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Mixed $C-$cosine families of bounded linear operators on non-Archimedean Banach spaces

A. Blali[1] A. El Amrani[2] and J. Ettayb[3][4]

Abstract

In this paper, we introduce and check some properties of $H-C-$cosine and mixed $C-$cosine families of bounded linear operators on non-Archimedean Banach spaces. We show some results for $H-C-$cosine and mixed $C-$cosine families of bounded linear operators on non-Archimedean Banach spaces. In contrast with the classical setting, the parameter of a given mixed $C-$cosine family of bounded linear operators belongs to a clopen ball Ω_r of the ground field $\mathbb{K}$. Examples are given to support our work.

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1 Introduction and Preliminaries

Throughout this paper, X is a non-Archimedean (n.a) Banach space over a (n.a) non trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, $B(X)$ denotes the set of all bounded linear operators on X, $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|.|_p$, $\mathbb{Z}_p$ denotes the ring of p-adic integers is the unit ball of $\mathbb{Q}_p$. For more details and related issues, we refer to [7], [8] and [9]. We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic absolute value $|\cdot|_p$ by $\mathbb{C}_p$ (see [7], p.45). Remember that a free Banach space X is a non-Archimedean Banach space for which there exists a family $(e_i)_{i\in\mathbb{N}}$ in $X\backslash\{0\}$ such that every element $x \in X$ can be written in the form of a convergent sum $x = \sum\limits_{i\in\mathbb{N}} x_i e_i$, $x_i \in \mathbb{K}$ and $\|x\| = \sup\limits_{i\in\mathbb{N}} |x_i|\|e_i\|$. The family $(e_i)_{i\in\mathbb{N}}$ is called an orthogonal basis. In a free Banach space X, each bounded linear operator A on X can be written in a unique fashion as a pointwise convergent series, that is, there exists an infinite matrix $(a_{i,j})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ with coefficients in $\mathbb{K}$ such that

$$A = \sum_{i,j\in\mathbb{N}} a_{i,j} e_j^{'} \otimes e_i, \; and \; \forall j \in \mathbb{N}, \;\; \lim_{i\to\infty} |a_{i,j}|\|e_i\| = 0,$$

[1]Ecole Normale Superieure, Department of Mathematics, Sidi Mohamed Ben Abdellah University, B. P. 5206 Bensouda-Fez, Morocco.
e-mail: aziz.blali@usmba.ac.ma, ORCID iD: orcid.org/0000-0001-5788-2399

[2]Department of mathematics and computer science, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Atlas Fez, Morocco.
e-mail: abdelkhalek.elamrani@usmba.ac.ma, ORCID iD: orcid.org/0000-0003-4841-8038

[3]Department of mathematics and computer science, Sidi Mohamed Ben Abdellah University, Faculty of Sciences Dhar El Mahraz, Atlas Fez, Morocco.
e-mail: jawad.ettayb@usmba.ac.ma, ORCID iD: orcid.org/0000-0002-4819-943X

[4]Corresponding author

where $(\forall j \in \mathbb{N})\ e_j^{'}(x) = x_j\ \left(e_j^{'} \text{ is the linear form associated with } e_j\right)$.
Moreover, for each $j \in \mathbb{N}$, $Ae_j = \sum_{i \in \mathbb{N}} a_{ij} e_i$ and its norm is defined by

$$\| A \| = \sup_{i,j} \frac{|a_{ij}| \|e_i\|}{\|e_j\|}.$$

For more details see [4] and [5]. Now, as in [6], take $r > 0$, Ω_r is the open ball of $\mathbb{K}$ centred at 0 with radius $r > 0$, that is $\Omega_r = \{k \in \mathbb{K} : |k| < r\}$. In the non-Archimedean context, the family $\{C(t),\ t \in \Omega_r\}$, $C : \Omega_r \to B(X)$ is called cosine family of bounded linear operators on X if

$$\text{for all } t,\ s \in \Omega_r,\ C(s+t) + C(s-t) = 2C(s)C(t)$$

and $C(0) = I$, where I is the identity operator on X. For more details, we refer to [1], [3] and [6]. Suppose that $\mathbb{K} = \mathbb{Q}_p$ and A is a bounded linear operator on a free Banach space X satisfying $\|A\| < r = p^{\frac{-1}{p-1}}$, then the function defined by for all $t \in \Omega_{\frac{-1}{p-1}}$, $f(t) = \left(\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} A^n\right) u_0$ for a fixed $u_0 \in X$ is the solution to homogeneous p-adic second order differential equation given by

$$\frac{d^2 u(t)}{dt^2} = Au(t),\ u(0) = u_0.$$

The aim of this work is to introduce the mixed $C-$cosine family of bounded linear operators on non-Archimedean Banach space and study some of its properties. We begin with the following definitions.

Definition 1.1 ([6], Definition 1.12)**.** Let $r > 0$ be a chosen real number such that $(T(t))_{t \in \Omega_r}$ are well defined. A one-parameter family $(T(t))_{t \in \Omega_r}$ of bounded linear operators from X into X is a group if

(i) $T(0) = I$, where I is the unit operator of X;

(ii) For all $t,\ s \in \Omega_r$, $T(t+s) = T(t)T(s)$.

The group $(T(t))_{t \in \Omega_r}$ will be called of class C_0 or strongly continuous if the following condition holds:

- For each $x \in X$, $\lim_{t \to 0} \|T(t)x - x\| = 0$.

A group of bounded linear operators $(T(t))_{t \in \Omega_r}$ is uniformly continuous if and only if $\lim_{t \to 0} \|T(t) - I\| = 0$.
The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists}\},$$

and

$$\text{for each } x \in D(A),\ Ax = \lim_{t \to 0} \frac{T(t)x - x}{t}$$

is called the infinitesimal generator of the group $(T(t))_{t \in \Omega_r}$.

Definition 1.2 ([2], Definition 2.3)**.** Let $r > 0$ be a real number. A family $(S(t))_{t\in\Omega_r}$ of bounded linear operators is said to be an $H - C_0-$group or a generalized C_0-group of bounded linear operators on X if

(i) $S(0) = I$; where I is the identity operator of X.

(ii) there is a C_0-group $(T(t))_{t\in\Omega_r}$ of bounded linear operators and $D \in B(X)$ such that for all $t,\ s \in \Omega_r$,

$$\begin{aligned} S(s+t) &= H\Big(S(s),\ S(t)\Big) \\ &= S(s)S(t) + D\Big(S(s) - T(s)\Big)\Big(S(t) - T(t)\Big); \end{aligned}$$

(iii) for each $x \in X, S(\cdot)x : \Omega_r \longrightarrow S(t)x$ is continuous on Ω_r.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists}\}$$

and

$$\text{for each } x \in D(A),\ Ax = \lim_{t\to 0} \frac{S(t)x - x}{t},$$

is called the infinitesimal generator of the $H - C_0-$group $(S(t))_{t\in\Omega_r}$.

From Definition 1.2, when $D = \alpha I$ for $\alpha \in \mathbb{K}$, we have the following definition.

Definition 1.3 ([2], Definition 2.5)**.** Let $r > 0$ be a real number. A family $(S(t))_{t\in\Omega_r}$ is said to be a mixed C_0-group of bounded linear operators on X if

(i) $S(0) = I$;

(ii) there is a C_0-group $(T(t))_{t\in\Omega_r}$ of bounded linear operators and $\alpha \in \mathbb{K}$ such that for all $s, t \in \Omega_r$,

$$\begin{aligned} S(s+t) &= H\Big(S(s),\ S(t)\Big) \\ &= S(s)S(t) + \alpha\Big(S(s) - T(s)\Big)\Big(S(t) - T(t)\Big); \end{aligned}$$

(iii) for each $x \in X,\ S(\cdot)x : \Omega_r \longrightarrow S(t)x$ is continuous on Ω_r.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists}\}$$

and

$$\text{for each } x \in D(A),\ Ax = \lim_{t\to 0} \frac{S(t)x - x}{t},$$

is called the infinitesimal generator of the mixed C_0-group $(S(t))_{t\in\Omega_r}$.

Set $A_1 = (1 + D)A - DA_0$, where A_0 is the infinitesimal generator of the $C-$group $\{T(t)\}_{t\in\Omega_r}$ and A is the infinitesimal generator of the $H - C-$group $\{S(t)\}_{t\in\Omega_r}$. Similarly, the proof of Theorem 2.10 of [2], we have.

Theorem 1.4. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $\{S(t)\}_{t\in\Omega_r}$ be an $H - C-$group family of bounded linear operators on X with for all $s \in \Omega_r$, $DS(s) = S(s)D$ and $T(s)D = DT(s)$. Set for all $t \in \Omega_r$, $T_1(t) = (I + D)S(t) - DT(t)$, we have*

(i) $\{T_1(t)\}_{t\in\Omega_r}$ is a $C-$group of bounded linear operators whose infinitesimal generator is an extension of A_1.

(ii) If $I + D$ is invertible, then for all $x \in X$, and $t \in \Omega_r$,

$$S(t)x = (1 + D)^{-1}T_1(t)x + D(I + D)^{-1}T(t)x.$$

From Example 2.7 of [2], we conclude the following example.

Example 1.5. Let $r = p^{\frac{-1}{p-1}}$, suppose that X is a non-Archimedean Banach space over $\mathbb{Q}_p$, let $A_0, A \in B(X)$ such that $AA_0 = A_0A$ and $\|A_0\| < r$. Set

$$\text{for all } t \in \Omega_r,\ S(t) = e^{tA_0} + t(A - A_0)e^{tA_0}.$$

Then one can see that with $D = -I$, $\{S(t)\}_{t\in\Omega_r}$ is a mixed C_0-group, where for all $t \in \Omega_r$, $T(t) = e^{tA}$. In this case for all $t,\ s \in \Omega_r,\ S(s)S(t) = S(t)S(s)$.

We have the following definition.

Definition 1.6 ([6], Definition 2.24)**.** Let $r > 0$ be a real number. A function $C : \Omega_r \longrightarrow B(X)$ is called a C_0 or strongly continuous operator cosine function on X if

(i) $C(0) = I$,

(ii) For every $t,\ s \in \Omega_r,\ C(t + s) + C(t - s) = 2C(t)C(s)$,

(iii) For each $x \in X,\ \ t \longrightarrow C(t)x$ is continuous on Ω_r.

A cosine family of bounded linear operators $(C(t))_{t\in\Omega_r}$ is uniformly continuous if $\lim\limits_{t\to 0} \|C(t) - I\| = 0$.
The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} 2\frac{C(t)x - x}{t^2} \text{ exists}\}$$

and

$$\text{for each } x \in D(A), Ax = \lim_{t\to 0} 2\frac{C(t)x - x}{t^2}$$

is called the infinitesimal generator of the cosine family $(C(t))_{t\in\Omega_r}$.

We begin with the following lemmas.

Lemma 1.7 ([6], Lemma 2.26)**.** *Let $(C(t))_{t\in\Omega_r}$ be a strongly continuous cosine family on X, then for each $t\in\Omega_r$, $C(2t)=2C(t)^2-I$.*

Remark 1.8. Let $\mathbb{K}=\mathbb{Q}_p$. By Lemma 1.7, if $p\neq 2$, we have for all $t\in\Omega_r,\ C(\frac{t}{2})^2=\frac{C(t)+I}{2}$.

Lemma 1.9 ([6], Lemma 2.27)**.** *Let $(C(t))_{t\in\Omega_r}$ be a strongly continuous cosine family on X, then:*

(i) For every $t\in\Omega_r$, $C(-t)=C(t)$,

(ii) For each $t,\ s\in\Omega_r$, $C(t)C(s)=C(s)C(t)$.

We have the following theorem.

Theorem 1.10 ([6], Theorem 2.32)**.** *Let $(C(t))_{t\in\Omega_r}$ be a strongly continuous cosine family satisfying : there is $M>0$ such that for each $t\in\Omega_r$ holds $\|C(t)\|\le M$, and let A be its infinitesimal generator. Then, for every $x\in D(A)$, $AC(s)x=C(s)Ax$ and $C(s)x\in D(A)$ for each $s\in\Omega_r$.*

Recall that $\mathbb{C}_p^+=\{a\in\mathbb{C}_p:|1-a|<1\}$. For each $a\in\mathbb{C}_p^+$ where $p\neq 2$, the element

$$\sqrt{a}=a^{\frac{1}{2}}=\sum_{n\in\mathbb{N}}\binom{\frac{1}{2}}{n}(a-1)^n \tag{1.1}$$

is the unique positive square root of a. For more details see [8], section 49, page 143.

Example 1.11 ([6], Example 2.28)**.** Let $\mathbb{K}=\mathbb{C}_p$ with $p\neq 2$. Consider the ball Ω_r of $\mathbb{C}_p$ with $r=p^{\frac{-1}{p-1}}$. Let X be a free $n.a.$ Banach space over $\mathbb{C}_p$ and $(e_i)_{i\in\mathbb{N}}$ be the canonical base of X. Define for each $q\in\Omega_r$ and for $x=\sum_{i\in\mathbb{N}}x_ie_i$ the family of linear operators $C(q)x=\sum_{i\in\mathbb{N}}x_i\cosh(\sqrt{\mu_i}q)e_i$, where $(\mu_i)_{i\in\mathbb{N}}\subset\mathbb{C}_p^+$ is a sequence of positive elements of $\mathbb{C}_p$. It is routine to check that the family $(C(q))_{q\in\Omega_r}$ is well defined.

Proposition 1.12 ([6], Proposition 2.29)**.** *The family $(C(q))_{q\in\Omega_r}$ of linear operators given above is a cosine family of bounded linear operators, whose infinitesimal generator is the bounded diagonal operator A defined by $Ax=\sum_{i\in\mathbb{N}}\sqrt{\mu_i}x_ie_i$ for each $x=\sum_{i\in\mathbb{N}}x_ie_i\in X$.*

Recall that k is the residue class field of $\mathbb{K}$. Througout this paper, we assume that $\mathbb{K}$ is a complete non-Archimedean valued field of characteristic zero $\Big(char(\mathbb{K})=0\Big)$ with $char(k)=p$ $\Big(p$ is a prime integer number$\Big)$. We have the following example.

Example 1.13 ([3], Example 2.1). Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$ such that $\|A\| < r \left(= p^{\frac{-1}{p-1}}\right)$; it is easy to check that for all $t \in \Omega_r$, $C(t) = \sum_{n\in\mathbb{N}} \frac{t^{2n}}{(2n)!} A^n$ is a strongly continuous cosine family of bounded operators of infinitesimal generator A on X.

We have the following lemma.

Lemma 1.14 ([3], Lemma 2.2). *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $(C(t))_{t\in\Omega_r}$ be a strongly continuous cosine family on X. Then for each $t \in \Omega_r$ and $n \in \mathbb{N}^*$ there exist $n+1$ constants $a_0, \cdots, a_n$ in $\mathbb{K}$ such that $C(nt) = a_0 I + a_1 C(t) + \cdots + a_n C(t)^n$.*

Theorem 1.15 ([3], Theorem 2.4). *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$ such that $\|A\| < r \left(r = p^{\frac{-1}{p-1}}\right)$. Then, A is the infinitesimal generator of a uniformly continuous cosine family of bounded operators $(C(t))_{t\in\Omega_r}$.*

We have the following proposition.

Proposition 1.16 ([3], Proposition 2.6). *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $(T(t))_{t\in\Omega_r}$ be a uniformly continuous group of bounded linear operators on X. Set for all $t \in \Omega_r$, $C(t) = \frac{T(t)+T(-t)}{2}$, $(C(t))_{t\in\Omega_r}$ is a uniformly continuous cosine family of bounded linear operators on X.*

We have the following proposition.

Proposition 1.17 ([3], Proposition 2.28). *There exists a Banach space X over $\mathbb{Q}_p$ and strongly continuous cosine family $(C(t))_{t\in\mathbb{Q}_p}$ of bounded linear operators on X satisfying: there exists $M > 0$ such that for all $z \in X$, $t \in \mathbb{Q}_p$, $\|C(t)z\| \leq (1 + |t|_p^2 M)\|z\|$.*

Definition 1.18 ([1], Definition 2.1). Let $r > 0$ and $C \in B(X)$ be invertible. A one parameter family $(C(t))_{t\in\Omega_r}$ of bounded linear operators from X into X is called a $C-$cosine family if

(i) $C(0) = C$;

(ii) For every $t,\ s \in \Omega_r$, $C\Big(C(t+s) + C(t-s)\Big) = 2C(t)C(s)$;

(iii) For each $x \in X$, $\Omega_r \longrightarrow C(t)x$ is continuous on Ω_r.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} 2\frac{C(t)x - Cx}{t^2} \text{ exists}\},$$

and

$$\text{for each } x \in D(A),\ Ax = C^{-1} \lim_{t\to 0} 2\frac{C(t)x - Cx}{t^2},$$

is called the infinitesimal generator of $(C(t))_{t\in\Omega_r}$.

We have the following remark.

Remark 1.19 ([1], Remark 2.1). Generally in Definition 1.18, if $C \in B(X)$ is just injective (not invertible), we have

$$D(A) = \{x \in X : \lim_{t \to 0} 2\frac{C(t)x - Cx}{t^2} \text{ exists in the range of } C\}.$$

We start with the following statements.

Lemma 1.20 ([1], Lemma 2.1)**.** *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $(C(t))_{t\in\Omega_r}$ be a $C-$cosine family on X, then for each $t \in \Omega_r$, $CC(2t) = 2C(t)^2 - C^2$.*

Remark 1.21 ([1], Remark 2.2). Suppose that $\mathbb{K} = \mathbb{Q}_p$. From Lemma 1.20, if $p \neq 2,$ we have for all $t \in \Omega_r,\ C(\frac{t}{2})^2 = \frac{CC(t)+C^2}{2}$.

Lemma 1.22 ([1], Lemma 2.2)**.** *Let $(C(t))_{t\in\Omega_r}$ be a $C-$cosine family on X, then:*

(i) For every $t \in \Omega_r$, $C(-t) = C(t)$,

(ii) For each $t,\ s \in \Omega_r$, $C(t)C(s) = C(s)C(t)$.

Proposition 1.23 ([1], Proposition 2.1)**.** *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $(C(t))_{t\in\Omega_r}$ be a C_1-cosine family with infinitesimal generator A and $C_2 \in B(X)$ be invertible such that for all $t \in \Omega_r, C_2C(t) = C(t)C_2$, then $(C_2C(t))_{t\in\Omega_r}$ is a C_1C_2-cosine family on X.*

We have the following theorem.

Theorem 1.24 ([1], Theorem 2.1)**.** *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $A \in B(X)$ such that $\|A\| < r = p^{\frac{-1}{p-1}}$. Then A is the infinitesimal generator of a uniformly $C-$cosine family of bounded linear operators $(C(t))_{t\in\Omega_r}$.*

We have the following theorem.

Theorem 1.25 ([1], Theorem 2.2)**.** *Let $(C(t))_{t\in\Omega_r}$ be a $C-$cosine family satisfying: there exists $M > 0$ such that for each $t \in \Omega_r$, $\|C(t)\| \leq M$, and let A be its infinitesimal generator. Then, for every $x \in D(A),\ t \in \Omega_r$, $C(t)x \in D(A)$, and $AC(t)x = C(t)Ax$.*

Definition 1.26 ([1], Definition 2.5)**.** Let $r > 0$ be a real number. A family $(S(t))_{t\in\Omega_r}$ of bounded linear operators is said to satisfy p-adic $H-$generalized cosine family of bounded linear operators on X if

$$\text{for all } t,\ s \in \Omega_r,\ S(s+t) + S(s-t) = H\Big(S(s),\ S(t)\Big),$$

where $H :\ B(X) \times B(X) \to B(X)$ is a function.

Remark 1.27 ([1], Remark 2.5). If $H\Big(S(s),\ S(t)\Big) = 2S(s)S(t)$, with $S(0) = I$, $(S(t))_{t\in\Omega_r}$ is a cosine family of bounded linear operators on X.

We have the following definition.

Definition 1.28 ([1], Definition 2.6)**.** Let $r > 0$ be a real number. A family $(S(t))_{t\in\Omega_r}$ of bounded linear operators is said to be an $H - C_0-$cosine family or a generalized C_0-cosine family of bounded linear operators on X if

(1) $S(0) = I$; where I is the identity operator of X.

(2) *For all* $t,\ s \in \Omega_r$,

$$\begin{aligned} S(s+t) + S(s-t) &= H\Big(S(s),\ S(t)\Big) \\ &= 2S(s)S(t) + 2D\Big(S(s) - C(s)\Big)\Big(S(t) - C(t)\Big), \end{aligned}$$

where $(C(t))_{t\in\Omega_r}$ is a C_0-cosine family of bounded linear operators with the infinitesimal generator A_0 and $D \in B(X)$.

(3) For each $x \in X$, $S(\cdot)x : \Omega_r \longrightarrow X$ is continuous on Ω_r.

The linear operator A defined by

$$D(A) = \{x \in X : 2 \lim_{t\to 0} \frac{S(t)x - x}{t^2} \text{ exists}\}$$

and

$$\text{for each } x \in D(A),\ Ax = 2 \lim_{t\to 0} \frac{S(t)x - x}{t^2},$$

is called the infinitesimal generator of the $H - C_0-$cosine family $(S(t))_{t\in\Omega_r}$.

2 Main results

We introduce the following definition.

Definition 2.1. Let $r > 0$ be a real number and $C \in B(X)$ be invertible. A family $(S(t))_{t\in\Omega_r}$ of bounded linear operators is said to satisfy a p-adic $H - C-$cosine Cauchy equation of bounded linear operators on X if

$$\text{for all } t,\ s \in \Omega_r,\ \big(S(s+t) + S(s-t)\big)C = H\Big(S(s),\ S(t)\Big),$$

where $H : B(X) \times B(X) \to B(X)$ is a function.

Remark 2.2. If $H\Big(S(s),\ S(t)\Big) = 2S(s)S(t)$ with $S(0) = C$, $(S(t))_{t\in\Omega_r}$ satisfies the first and second conditions of $C-$cosine family of bounded linear operators on X.

Definition 2.3. Let $r > 0$ be a real number and $C \in B(X)$ be invertible. A family $(S(t))_{t\in\Omega_r}$ of bounded linear operators will be called an $H - C-$cosine family or a generalized $C-$cosine family of bounded linear operators on X if

(i) $S(0) = C$;

(ii) there is a $C-$cosine family $(C(t))_{t\in\Omega_r}$ of bounded linear operators and $D \in B(X)$ such that for all $t,\ s \in \Omega_r$,

$$\begin{array}{rcl} C\big(S(s+t)+S(s-t)\big) & = & H\Big(S(s),\ S(t)\Big) \\ & = & 2S(s)S(t) + 2D(S(s)-C(s))(S(t)-C(t)); \end{array}$$

(iii) for each $x \in X, S(\cdot)x : \Omega_r \longrightarrow S(t)x$ is continuous on Ω_r.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} 2\frac{S(t)x - Cx}{t^2} \text{ exists}\}$$

and

$$\text{for each } x \in D(A),\ Ax = 2C^{-1} \lim_{t\to 0} \frac{S(t)x - Cx}{t^2},$$

is called the infinitesimal generator of the $H - C-$cosine family $(S(t))_{t\in\Omega_r}$.

Remark 2.4. Let $(S(t))_{t\in\Omega_r}$ be a generalized $C-$cosine family on X, if $D = 0$, then $(S(t))_{t\in\Omega_r}$ is a $C-$cosine family of bounded linear operators on X.

2.1 Question

Can you characterize the infinitesimal generator of an $H - C-$cosine family of bounded linear operators on infinite dimensional non-Archimedean Banach space ?

Definition 2.5. Let $r > 0$ be a real number and $C \in B(X)$ be invertible. A family $(S(t))_{t\in\Omega_r}$ is said to be a mixed $C-$cosine family of bounded linear operators on X if

(i) $S(0) = C$;

(ii) there is a $C-$cosine family $(C(t))_{t\in\Omega_r}$ of bounded linear operators with infinitesimal generator A_0 and $\alpha \in \mathbb{K}$ such that for all $s, t \in \Omega_r$,

$$\begin{array}{rcl} C(S(s+t)+S(s-t)) & = & H\Big(S(s),\ S(t)\Big) \\ & = & 2S(s)S(t) + 2\alpha(S(s)-C(s))(S(t)-C(t)); \end{array}$$

(iii) for each $x \in X,\ S(\cdot)x : \Omega_r \longrightarrow S(t)x$ is continuous on Ω_r.

The linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} 2\frac{S(t)x - Cx}{t^2} \text{ exists}\}$$

and

$$\text{for each } x \in D(A),\ Ax = 2C^{-1} \lim_{t\to 0} \frac{S(t)x - Cx}{t^2},$$

is called the infinitesimal generator of the mixed $C-$cosine family $(S(t))_{t\in\Omega_r}$.

Remark 2.6. Let $(S(t))_{t\in\Omega_r}$ be a mixed $C-$cosine family on X, if $\alpha = 0$, then $(S(t))_{t\in\Omega_r}$ is a $C-$cosine family of bounded linear operators on X.

We have the following example.

Example 2.7. Let $r = p^{\frac{-1}{p-1}}$, suppose that X is a non-Archimedean Banach space over $\mathbb{C}_p$, $A, C \in B(X)$ such that C is invertible, $AC = CA$ and $\|A\| < r$. Set for all $t \in \Omega_r$, $S(t) = Cch(tA) + tACsh(tA)$, where $ch(tA) = \sum_{n\in\mathbb{N}} \frac{t^{2n}}{(2n)!}A^{2n}$ and $sh(tA) = \sum_{n\in\mathbb{N}} \frac{t^{2n+1}}{(2n+1)!}A^{2n+1}$. Then one can see that with $D = -I$, $\{S(t)\}_{t\in\Omega_r}$ is an $H-C-$cosine family, where for all $t \in \Omega_r$, $C(t) = Cch(tA)$. In this case for all $t,\ s \in \Omega_r,\ S(s)S(t) = S(t)S(s)$.

We have the following lemma.

Lemma 2.8. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $\{S(t)\}_{t\in\Omega_r}$ be an $H-C-$cosine family on non-Archimedean Banach space X, then for all $t \in \Omega_r,\ S(-t) = S(t)$ and $CS(t) = S(t)C$.*

Proof. By Definition 2.5 and $s = 0$, we have for all $t \in \Omega_r,\ CS(-t) = CS(t)$. Since C is invertible, we get for all $t \in \Omega_r,\ S(-t) = S(t)$. It is easy to see that $t \in \Omega_r,\ CS(t) = S(t)C$. □

The following proposition gives a condition under which an $H-C-$cosine family commutes.

Proposition 2.9. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $\{S(t)\}_{t\in\Omega_r}$ be an $H-C-$cosine family on X. If $I+D$ is injective and for all $t,\ s \in \Omega_r,\ C(s)S(t) = S(t)C(s)$, then for all $t,\ s \in \Omega_r,\ S(s)S(t) = S(t)S(s)$.*

Proof. Assume that $I+D$ is injective and for all $t,\ s \in \Omega_r,\ C(s)S(t) = S(t)C(s)$, then for all $t,\ s \in \Omega_r$,

$$\begin{aligned} & 2S(s)S(t) + 2D\Big(S(s) - C(s)\Big)\Big(S(t) - C(t)\Big) \\ = \quad & C\big(S(s+t) + S(s-t)\big) \\ = \quad & C\big(S(t+s) + S(t-s)\big) \\ = \quad & 2S(t)S(s) + 2D\Big(S(t) - C(t)\Big) \times \Big(S(s) - C(s)\Big). \end{aligned}$$

Thus, $(I+D)\Big(S(t)S(s) - S(s)S(t)\Big) = 0$, then for all $t,\ s \in \Omega_r,\ S(s)S(t) = S(t)S(s)$. □

From Proposition 2.9, we conclude the following proposition.

Proposition 2.10. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $\{S(t)\}_{t\in\Omega_r}$ be a mixed $C-$cosine family on X with $\{C(t)\}_{t\in\Omega_r}$ be a $C-$cosine family and $\alpha \in \mathbb{K}\backslash\{-1\}$ such that for all $t,\ s \in \Omega_r,\ C(s)S(t) = S(t)C(s)$, then for all $t,\ s \in \Omega_r,\ S(s)S(t) = S(t)S(s)$.*

Set $A_1 = (1+\alpha)A - \alpha A_0$, where $\alpha \in \mathbb{K}\backslash\{-1\}$ and A_0 is the infinitesimal generator of the $C-$cosine family $\{C(t)\}_{t\in\Omega_r}$ and A is the infinitesimal generator of a mixed $C-$cosine family $\{S(t)\}_{t\in\Omega_r}$. We have the following theorem.

Theorem 2.11. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $\{S(t)\}_{t\in\Omega_r}$ be a mixed $C-$cosine family of bounded linear operators on X with $\alpha \in \mathbb{K}\backslash\{-1\}$. Set for all $t \in \Omega_r$, $C_1(t) = (1+\alpha)S(t) - \alpha C(t)$, then $\{C_1(t)\}_{t\in\Omega_r}$ is a $C-$cosine family of bounded linear operators whose infinitesimal generator is an extension of A_1. Furthermore, for all $x \in X$, and $t \in \Omega_r$,*

$$S(t)x = \frac{1}{1+\alpha}C_1(t)x + \frac{\alpha}{1+\alpha}C(t)x.$$

Proof.

(i) Trivially, $C_1(0) = (1+\alpha)S(0) - \alpha C(0) = C$.

(ii) For all $t,\ s \in \Omega_r,\ x \in X$, we have

$$\begin{aligned}
&C\Big(C_1(s+t) + C_1(s-t)\Big)x \\
&= (1+\alpha)\Big(S(s+t) + S(s-t)\Big)x - \alpha\Big(C(s+t) + C(s-t)\Big)x \\
&= (1+\alpha)\Big(2S(s)S(t) + 2\alpha(S(s) - C(s)) \times (S(t) - C(t))\Big)x - 2\alpha C(s)C(t)x \\
&= 2(1+\alpha)S(s)S(t)x + 2\alpha(1+\alpha)S(s)S(t)x \\
&\quad -2\alpha(1+\alpha)S(s)C(t)x - 2\alpha(1+\alpha)C(s)S(t)x \\
&\quad +2\alpha(1+\alpha)C(s)C(t)x - 2\alpha C(s)C(t)x \\
&= 2(1+\alpha)^2S(s)S(t)x - 2\alpha(1+\alpha)S(s)C(t)x \\
&\quad -2\alpha(1+\alpha)C(s)S(t)x + 2\alpha(1+\alpha)C(s)C(t)x \\
&\quad -2\alpha C(s)C(t)x \\
&= 2\Big((1+\alpha)S(s) - \alpha C(s)\Big) \times \Big((1+\alpha)S(t) - \alpha C(t)\Big)x \\
&= 2C_1(s)C_2(t)x.
\end{aligned}$$

Since $(C(t))_{t\in\Omega_r}$ and $(S(t))_{t\in\Omega_r}$ are continuous, then $(C_1(t))_{t\in\Omega_r}$ is continuous. Thus, $(C_1(t))_{t\in\Omega_r}$ is a $C-$ cosine family of bounded linear operators on X.

(iii) Finally, we show that an extension of A_1 is the infinitesimal generator of $\{C_1(t)\}_{t\in\Omega_r}$, Let B be the infinitesimal generator of $\{C_1(t)\}_{t\in\Omega_r}$ and let $x \in D(A_1) = D(A) \cap D(A_0)$. By definition of $D(A)$ and $D(A_0)$, we have

$2C^{-1}\lim_{t\to 0}\Big(\frac{S(t)x-x}{t^2}\Big)=Ax$ and $2C^{-1}\lim_{t\to 0}\Big(\frac{C(t)x-x}{t^2}\Big)=A_0x$. Then,

$$\begin{aligned}
&2C^{-1}\lim_{t\to 0}\Big(\frac{C_1(t)x-x}{t^2}\Big)\\
&= 2C^{-1}\lim_{t\to 0}\Big(\frac{(1+\alpha)S(t)x-\alpha C(t)x-x}{t^2}\Big)\\
&= 2(1+\alpha)C^{-1}\lim_{t\to 0}\Big(\frac{S(t)x-x}{t^2}\Big)\\
&\quad -2\alpha C^{-1}\lim_{t\to 0}\Big(\frac{C(t)x-x}{t^2}\Big)
\end{aligned}$$

exists in X. It follows that $x\in D(B)$ and $A_1x=Bx$, then the infinitesimal generator of $(C_1(t))_{t\in\Omega_r}$ is an extension of A_1. □

Put $A_1=(1+D)A-DA_0$, where $D\in B(X)$ and A_0 is the infinitesimal generator of the $C-$cosine family $\{C(t)\}_{t\in\Omega_r}$ and A is the infinitesimal generator of an $H-C-$cosine family $\{S(t)\}_{t\in\Omega_r}$, similar the proof of Theorem 2.11, we conclude the following theorem.

Theorem 2.12. *Let X be a non-Archimedean Banach space over $\mathbb{K}$, let $\{S(t)\}_{t\in\Omega_r}$ be a commuting $H-C-$cosine family of bounded linear operators on X with for all $s\in\Omega_r$, $DT(s)=T(s)D$ and $DS(s)=S(s)D$. Set for all $t\in\Omega_r$, $C_1(t)=(I+D)S(t)-DC(t)$, we have*

(i) $\{C_1(t)\}_{t\in\Omega_r}$ is a $C-$cosine family of bounded linear operators whose infinitesimal generator is an extension of A_1.

(ii) If $I+D$ is invertible, then for all $x\in X$, and $t\in\Omega_r$,

$$S(t)x=(1+D)^{-1}C_1(t)x+D(I+D)^{-1}C(t)x.$$

References

[1] Blali, A., Amrani, A. E., and Ettayb, J. $c-$cosine and mixed c_0-cosine families of bounded linear operators on non-Archimedean Banach spaces. *Novi Sad J. Math. 53*, 2 (2023), 207–222.

[2] Blali, A., Amrani, A. E., and Ettayb, J. On two-parameter $c-$groups of bounded linear operators on non-Archimedean Banach spaces. *Novi Sad J. Math. 53*, 1 (2023), 133–142.

[3] Blali, A., amrani, A. E., Ettayb, J., and Hassani, R. A. Cosine families of bounded linear operators on non-Archimedean Banach spaces. *Novi Sad J. Math. 52*, 1 (2022), 173–184.

[4] Diagana, T. C_0-semigroups of linear operators on some ultrametric Banach spaces. *Int. J. Math. Math. Sci.* (2006), Art. ID 52398, 1–9.

[5] Diagana, T., and Ramaroson, F. *Non-Archimedean operator theory.* SpringerBriefs in Mathematics. Springer, Cham, 2016.

[6] El Amrani, A., Blali, A., Ettayb, J., and Babahmed, M. A note on C_0-groups and $C-$groups on non-archimedean Banach spaces. *Asian-Eur. J. Math. 14*, 6 (2021), Paper No. 2150104, 19.

[7] Koblitz, N. *p-adic analysis: a short course on recent work*, vol. 46 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge-New York, 1980.

[8] Schikhof, W. H. *Ultrametric calculus*, vol. 4 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2006. An introduction to p-adic analysis, Reprint of the 1984 original [MR0791759].

[9] van Rooij, A. C. M. *Non-Archimedean functional analysis*, vol. 51 of *Monographs and Textbooks in Pure and Applied Math.* Marcel Dekker, Inc., New York, 1978.

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