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Complex Hermite polynomials: their generating functions via Lie algebra representation and operational methods

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Abstract. The purpose of this paper is to get new generating relations of the complex Hermite polynomials $H_{p,q}(z,z^*)$ by extending the realization $\uparrow_{\omega,\mu}$ to study multiplier representations of a Lie group G(0,1). Also, we establish new operational formulas involving the polynomials $H_{p,q}(z,z^*)$ and we will use them in a simple way to obtain new generating functions for the polynomials $H_{p,q}(z,z^*)$. Also, we derive some special cases which are worth interest.

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1. Introduction

As an interesting extension of the classical real Hermite polynomials $H_n(x)$ (see e.g. [1] and [4]):

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{\partial}{\partial x}\right)^n e^{-x^2}, n = 0,1,2,\cdots,$$

which are extensively studied, are the complex Hermite polynomials (see [10]):

$$H_{p,q}(z,z^*) = p!q! \sum_{n=0}^{\min(p,q)} \frac{(-1)^n z^{p-n} z^{*q-n}}{n!(p-n)!(q-n)!}, \tag{1.1}$$

where $z = x + iy$ a complex variable and $z^* = x - iy$ denotes its conjugate, $x, y \in \mathbb{R},\ p, q = 0,1,2\cdots$.

The known operational representations and generating functions concerning the real Hermite polynomials are extended to the complex case as follows[6]:

$$H_{p,q}(z,z^*) = (-1)^{p+q} e^{zz^*} \frac{\partial^{p+q}}{\partial z^{*p} \partial z^q} e^{-zz^*}, (p,q) > 0,$$

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(1.2)
$$H_{p,q}(z,z^*) = (-\partial_{z^*} + z)^p \ z^{*q} = (-\partial_z + z^*)^q \, z^p,$$

$$\sum_{q=0}^{\infty} H_{p,q}(z,z^*) \frac{t^q}{q!} = (z-t)^p \, e^{tz^*},$$

$$\sum_{p=0}^{\infty} H_{p,q}(z,z^*) \frac{t^p}{p!} = (z^*-t)^q \, e^{tz}.$$

Further, the complex Hermite polynomials $H_{p,q}(z,z^*)$ can be expressed in terms of $H_n(x)$ as (see[6]):

$$H_{p,q}(z,z^*) = p!q! \left(\frac{i}{2}\right)^{\left(\frac{p+q}{2}\right)} \sum_{j=0}^{p}\sum_{k=0}^{q} (-1)^{q+j} \frac{i^{j+k}}{j!k!} \frac{H_{j+k}(x) \ H_{p+q-j-k}(y)}{(p-j)!\,(q-k)!},$$

or in terms of the associated Laguerre polynomials (see [1] or [3]):

(1.3)
$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n} \frac{(-1)^k \Gamma(\alpha+n+1) z^k}{k!(n-k)!\Gamma(\alpha+k+1)}.$$

as follows

(1.4)
$$H_{p,q}(z,z^*) = (-1)^{\min(p+q)} (\min(p,q))! \, |z|^{|p-q|} \, e^{i(p-q)\arg(z)} L_{\min(p,q)}^{(|p-q|)}\left(|z|^2\right).$$

So that for $q = p + s$, the polynomials $H_{p,q}(z,z^*)$ are related to Laguerre polynomials via

$$H_{p,q}(z,z^*) = (-1)^p \, p! z^{*s} L_p^{(s)}\left(|z|^2\right).$$

The complex polynomials $H_{p,q}(z,z^*)$ have many applications to physical problems, (see [16] - [17]). Mathematical properties of these polynomials have been developed in Ghanmi[5, 6]. A multilinear generating function of Kibble-Slepian type [11] is proved in [8], Ismail and Zeng [9] gave a detailed study of a one-parameter generalization of the complex Hermite polynomials and derive linear and bilinear generating functions. So we can see that generalization of the Hermite polynomials to many variables and/or to the complex domain has been located in mathematical and physical literature for some decades. Recently Gorska et al. [7] for the first time investigated holomorphic Hermite polynomials in two variables by developing their algebraic and analytic properties.

In comparison with the above-said works, in this article, we will consider the problem of framing the two-variable complex Hermite polynomials $H_{p,q}(z,z^*)$ into the context of the representation $\uparrow_{\omega,\mu}$ to study multiplier representations of a Lie group G(0,1) and the representations of operational identities. Bilateral, bilinear, and linear generating relations involving $H_{p,q}(z,z^*)$ are obtained by using Miller's operational technique.

The rest of this paper is organized as follows. Section 2 is devoted to derive some expansion series and relationship of the complex Hermite polynomials

$H_{p,q}(z,z^*)$ in terms of certain special functions. Also, we establish several integral representations for the polynomials $H_{p,q}(z,z^*)$. In Section 3, we follow the approach of Miller [14] and obtain generating relations of the complex Hermite polynomials $H_{p,q}(z,z^*)$ by extending the realizations of $\uparrow_{\omega,\mu}$ to multiplier representations of a Lie group G(0,1). Finally, in Section 4, we show how readily new generating functions for the polynomials $H_{p,q}(x,y)$ can be derived from the operational representations obtained in Section 1.

2. Basic properties

First of all, by exploiting the results (see [1] or [4]):

$$(-n)_k = \begin{cases} \frac{(-1)^k n!}{(n-k)!}, & n \leq k \leq 0, \\ 0, & k > n, \end{cases} \tag{2.1}$$

and

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n},$$

where the Pochhammer symbol $(\lambda)_n$ is defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n \in \mathbb{N}, \end{cases}$$

Γ: Gamma function, and using the definition of the hypergeometric function ${}_2F_0$ (see [1]):

$${}_2F_0[a,b;-;z] = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{n!},$$

we find from (1.1) that

$$H_{p,q}(z,z^*) = z^{*q} z^p {}_2F_0\left[-p;-q;-;\frac{-1}{x^2+y^2}\right], \tag{2.2}$$

where $z = x+iy$, $z^* = x-iy$ and $x^2+y^2 \neq 0$.

Now, by the assertion (2.2) and with help of the representations of the hyper-geometric function ${}_2F_0$(see [3])

$$\begin{aligned} {}_2F_0\left[-n,a;-;z\right] &= (a)_n(-z)^n {}_1F_1\left[-n;1-a-n;-z^{-1}\right] \\ &= n! z^n L_n^{(-a-n)}(-z^{-1}), \end{aligned}$$

we can easily establish the explicit representations

$$H_{p,q}(z,z^*) = z^{*q} z^p (-q)_p (x^2+y^2)^{-p} \; {}_1F_1\left[-p;1+q-p;x^2+y^2\right], \tag{2.3}$$

$$H_{p,q}(z,z^*) = z^{*q} z^p (-p)_q (x^2+y^2)^{-q} \; {}_1F_1\left[-q;1+p-q;x^2+y^2\right],$$

or equivalently

$$H_{p,q}(z,z^*) = (-1)^p z^{*q} z^p p! \left(x^2+y^2\right)^{-p} L_p^{(q-p)}\left(x^2+y^2\right), \tag{2.4}$$

$$H_{p,q}(z,z^*) = (-1)^q z^{*q} z^p q! \left(x^2+y^2\right)^{-q} L_q^{(p-q)}\left(x^2+y^2\right), \tag{2.5}$$

where ${}_1F_1$ is the confluent hypergeometric function [4]:

$${}_1F_1[a;c;z] = \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!},$$

and $L_n^{(\alpha)}(x)$ is the associated Laguerre polynomials (see Eq. (1.3)). Since the polynomials $H_{p,q}(z,z^*)$ can be expressed in terms of representation involving the confluent hypergeometric function ${}_1F_1$ and the Laguerre polynomials $L_n^{(\alpha)}(x)$, the properties of these function and polynomials assume noticeable importance. Indeed, each of these properties will naturally lead to various other needed properties for the polynomials $H_{p,q}(z,z^*)$.

Next, according to the relation between Laguerre polynomials $L_n^{(\alpha)}(x)$ and Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ (see [1], p.189, Eq. (5.97)):

$$L_n^{(\alpha)}(z) = \lim_{\lambda\to\infty} P_n^{(\alpha,\lambda)}\left(1-\frac{2z}{\lambda}\right),$$

and with the aid of the assertion (2.5), we can obtain the explicit relation

$$H_{p,q}(z,z^*) = (-1)^q z^{*q} z^p q! \left(x^2+y^2\right)^{-q} \lim_{\lambda\to\infty} P_p^{(p-q,\lambda)}\left(1-\frac{2\left(x^2+y^2\right)}{\lambda}\right),$$

or equivalently (see Eq. (2.4))

$$H_{p,q}(z,z^*) = (-1)^p z^{*q} z^p p! \left(x^2+y^2\right)^{-p} \lim_{\lambda\to\infty} P_q^{(q-p,\lambda)}\left(1-\frac{2\left(x^2+y^2\right)}{\lambda}\right).$$

Further, the Laguerre polynomials $L_n^{(\alpha)}(x)$ have the following asymptotic representation which describes their behavior for a large value of the degree n [[12], p.87, Eq. (4.22.18)]; see also [13]:

$$L_n^{(\alpha)}(x) \approx \frac{\Gamma(\alpha+n+1)}{n!} e^{\frac{x}{2}} (Nx)^{\frac{-\alpha}{2}} J_\alpha(2\sqrt{(Nx)}), \tag{2.6}$$

$$n\to\infty, N = n+\frac{\alpha+1}{2},$$

where $J_n(x)$ is the Bessel function defined by [4]

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!(k+n)!}, -\infty < x < \infty.$$

In view of the explicit representation (2.5), it follows from (2.6) that

$$H_{p,q}(z,z^*) \approx (-1)^q z^p z^{*q} \Gamma(p+1) \left(x^2+y^2\right)^{-q} e^{\frac{\left(x^2+y^2\right)}{2}} \left(N(x^2+y^2)\right)^{\frac{q-p}{2}}$$
$$\times J_{q-p}\left[2\sqrt{(N(x^2+y^2))}\right], q \to \infty, N = \frac{p+q+1}{2}.$$

Since

$$\left(\frac{\partial}{\partial z}\right)^r z^m = \frac{m!}{(m-r)!} z^{m-r},$$

and in view of the result (2.1), we infer from the series representation (1.1) the operational formula

$$H_{p,q}(z,z^*) = \left(1 - z^{-1}\frac{\partial}{\partial z^*}\right)^p z^p z^{*q}. \tag{2.7}$$

Next, using the result

$$\left(\frac{\partial}{\partial z}\right)^r z^r = r!,$$

and the definition of the classical Laguerre polynomials $L_n(x)$ [4]:

$$L_n(x) = \sum_{k=0}^{n} \frac{(-1)^k n! x^k}{k!(n-k)!},$$

we can establish the following operational relation

$$H_{p,q}(z,z^*) = L_p\left(\frac{\partial^2}{\partial t \partial z^*} t/z\right) z^p z^{*q}. \tag{2.8}$$

Now, we consider some integral representations for the polynomials $H_{p,q}(z,z^*)$. To obtain integral representations, we first recall the following results (see [[1], p.300, Eq. (9.13) and p.306, Eq. (17)], [[4], Section (6.11.1), Eq. (3)]):

$${}_1F_1[a;c;x] = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt} t^{a-1}(1-t)^{c-a-1} dt, \tag{2.9}$$

$${}_1F_1[a;c;x] = \frac{\Gamma(c)}{\Gamma(c-a)} e^x x^{\left(\frac{1-c}{2}\right)} \int_0^\infty e^{-t} t^{\frac{1}{2}(c-1)-a} J_{c-1}(2\sqrt{(xt)}) dt, \tag{2.10}$$

and

$${}_1F_1[a;c;x] = \frac{\Gamma(\gamma)\Gamma(1-a)}{2\pi i \Gamma(c-a)} \oint_\gamma e^{xs} \left(\frac{s}{s-1}\right)^a (1-s)^{c-1} \frac{ds}{s}, \tag{2.11}$$

where c is a positive integer and the contour γ starts and ends at the point $s = 1$ on the s-axis and encircles the origin in a positive direction and that $\Re(c) > \Re(a)$. Also, a fourth representation can be obtained from [[4], Section

(6.11.1), Eq. (7)], for $\Re(c) > 0, \gamma > 1$ and $a \neq 1, 2, 3, \cdots, c-1$, with $b = c = n+1, n = 0, 1, \cdots$, where the integrand is a one-valued function of the parameter s and the path of integration may be replaced by a contour, for instance a circle $|s| = \rho > 1$. This representation is given by (see [4]):

$$ {}_1F_1[a; c; x] = \frac{\Gamma(c)}{2\pi i x^{c-1}} \oint_\gamma e^{xs} \left(\frac{s}{s-1} \right)^a \frac{ds}{s^c}. \tag{2.12} $$

Directly from the results (2.9) to (2.12) and based on the definition (2.3), we can establish the following integral representations.

Theorem 2.1. *For the complex Hermite polynomials $H_{p,q}(z, z^*)$ the following integral representations hold, true:*

$$ \begin{aligned} H_{p,q}(z, z^*) &= \frac{\Gamma(1+q-p)}{\Gamma(-p)\Gamma(1+q)} z^{*q} z^p (-q)_p (x^2+y^2)^{-p} \\ &\times \int_0^1 e^{(x^2+y^2)t} t^{-(p+1)} (1-t)^q dt, \end{aligned} \tag{2.13} $$

$$ \begin{aligned} H_{p,q}(z, z^*) &= \frac{\Gamma(1+q-p)}{\Gamma(1+q)} z^{*q} z^p (-q)_p (x^2+y^2)^{\frac{1}{2}(p-q)} e^{(x^2+y^2)} \\ &\times \int_0^\infty e^{-t} t^{\frac{1}{2}(3p-q)} J_{q-p}(2\sqrt{(x^2+y^2)t})) dt, \end{aligned} \tag{2.14} $$

$$ \begin{aligned} H_{p,q}(z, z^*) &= \frac{\Gamma(\gamma)\Gamma(1+p)}{2\pi i\ \Gamma(q+1)} z^{*q} z^p (-q)_p (x^2+y^2)^{-p} \\ &\times \oint_\gamma e^{(x^2+y^2)s} \left(\frac{s}{s-1} \right)^{-p} (1-s)^{q-p} \frac{ds}{s}, \end{aligned} \tag{2.15} $$

$$ \begin{aligned} H_{p,q}(z, z^*) &= \frac{\Gamma(1+q-p)}{2\pi i} z^{*q} z^p (-q)_p (x^2+y^2)^{2p-q} \\ &\times \oint_\gamma e^{(x^2+y^2)s} \left(\frac{s}{s-1} \right)^{-p} \frac{ds}{s^{1+q-p}}. \end{aligned} \tag{2.16} $$

Proof. By replacing the function ${}_1F_1$ in (2.3) by its integral representation (2.9), we get

$$ \begin{aligned} H_{p,q}(z, z^*) &= z^{*q} z^p (-q)_p (x^2+y^2)^p \frac{\Gamma(1+q-p)}{\Gamma(-p)\Gamma(1+q)} \\ &\times \int_0^1 e^{(x^2+y^2)t} t^{-(p+1)} (1-t)^q dt, \end{aligned} $$

which gives us the formula (2.13). Similarly, by employing the relations (2.10), (2.11) and (2.12) and exploiting the same procedure leading to (2.13), one can derive the formulas (2.14), (2.15) and (2.16), respectively. □

3. Representation $\uparrow_{\omega,\mu}$ of $\ell(0,1)$ and generating relations

We note that the following isomorphism [[14], p.36]

$$\ell(0,1)\cong L[G(0,1)],$$

where $\ell(0,1)\cong L[G(0,1)]$ is the Lie algebra of the complex four-dimensional harmonic oscillator Lie group G(0,1) [[15], Chapter 10], multiplicative matrix group with elements [[14], p.9]

$$g(a,b,c,\tau)=\begin{pmatrix} 1 & ce^{\tau} & a & \tau \\ 0 & e^{\tau} & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where a,b,c,$\tau \in \mathbb{C}$.

A basis for L[G(0,1)] is provided by the matrices [[14], p.9]:

$$j^{+}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ j^{-}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$j^{3}=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ \varepsilon=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with commutation relations

$$[j^3,j^{\pm}]=\pm j^{\pm},\ [j^+,j^-]=-\varepsilon,\ [\varepsilon,j^{\pm}]=[\varepsilon,j^3]=0.$$

The machinery constructed in [[14], Chapter 1,2 and 4] will be applied to find realization of the irreducible representation $\uparrow_{\omega,\mu}$ of $\ell(0,1)$ where $\{\omega,\mu\}\in\mathbb{C}$ such that $\mu\neq 0$. The spectrum S of $\uparrow_{\omega,\mu}$ is the set S=$\{-\omega+k, k$ is nonnegative integer$\}$.

In particular we are looking for the function

$$f_{m,n}(z,z^*,p,s)=Z_{m,n}(z,z^*)p^m s^n,$$

such that

$$\begin{aligned} J^3 f_{m,n} &= m f_{m,n}, E f_{m,n}=\mu f_{m,n},\\ J^+ &= \mu f_{m+1,n}, J^- f_{m,n}=(m+\omega) f_{m-1,n},\\ C_{0,1} f_{m,n} &= (J^+J^- - EJ^3) f_{m,n}=\mu\omega f_{m,n}, \end{aligned}\tag{3.1}$$

for all $m\in$ S, where the operator

$$C_{0,1}=j^+j^- - Ej^3,$$

is known as the Casimir operator (see [[14], p.32]). The commutation relations satisfied by the operators $J^{\pm},J^3,E$ are

$$[J^3,J^{\pm}]=\pm J^{\pm},\ [J^+,J^-]=-E,\ [J^{\pm},E]=[J^3,\ E]=0.\tag{3.2}$$

The number of possible solutions of the assertion (3.2) is tremendous . We assume that these operators take the form

$$\begin{aligned} J^{+} &= p\left(z - \frac{\partial}{\partial z^*}\right), \\ J^{-} &= \frac{1}{p}\frac{\partial}{\partial z}, \\ J^{3} &= p\frac{\partial}{\partial p}, \\ E &= 1, \end{aligned}$$

where $(z, p) \in \mathbb{C}$ and note these operators satisfy the commutation relations (3.2).

We can assume that $\omega = 0$ and $\mu = 1$ without any loss of generality for the theory of special functions. In terms of the functions $Z_{m,n}(z, z^*)$ equations (3.1) become

$$\begin{aligned} \left(z - \frac{\partial}{\partial z^*}\right) Z_{m,n}(z, z^*) &= Z_{m+1,n}(z, z^*), \\ \left(\frac{\partial}{\partial z}\right) Z_{m,n}(z, z^*) &= mZ_{m-1,n}(z, z^*), \\ \left(-\frac{\partial^2}{\partial z^* \partial z} + z\frac{\partial}{\partial z} - m\right) Z_{m,n}(z, z^*) &= 0, \\ m &= 0, 1, 2, \cdots . \end{aligned} \tag{3.3}$$

Again, if we take the function

$$f_{m,n}(z, z^*, p, s) = Z_{m,n}(z, z^*)p^m s^n,$$

such that

$$\begin{aligned} J^{3'} f_{m,n} &= n f_{m,n}, E' f_{m,n} = \mu f_{m,n}, \\ J^{+'} &= \mu f_{m,n+1}, J^{-'} f_{m,n} = (n + \omega) f_{m,n-1}, \\ C'_{0,1} f_{m,n} &= (J^{+'} J^{-'} - E' J^{3'}) f_{m,n} = \mu\omega f_{m,n}, \end{aligned} \tag{3.4}$$

for all $n \in$S, then the differential operators $J^{\pm'}, J^{3'}$ and E' are given by

$$\begin{aligned} J^{+'} &= s\left(z^* - \frac{\partial}{\partial z}\right), \\ J^{-'} &= \frac{1}{s}\frac{\partial}{\partial z^*}, \\ J^{3'} &= s\frac{\partial}{\partial s}, \\ E' &= 1, \end{aligned} \tag{3.5}$$

where $(z^*, s) \in \mathbb{C}$ and satisfy the commutation relations identical to (3.2).

Just like before taking $\omega = 0$ and $\mu = 1$, equations (3.4) become

$$\begin{aligned} \left(z^* - \frac{\partial}{\partial z}\right) Z_{m,n}(z, z^*) &= Z_{m,n+1}(z, z^*), \\ \left(\frac{\partial}{\partial z^*}\right) Z_{m,n}(z, z^*) &= n Z_{m,n-1}(z, z^*), \\ \left(-\frac{\partial^2}{\partial z^* \partial z} + z^* \frac{\partial}{\partial z^*} - n\right) Z_{m,n}(z, z^*) &= 0, \\ n &= 0, 1, 2, \cdots . \end{aligned} \tag{3.6}$$

We see from (3.3) and (3.7) that

$$Z_{m,n}(z, z^*) = H_{m,n}(z, z^*),$$

where $H_{m,n}(z, z^*)$ is given by (1.2).

Functions

$$f_{m,n}(z, z^*, p, s) = H_{m,n}(z, z^*) p^n s^n, m \in \mathrm{S},$$

form a basis for a realization of the representation $\uparrow_{0,1}$ of $\ell(0,1)$. This realization of $\ell(0,1)$ can be extended to a local multiplier representation $T(g)$,$g \in G(0,1)$ defined on F the space of all functions analytic in a neighborhood of the point

$$(z^0, z^{*0}, p^0, s^0) = (1, 1, 1, 1).$$

Using operators (3.5), the multiplier representation ([14], p.17) takes the form

$$\begin{aligned} [T(\exp a\varepsilon) f](z, z^*, p, s) &= \exp(a) f(z, z^*, p, s), \\ [T(\exp b j^+) f](z, z^*, p, s) &= \exp(bzp) f(z, z^* - bp, p, s), \\ [T(\exp c j^-) f](z, z^*, p, s) &= f\left(z + \frac{c}{p}, z^*, p, s\right), \\ [T(\exp \tau j^3) f](z, z^*, p, s) &= f(z, z^*, pe^\tau, s), \end{aligned}$$

for $f \in$ F. If $g \in G(0,1)$ has parameters (a, b, c, τ), then

$$T(g) = T(\exp a\varepsilon) T(\exp b j^+) T(\exp c j^-) T(\exp \tau j^3),$$

and therefore we obtain

$$[T(g) f](z, z^*, p, s) = \exp(a + bzp) f\left(z + \frac{c}{p}, z^* - bp, pe^\tau, s\right).$$

The matrix element of $T(g)$ with respect to the analytic basis

$$f_{m,n}(z, z^*, p, s) = H_{m,n}(z, z^*) p^m s^n,$$

are the functions $A_{lk}(g)$ uniquely determined by $\uparrow_{\omega,\mu}$ of $\ell(0,1)$ and we obtain relations

$$[T(g) f_{k,n}](z, z^*, p, s) = \sum_{l=0}^{\infty} A_{lk}(g) f_{l,n}(z, z^*, p, s), k = 0, 1, 2, \cdots ,$$

which simplify to the identity

$$\exp(a+\tau k+bzp)H_{k,n}\left(z+\frac{c}{p},z^{*}-bp\right)=\sum_{l=0}^{\infty}A_{lk}(g)H_{l,n}(z,z^{*})p^{l-k}, \tag{3.7}$$

$k=0,1,2,\cdots$, and the matrix elements $A_{lk}(g)$ are given by [[14], p.87, Eq.(4.26)]

$$A_{lk}(g)=\exp(a+\tau k)c^{k-l}L_{l}^{k-l}(-bc), k,l\geq 0. \tag{3.8}$$

Substituting (3.7) into (3.8), we obtain the generating relation

$$\begin{aligned}\exp(bzp)\ H_{k,n}\left(z+\frac{c}{p},z^{*}-bp\right)&=\sum_{l=0}^{\infty}c^{k-l}L_{l}^{k-l}(-bc)H_{l,n}(z,z^{*})p^{l-k},\\ &\qquad b,c,p\in\mathbb{C},\ n,k=0,1,2,\cdots.\end{aligned} \tag{3.9}$$

Again, taking the operators (3.5) and proceeding exactly as before, we get

$$\begin{aligned}\exp(b^{'}zs)H_{m,q}\left(z-b^{'}s,z^{*}+\frac{c^{'}}{s}\right)&=\sum_{i=0}^{\infty}(c^{'})^{q-i}L_{i}^{q-i}(-bc)\ H_{m,i}(z,z^{*})s^{i-q},\\ &\qquad b^{'},c^{'},s\in\mathbb{C},m,q=0,1,2,\cdots.\end{aligned} \tag{3.10}$$

Now, we turn to some applications of the generating relations obtained in this section, which yield many new and known relations for the polynomials related to complex Hermite polynomials.

I. Making use of (1.4) in (3.9) we get

$$\begin{aligned}&k!\exp(bzp)\left(1-\frac{bp}{z^{*}}\right)^{n-k}L_{k}^{(n-k)}\left(\left(z+\frac{c}{p}\right)(z^{*}-bp)\right)\\ &=\sum_{l=0}^{\infty}c^{k-l}L_{l}^{(k-l)}(-bc)L_{l}^{(n-l)}(zz^{*})(z^{*})^{k-l}(-p)^{l-k}l!,(b,c,p\in\mathbb{C},n,k=0,1,2,\cdots).\end{aligned} \tag{3.11}$$

Now in particular, taking $z=1$ in equation (3.11) and replacing z^{*},p and n by $u,-t$ and q, respectively, we get the following result of Miller [[14], p.112(4.94)]:

$$\begin{aligned}&k!e^{-bt}(1+bt/u)^{q-k}L_{k}^{(q-k)}[u(1+bt/u)(1-c/t)]t^{t}k\\ &=\sum_{l=0}^{\infty}c^{k-l}L_{l}^{k-l}(-bc)l!u^{k-l}L_{l}^{(q-l)}(u)t^{l}.\end{aligned}$$

II. Again making use of (1.4) in (3.10) we get

$$\begin{aligned}&\exp(b^{'}zs)\left(z^{*}+\frac{c^{'}}{s}\right)^{q-m}L_{m}^{(q-m)}\left(\left(z-b^{'}s\right)\left(z^{*}+\frac{c^{'}}{s}\right)\right)\\ &=\sum_{i=0}^{\infty}(c^{'})^{q-i}L_{i}^{q-i}(-bc)(z^{*})^{i-m}L_{m}^{(i-m)}(zz^{*})s^{i-q},(b^{'},c^{'},s\in\mathbb{C},\ m,q=0,1,2,\cdots.)\end{aligned} \tag{3.12}$$

Next, in particular taking $s = 1$ in assertion (3.12) and replacing $z, z^*, b^{'}, c^{'}, m, q$ and r by $b_1, c_1, -b_2, c_2, l, l+n$ and j, respectively, we get a known result of Miller [[14], p.88(4.28)]:

$$e^{-c_1b_2}(c_1+c_2)^n L_l^{(n)}[(b_1+b_2)(c_1+c_2)] = \sum_{j=0}^{\infty} c_1^{j-l} L_l^{(j-l)}(b_1c_1)c_2^{l+n-j} L_j^{(l+n-j)}(b_2c_2).$$

4. Generating relations via operational methods

In this section, we show how readily new generating functions for the polynomials $H_{p,q}(z, z^*)$ can be derived from the operational representations obtained in Section 1. First, in the identity (2.8) multiply throughout by $\frac{u^p}{p!}$ and sum, to get

$$\sum_{p=0}^{\infty} L_p\left(\frac{\partial^2}{\partial t \partial z^*} t/z\right) \frac{(zu)^p}{p!} z^{*q} = \sum_{p=0}^{\infty} H_{p,q}\left(z, z^*\right) \frac{u^p}{p!}.$$

Now, if we replace the left-hand side of above equation by the left-hand side of the well-known generating function [1]:

$$e^t {}_0F_1\left(-; 1; -tx\right) = \sum_{p=0}^{\infty} L_p(x) \frac{t^p}{p!}, \tag{4.1}$$

we get the desired generating function:

$$e^{uz} {}_0F_1\left[-; 1; -u\left(\frac{\partial}{\partial t}\frac{\partial}{\partial z^*}\ t\right)\right]\ z^{*q} = \sum_{p=0}^{\infty} H_{p,q}(z, z^*) \frac{u^p}{p!}. \tag{4.2}$$

In the same manner, from the operational identity in (4.2), one can derive the following unilateral generating functions

$$\sum_{p=0}^{\infty} H_{p,q}\left(z, z^*\right) \frac{u^p}{p!} = \exp\left(\left(-\partial_{z^*} + z\right) u\right)\ z^{*q},$$

and

$$\sum_{q=0}^{\infty} H_{p,q}\left(z, z^*\right) \frac{u^q}{q!} = \exp\left(\left(-\partial_{z} + z^*\right) u\right)\ z^{p}.$$

Again, by starting from equation (2.7) multiplying throughout by

$$(\lambda)_{p+q} \frac{u^p v^q}{p!q!},$$

here and elsewhere $\lambda \in \mathbb{C}$, and exploiting the previously outlined method, we can show that

$$\left[1 - zu - \left(1 - z^{-1}\frac{\partial}{\partial z^*}\right) z^* v\right]^{-\lambda} = \sum_{p,q=0}^{\infty} (\lambda)_{p+q}\ H_{p,q}(z, z^*) \frac{u^p}{p!}\frac{v^q}{q!}.$$

The previously outlined procedure offers a useful tool for the derivation of other families of generating functions for the polynomials $H_{p,q}(x,y)$. For instance, let us consider the generating relation

$$f(z,z^*,w,w^*;u,v)=\sum_{p,q=0}^{\infty}(\lambda)_{p+q}\ H_{p,q}(z,z^*)\times H_{p,q}(w,w^*)\frac{u^p}{p!}\frac{v^q}{q!},$$

which according to the assertion (2.13) yields

$$f(z,z^*,w,w^*;u,v)=\sum_{p,q=0}^{\infty}(\lambda)_{p+q}\ H_{p,q}(z,z^*)\times H_{p,q}(w,w^*)\frac{u^p}{p!}\frac{v^q}{q!}$$

$$=\sum_{p,q=0}^{\infty}(\lambda)_{p+q}\left(1-z^{-1}\frac{\partial}{\partial z^*}\right)^p\left(1-w^{-1}\frac{\partial}{\partial w^*}\right)^p(zwu)^p(z^*w^*v)^q\frac{u^p}{p!}\frac{v^q}{q!}.$$

Now, on using the multinomial expression

$$(1-x-y)^n=\sum_{p,q=0}^{\infty}(-n)_{p+q}\frac{x^py^q}{p!q!},$$

we obtain the following bilinear generating function:

$$\left[1-\left(1-z^{-1}\frac{\partial}{\partial z^*}\right)\left(1-w^{-1}\frac{\partial}{\partial w^*}\right)zwu-z^*w^*v\right]^{-\lambda}$$

$$=\sum_{p,q=0}^{\infty}(\lambda)_{p+q}H_{p,q}(z,z^*)\times H_{p,q}(w,w^*)\frac{u^p}{p!}\frac{v^q}{q!}. \tag{4.3}$$

In [2] the following 2D-Laguerre-Konhauser polynomials have been introduced

$${}_kL_n^{(\alpha,\beta)}(x,y)=n!\sum_{s}^{n}\sum_{r}^{n-s}\frac{(-1)^{s+r}x^{\alpha+r}y^{\beta+ks}}{s!r!(n-s-r)!\Gamma(\alpha+r+1)\Gamma(ks+\beta+1)}.$$

where $k=1,2,\cdots$, together with the operational identity

$$\left(1-\left(\frac{\partial}{\partial x}\right)^{-1}-\left(\frac{\partial}{\partial y}\right)^{-k}\right)^n\frac{x^\alpha y^\beta}{\Gamma(\alpha+1)\Gamma(\beta+1)}={}_kL_n^{(\alpha,\beta)}(x,y). \tag{4.4}$$

Let us consider the generating relation

$$f(z,z^*,w,w^*;u,v)=\sum_{p,q=0}^{\infty}(\lambda)_{p+q}\ H_{p,q}(z,z^*)\times{}_kL_p^{(\beta,\gamma)}(w,w^*)\frac{u^p}{p!}\frac{v^q}{q!}. \tag{4.5}$$

Now, directly from (2.7) and (4.4) by employing the previously outlined method leading to the bilinear generating function (4.3), we obtain from (4.5) the following bilateral generating function:

$$\left[1-\left(1-z^{-1}\frac{\partial}{\partial z^*}\right)\left(1-\left(\frac{\partial}{\partial w}\right)^{-1}-\left(\frac{\partial}{\partial w^*}\right)^{-k}\right)zt\right]^{-\lambda}\frac{w^\alpha w^{*\beta}z^{*q}}{\Gamma(\alpha+1)\Gamma(\beta+1)}$$

$$=\sum_{p,q=0}^{\infty}(\lambda)_{p+q}H_{p,q}(z,z^*)\times {}_kL_p^{(\beta,\gamma)}(w,w^*)\frac{u^p}{p!}\frac{v^q}{q!}.$$

It is worthy to note that the obtained generating functions in this section are modification and generalization of several known results. For instance, the result (4.2) is a generalization of the formula (4.1).

5. Conclusion

We have considered the problem of framing complex Hermite polynomials $H_{p,q}(z,z^*)$ into the context of the representation $\uparrow_{\omega,\mu}$ of the Lie algebra $\ell(0,1)$ of the complex harmonic group $G(0,1)$. Generating relations involving the complex Hermite polynomials are obtained by using Miller's and operational technique. Some (known and new) relations for the products of Hermite and Laguerre polynomials and identities of Miller are also obtained as special cases.

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