditions, we prove that this class has at most one non algebraic crossing limit cycle explicitly given, and to illustrate our results we present an example.

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1. Introduction

A limit cycle is a periodic orbit of a differential system in $\mathbb{R}^2$, isolated in the set of all periodic orbits of that system. The existence and number of limit cycles of planar polynomial differential systems is one of the open questions proposed by D. Hilbert at the International Congress of Mathematicians in Paris (1900). These problems are known as the "second part of Hilbert's 16th problem", see for example [8, 10].

The study of piecewise linear differential systems goes back to to Andronov, Vitt and Khaikin [1], and today these systems are still of interest to many specialists. These piecewise differential systems are also called Filippov systems, for more details see [7]. They have several applications in the modeling of processes which appear in electronics, mechanics, economics, etc..., see for example the books of M. di Bernardo [5] and Simpson [17], the survey of Makarenkov and Lamb [13], and the references cited in these last three works.

In the planar discontinuous piecewise linear differential systems the limit cycles are of two kinds, the crossing and sliding ones. The crossing limit cycles

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only contain isolated points of the lines of discontinuity. The sliding limit cycles contain some segment of the lines of discontinuity that separate the different linear differential systems (see for more details [15]).

Limit cycles of discontinuous piecewise linear differential systems have been studied by many authors (see for instance [6, 9, 2, 3]).

One of the interesting classes of planar differential systems is a quadratic differential system because they are the simplest ones after the linear systems and they have been studied intensively, there are many papers have been published on those systems, see the books of Ye Yanquian and al [18], Reyn [16] and the references quoted therein. Up to now it is known that there are quadratic systems having algebraic limit cycles of degrees 2, 4, 5 or 6, see [4, 11] and their references, but there are no quadratic systems that have non algebraic limit cycles. We say that a limit cycle is algebraic if it is contained in an algebraic curve of the plane, otherwise it is called non-algebraic.

In this paper we will study the crossing limit cycles of discontinuous piecewise differential systems separated by a straight line $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0\}$, and formed by linear differential system without equilibrium points and a quadratic isochronous center. In a real planar polynomial differential system we recall that a center is isochronous center if there exists a neighborhood such that all periodic orbits in this neighborhood have the same period.

The quadratic polynomial differential systems with an isochronous center were classified into four classes by Loud [12]. Using the notation of [14] namely S_1, S_2, S_3, and S_4, we consider the four classes of quadratic isochronous centers and their first integrals as follows

S_1) $\dot{x} = -y + x^2, \qquad \dot{y} = x + yx,$ with the first integral

$$H_1(x, y) = \frac{x^2 + y^2}{(1 + y)^2}. \tag{1.1}$$

S_2) $\dot{x} = -y + \frac{1}{4}x^2, \qquad \dot{y} = x + xy,$ with the first integral

$$H_2(x, y) = \frac{\left(x^2 + 4y + 8\right)^2}{1 + y}. \tag{1.2}$$

S_3) $\dot{x} = -y + \frac{1}{2}x^2 - \frac{1}{2}y^2, \qquad \dot{y} = x + yx,$ with the first integral

$$H_3(x, y) = \frac{x^2 + y^2}{1 + y}. \tag{1.3}$$

S_4) $\dot{x} = -y + 2x^2 - \frac{1}{2}y^2, \qquad \dot{y} = x + xy,$ with the first integral

$$H_4(x, y) = \frac{4x^2 - 2(y + 1)^2 + 1}{(1 + y)^2}. \tag{1.4}$$

2. Preliminaries and Main results

In order to state precisely our results we introduce first some notations and definitions. Consider the piecewise differential system

$$(\dot{x},\dot{y}) = F_{\pm}(x,y) = (f_{\pm}(x,y), g_{\pm}(x,y)) \tag{2.1}$$

being bi-valued on the separation line $y = 0$. Following Filippov [7], a point $(x,0)$ is a crossing point if $g_{-}(x,0)g_{+}(x,0) > 0$. If there exists a periodic orbit of the discontinuous differential system (2.1) having exactly two crossing points, then we call it a crossing periodic orbit. A crossing limit cycle is an isolated periodic orbit in the set of all crossing periodic orbits of system (2.1). In what follows for simplicity, we shall say limit cycle instead of crossing limit cycle.

In the next lemma we present a linear differential system without equilibrium points and the explicit expression of its first integral. For a proof of the following lemma see [3].

Lemma 2.1. *A linear system without equilibrium points can be written as*

$$\dot{x} = ax + by + c, \qquad \dot{y} = \lambda ax + \lambda by + d, \tag{2.2}$$

where a, b, c, λ and d are real constants such that $d \neq \lambda c$ and $\lambda \neq 0$. Moreover this system is integrable and has the first integral

(2.3)

$$H_L(x,y) = \begin{cases} b\lambda^2x^2 - 2b\lambda xy - 2dx + by^2 + 2cy & \text{if } a + b\lambda = 0 \\ ((a+b\lambda)(ax+by) + ac + bd)\, e^{\frac{a+b\lambda}{d-c\lambda}(\lambda x - y)} & \text{if } a + b\lambda \neq 0 \end{cases}.$$

The objective of this paper is to study the crossing limit cycles of the planar discontinuous piecewise differential systems separated by the straight line $y = 0$ having in $y < 0$ a linear differential system (2.2) and in the half–plane $y > 0$ one of the previous quadratic isochronous differential systems (S_j), $j = 1, 2, 3, 4$, using their first integrals, we determine sufficient conditions for a discontinuous piecewise differential systems to possess at most one explicit non-algebraic limit cycle. Finally, we give an example, to illustrate our results.

Our main results is given by the following theorem

Theorem 2.2. *A discontinuous piecewise polynomial differential system separated by one straight line with two differential systems, such that one of them is a quadratic isochronous center at the origin and the second is a linear system without equilibrium points, neither real nor virtual, can have at most one limit cycle. Moreover, this limit cycle, if it exists, is non algebraic.*

Proof. Since the discontinuous piecewise differential system $(2.2) - (S_j)$ has a crossing limit cycle it must intersect the line $y = 0$ in exactly two points $(x_0, 0)$ and $(x_1, 0)$ with $x_0 < 0$ and $x_1 > 0$. Since H_L and H_j, $j = 1, 2, 3, 4$ are the first integrals of linear system and quadratic isochronous centers, respectively, they must satisfy the following system

$$\begin{cases} L(x_0, x_1) := H_L(x_0, 0) - H_L(x_1, 0) = 0, \\ K_j(x_0, x_1) := H_j(x_0, 0) - H_j(x_1, 0) = 0. \end{cases} \tag{2.4}$$

The second equations $K_j(x_0, x_1) = 0$, $j = 1, 2, 3, 4$ of system (2.4) are equivalent to

$$\begin{aligned} K_1(x_0, x_1) &= (x_0 - x_1)(x_0 + x_1) = 0, \\ K_2(x_0, x_1) &= (x_0 + x_1)(x_0 - x_1)\left(x_0^2 + x_1^2 + 16\right) = 0, \\ K_3(x_0, x_1) &= (x_0 - x_1)(x_0 + x_1) = 0, \\ K_4(x_0, x_1) &= 4(x_0 - x_1)(x_0 + x_1) = 0. \end{aligned}$$

From the equations $K_j(x_0, x_1) = 0$ the unique solution satisfying $x_1 \neq x_0$, is $x_1 = -x_0$. Now, it is easy to see that the existence of crossing periodic solutions of discontinuous piecewise differential systems is equivalent to the existence of the isolated values of x_0 which satisfy

$$L(x_0) := H_L(x_0, 0) - H_L(-x_0, 0) = 0. \tag{2.5}$$

Here, we must separate the proof of Theorem **2.2** in two cases.

Case 1 : $a + b\lambda = 0$, in this case the equation $L(x_0) = 0$ becomes $-4dx_0 = 0$. Then the system (2.4) has no solution satisfying $x_1 \neq x_0$. Consequently the discontinuous piecewise differential systems formed by a linear system (2.2) and one of quadratic isochronous centers (S_j), $j = 1, 2, 3, 4$ has no limit cycles.

Case 2 : $a + b\lambda \neq 0$, in this case the equation (2.5) becomes

$$((a + b\lambda) a x_0 + ac + bd) e^{\frac{a+b\lambda}{d-c\lambda}\lambda x_0} - ((a + b\lambda)(-a x_0) + ac + bd) e^{-\frac{a+b\lambda}{d-c\lambda}\lambda x_0} = 0,$$

equivalent to

$$((a + b\lambda)(a x_0) + ac + bd) e^{\frac{a+b\lambda}{d-c\lambda}(2\lambda x_0)} - ((a + b\lambda)(-a x_0) + ac + bd) = 0. \tag{2.6}$$

We put

$$f(x) := ((a + b\lambda)(ax) + ac + bd) e^{\frac{a+b\lambda}{d-c\lambda}(2\lambda x)} - ((a + b\lambda)(-ax) + ac + bd) \quad (x \in \mathbb{R}). \tag{2.7}$$

Finding the solutions of equation (2.6) is equivalent to finding the roots of equation $f(x) = 0$. Since f is a differentiable function in $\mathbb{R}$, then we use the first two derivatives of the function f.

$$f'(x) = \frac{a + b\lambda}{d - c\lambda}(2ax\lambda(a + b\lambda) + ca\lambda + da + 2bd\lambda) e^{\frac{a+b\lambda}{d-c\lambda}(2\lambda x)} + a(a + b\lambda)$$

and

$$f''(x) = 4\lambda \frac{(a + b\lambda)^3}{(d - c\lambda)^2}(d + ax\lambda) e^{\frac{a+b\lambda}{d-c\lambda}(2\lambda x)}$$

It is easy to see that f' and f'' are continuous functions. We remark that $f''(x) = 0$ has at most one zero, thus the equation $f'(x) = 0$ has at most two zeros and the equation $f(x) = 0$ has at most three zeros. We note that $f(0) = 0$, and so there remain two more zeros of $f(x)$, but if we have x_0 is a solution of $f(x) = 0$ then $-x_0$ is also a solution of $f(x) = 0$. Moreover, we can choose the appropriate parameters in such a way that $f(x) = 0$ has exactly one real negative root x_0, and $x_1 = -x_0$ that can provide at most one limit cycle for the discontinuous piecewise differential systems $(2.2) - (S_j), j = 1, 2, 3, 4$ given by $\Gamma = \Gamma_L \cup \Gamma_j, j = 1, 2, 3, 4$ where

$$\Gamma_L = \left\{ \begin{array}{l} ((a + b\lambda)(ax + by) + ac + bd)\, e^{\frac{a+b\lambda}{d-c\lambda}(\lambda x - y)} \\ = ((a + b\lambda)(ax_0) + ac + bd)\, e^{\frac{a+b\lambda}{d-c\lambda}(\lambda x_0)}, \quad y < 0 \end{array} \right\}$$

and

$$\Gamma_1 = \left\{ \frac{x^2 + y^2}{(1 + y)^2} = x_0^2,\ y > 0 \right\}, \Gamma_2 = \left\{ \frac{\left(x^2 + 4y + 8\right)^2}{1 + y} = \left(x_0^2 + 8\right)^2,\ y > 0 \right\},$$

$$\Gamma_3 = \left\{ \frac{x^2 + y^2}{1 + y} = x_0^2,\ y > 0 \right\}, \Gamma_4 = \left\{ \frac{4x^2 - 2(y + 1)^2 + 1}{(1 + y)^2} = 4x_0^2 - 1,\ y > 0 \right\}.$$

For which

$$((a + b\lambda)\, ax_0 + ac + bd)\, e^{\frac{a+b\lambda}{d-c\lambda}(2\lambda x_0)} - (-(a + b\lambda)\, ax_0 + ac + bd) = 0 \text{ holds.}$$

□

The next proposition shows that there are discontinuous piecewise differential systems $(2.2) - (S_j)$ separated by the straight line $y = 0$ and formed by linear system without equilibria and one of the quadratic isochronous centers $(S_j), j = 1, 2, 3, 4$; with one crossing non algebraic limit cycle.

Proposition 2.3. *Assume $\lambda d \neq 0$ and let $a = -\frac{d}{3\lambda}, b = -\frac{1}{3d\lambda^2}\left(9\lambda^2 - d^2\right), c = \frac{1}{d\lambda}\left(3\lambda^2 + d^2\right)$, the linear differential system* (2.2) *becomes*

$$\dot{x} = \left(-\tfrac{d}{3\lambda}\right) x - \frac{\left(9\lambda^2 - d^2\right)}{3d\lambda^2} y + \frac{\left(3\lambda^2 + d^2\right)}{d\lambda}, \quad \dot{y} = -\tfrac{1}{3} dx - \frac{\left(9\lambda^2 - d^2\right)}{3d\lambda} y + d. \tag{2.8}$$

Then the discontinuous piecewise polynomial differential systems $(2.8){-}(S_j), j = 1, 2, 3, 4$ *when $\lambda \neq 0$ and $d < 0$, has one explicit non-algebraic crossing limit cycle given by*

$$\Gamma = \Gamma_L \cup \Gamma_j, j = 1, 2, 3, 4$$

where

$$\Gamma_L = \left\{ -\frac{\left(\left(-d^2\lambda\right)x + \left(d^2 - 9\lambda^2\right)y + 4d^2\lambda\right)}{d^2\lambda} e^{-\frac{1}{\lambda}(y - x\lambda)} = -0.147\,02,\ y < 0 \right\}$$

$$\Gamma_1 = \left\{ \frac{x^2 + y^2}{(1+y)^2} = 15.978,\ y > 0 \right\}$$

$$\Gamma_2 = \left\{ \frac{\left(x^2 + 4y + 8\right)^2}{1+y} = 574.96,\ y > 0 \right\}$$

$$\Gamma_3 = \left\{ \frac{x^2 + y^2}{1+y} = 15.978,\ y > 0 \right\}$$

$$\Gamma_4 = \left\{ \frac{4x^2 - 2(y+1)^2 + 1}{(1+y)^2} = 62.914,\ y > 0 \right\}$$

Proof. The linear system (2.8) has no equilibria, neither real nor virtual, and it has the first integral given by

$$H_L(x,y) = -\frac{1}{d^2\lambda}\left(\left(-d^2\lambda\right)x + y\left(d^2 - 9\lambda^2\right) + 4d^2\lambda\right)e^{-\frac{1}{\lambda}(y - x\lambda)}$$

For studying the existence of crossing limit cycle for discontinuous differential systems $(2.8) - (S_j)$ we determine the solutions of (2.7). In this case the function f in (2.7) can be written as follows

$$f(x) := x + 4 + (x-4)e^{2x} \quad (x \in \mathbb{R}).$$

The negative root of this function is approximately $x_0 = -3.9973$. From this value of x_0, we get the value of $x_1 = 3.9973$, that can provide one crossing limit cycle passing through the crossing points $(-3.997, 0)$ and $(3.9973, 0)$, and given by the following expressions $\Gamma = \Gamma_L \cup \Gamma_j$, $j = 1,2,3,4$, where Γ_L and Γ_j, $j = 1,2,3,4$ are defined in the previous proposition. This completes the proof of Proposition 2.3. □

Remark 2.4. The assumption $d < 0$ in Proposition 2.3 is a necessary condition for the existence of crossing limit cycles of system because the crossing region of these systems is given by $-\frac{1}{3}dx(x-3) > 0$ then this last inequality implies that the crossing region is an open interval $(0,3)$ of the line $y = 0$ if $d > 0$ and is an open interval $(-\infty, 0) \cup (3, \infty)$ of the line $y = 0$ if $d < 0$. Since the intersection points $x_0 = -3.9973$ and $x_1 = 3.9973$ are located in $(-\infty, 0) \cup (3, \infty)$ we must choose that $d < 0$.

Example 2.5. If we take $d = -1$ and $\lambda = 2$; system (2.8) reads as follows

$$\dot{x} = \tfrac{1}{6}x + \tfrac{35}{12}y - \tfrac{13}{2}, \qquad \dot{y} = \tfrac{1}{3}x + \tfrac{35}{6}y - 1. \quad y < 0 \tag{2.9}$$

Then the discontinuous differential systems formed by this linear system in the half-plan $y < 0$ and one of the quadratic isochronous systems (S_j), $j = 1,2,3,4$

in $y > 0$ has a unique non-algebraic crossing limit cycle which intersects the line $y = 0$ at two points $x_0 = -3.9973$, $x_1 = 3.9973$ and given by $\Gamma = \Gamma_L \cup \Gamma_j$, $j = 1, 2, 3, 4$, where

$$\Gamma_L = \left\{ (2x + 35y - 8)\, e^{x - \frac{1}{2}y} = -0.29404, \quad y < 0 \right\}$$

and Γ_j, $j = 1, 2, 3, 4$ are defined in Proposition 2.3. See the following figures.

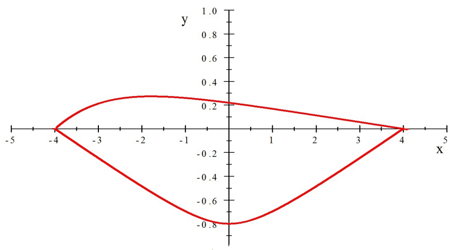

Figure 1: $\Gamma = \Gamma_L \cup \Gamma_1$.

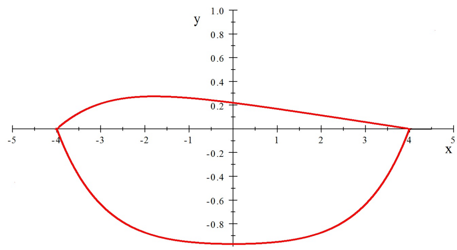

Figure 2: $\Gamma = \Gamma_L \cup \Gamma_2$.

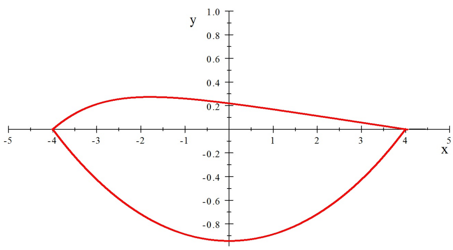

Figure 3: $\Gamma = \Gamma_L \cup \Gamma_3$.

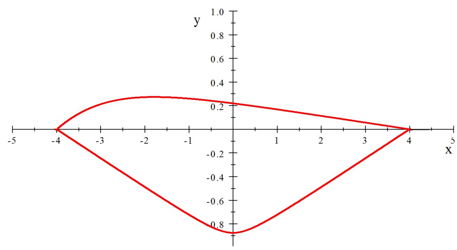

Figure 4: $\Gamma = \Gamma_L \cup \Gamma_4$.

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