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Kenmotsu manifolds endowed with the semi-symmetric non-metric ϕ-connection to its tangent bundle

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Abstract. The goal of this paper is to study the complete lift of the semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold to its tangent bundle TM and to obtain a relation between the semi-symmetric non-metric ϕ-connection $\bar{D}^C$ on Kenmotsu manifolds with respect to Levi-Civita connection D^C by utilizes specialized mathematical operators. on TM. Next, the complete lifts of the curvature tensor, the scalar curvature and the Ricci tensor on TM are constructed and show that Ricci tensor is symmetric on TM. Finally, a study of the complete lift of curvature tensor concerning the semi-symmetric non-metric ϕ-connection to its tangent bundle TM is done which shows that if the complete lift of the curvature tensor of $\bar{D}^C$ vanishes on TM, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(1)$ on TM.

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1. Introduction

The differential geometry of the tangent bundle have been substantially studied by Yano and Kobayashi [27], Yano and Ishihara [26], Tani [24], Pandey and Chaturvedi [7, 22] and the first author [18, 17]. The vertical, complete and horizontal lifts of tensors and connections from manifold to its tangent bundles were developed by Yano and Ishihara [26]. Tangent bundle immersed with quarter-symmetric and semi-symmetric non-metric connections on an almost Hermitian and a Kähler manifolds was studied by the first author [17, 15]. Recently, Akpinar [2], studied the complete lift of Weyl connection to the tangent bundle of hypersurface.

On the other hand, the study of semi-symmetric metric and linear connections on differential manifold were started in early 1930 by Friedman and Schouton [10] and Hayden [11]. The notion of semi-symmetric metric ϕ-connection

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was introduced by Yano and Imai [25]. Most recently, Barman et al [5] studied a special type of semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold and they proved that if the curvature tensor of Kenmotsu manifolds admitting a special type of semi-symmetric non-metric ϕ-connection $\bar{D}$ vanishes, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(1)$. Numerous investigators [8, 20, 21, 23] have studied Kenmotsu manifolds and gave their theories.

Let M be an n-dimensional differentiable manifold of class C^∞ with the Levi-Civita connection D. Let U and W be vector fields on M. Then a linear connection $\bar{D}$ is known as symmetric connection on M if the torsion tensor T of $\bar{D}$ defined by

$$T(U,W) = \bar{D}_U W - \bar{D}_W U - [U,W] \tag{1.1}$$

vanishes, otherwise it is non-symmetric. If the torsion tensor T satisfies

$$T(U,W) = \pi(W)U - \pi(U)W \tag{1.2}$$

and

$$(\bar{D}_U g)(W,Z) = -\pi(W)g(U,Z) - \pi(Z)g(U,W) \tag{1.3}$$

where

$$\pi(U) = g(P,U), \tag{1.4}$$

π is 1-form, g is a Riemannian metric and P is a vector field, then a linear connection $\bar{D}$ is called a semi-symmetric non-metric connection [1].

The present paper is organized as: Section 2 presents basic definitions of the tangent bundle, lifts, Kenmotsu manifold and semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold. Section 3 discusses the complete lift of the semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold to its tangent bundle TM and to obtain a relation between the semi-symmetric non-metric ϕ-connection $\bar{D}^C$ on Kenmotsu manifolds with respect to Levi-Civita connection D^C on TM. In Section 4, the complete lift of the curvature tensor, the scalar curvature and the Ricci tensor on the tangent bundle is constructed and shows that Ricci tensor is symmetric. Moreover, a study of the complete lift of curvature tensor concerning the semi-symmetric non-metric ϕ-connection to its tangent bundle is done which shows that if the complete lift of the curvature tensor of $\bar{D}^C$ vanishes on TM, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(1)$ on TM.

2. Preliminaries

Let M be a differentiable manifold and $TM = \bigcup_{p\in M} T_pM$ be the tangent bundle, where T_pM is the tangent space at point $p \in M$ and $\pi : TM \to M$ is the natural bundle structure of TM over M. For any coordinate system (Q, x^h) in M, where (x^h) is local coordinate system in the neighborhood Q, then $(\pi^{-1}(Q), x^h, y^h)$ is coordinate system in TM, where (x^h, y^h) is induced coordinate system in $\pi^{-1}(Q)$ from (x^h) [26].

2.1. Vertical lifts

If f is a function in M, then f^V is the vertical lift of function f in TM such that $f^V = f \circ \pi$. Suppose that η is a 1-form in M. Then vertical lift η^V of the 1-form η is given by $\eta^V = (\eta_i)^V (dx^i)^V$, where η_i are local components of η. Let U and ϕ be the vector field and tensor field of type (1,1) in M. Then U^V and ϕ^V be the vertical lift of U and ϕ, respectively, in TM and given by the form of components

$$U^V : \begin{pmatrix} 0 \\ x^h \end{pmatrix} \tag{2.1}$$

and

$$\phi^V : \begin{pmatrix} 0 & 0 \\ \phi_i^h & 0 \end{pmatrix}, \tag{2.2}$$

where x^h and ϕ_i^h are local components of U and ϕ, respectively.

2.2. Complete lifts

If f is a function in M, then the complete lift f^C of the function f in TM has components of the form $f^C = y^i \partial_i f = \partial f$, where ∂f is exterior derivative of f. Suppose that η is a 1-form in M. The complete lift η^C of η in TM has components of the form $\eta^C = (\partial\eta_i, \eta_i)$, where $\partial\eta_i$ is exterior derivative of η_i. Let U and ϕ be the vector field and tensor field of type (1,1) in M. Then U^C and ϕ^C are the complete lifts of U and ϕ, respectively, in TM and given by the form of components

$$U^C : \begin{pmatrix} x^h \\ \partial x^h \end{pmatrix} \tag{2.3}$$

and

$$\phi^V : \begin{pmatrix} \phi_i^h & 0 \\ \partial\phi_i^h & \phi_i^h \end{pmatrix}, \tag{2.4}$$

where x^h andϕ_i^h are local components of U and ϕ, respectively.

The complete lift of D on TM is denoted by D^C and given by

$$D^C_{U^C} W^C = (D_U W)^C, \qquad D^C_{U^C} W^V = (D_U W)^V. \tag{2.5}$$

Proposition 2.1. *[17, 26] Let M be the differentiable manifold and TM its tangent bundle. Then*

$$(fU)^V = f^V U^V, (fU)^C = f^C U^V + f^V U^C, \tag{2.6}$$

$$U^V f^V = 0, U^V f^C = U^C f^V = (Uf)^V, U^C f^C = (Uf)^C, \tag{2.7}$$

$$\eta^V(f^V) = 0, \eta^V(U^C) = \eta^C(U^V) = \eta(U)^V, \eta^C(U^C) = \eta(U)^C, \tag{2.8}$$

$$\phi^V U^C = (\phi U)^V, \phi^C U^C = (\phi U)^C, \tag{2.9}$$

$$[U, W]^V = [U^C, W^V] = [U^V, W^C], [U, W]^C = [U^C, W^C]. \tag{2.10}$$

Moreover, let L and K be arbitrary tensor fields in the manifold M. Then, by definition

$$(L \otimes K)^V = L^V \otimes K^V, (L \otimes K)^C = L^C \otimes K^V + L^V \otimes K^C,$$

$$(L + K)^V = L^V + K^V, (L + K)^C = L^C + K^C,$$

where L^C and K^C are the complete lifts of arbitrary tensor fields of L and K on TM.

2.3. Kenmotsu manifolds

A $(2n+1)$-dimensional differentiable manifold M is called an almost contact structure and denoted by (ϕ, ξ, η) if there are given a tensor field ϕ of type (1, 1), a vector field ξ and a 1-form η on M satisfying [3, 6]

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0 \quad \eta \circ \phi = 0. \tag{2.11}$$

Let g be a Riemannian metric such that

$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W) \tag{2.12}$$

and

$$g(U, \xi) = \eta(U), \tag{2.13}$$

for arbitrary vector fields U and W on M, then the structure (ϕ, ξ, η, g) is called the almost contact metric manifold.

Let Ω be 2-form on M given by $\Omega(U, W) = g(U, \phi W)$. The almost contact metric manifold is said to be an almost Kenmotsu manifold if

$$d\eta = 0; d\Omega = 2\eta \wedge \Omega.$$

Any normal almost Kenmotsu manifold is a Kenmotsu manifold M. An almost contact metric structure (ϕ, ξ, η, g) is a Kenmotsu manifold M [12] iff

$$(D_U \phi)W = g(\phi U, W)\xi - \eta(W)\phi U. \tag{2.14}$$

Let R and S be the curvature tensor and the Ricci tensor of M, respectively. The Kenmotsu manifold M has the following properties [4, 9]:

$$D_U \xi = U - \phi(U)\xi, \tag{2.15}$$

$$(D_U \eta)W = g(U, W)\xi - \eta(U)\eta(W) \tag{2.16}$$

$$R(U, W)\xi = \eta(U)W - \eta(W)U, \tag{2.17}$$

$$R(\xi, U)W = \eta(U)W - g(U, W)\xi, \tag{2.18}$$

$$\eta(R(U, W)Z) = g(U, W)\eta(W) - g(W, Z)\eta(U), \tag{2.19}$$

$$S(U, \xi) = -2n\eta(U). \tag{2.20}$$

2.4. Semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold

Let (M^{2n+1}, g) be a Kenmotsu manifold with a Riemannian metric g and the Levi-Civita connection D. The linear connection $\bar{D}$ on M is defined by

$$\bar{D}_U W = D_U W - \eta(W)U - 2\eta(U)W + g(U, W)\xi, \tag{2.21}$$

for arbitrary vector fields U and W on M. With the help of (1.1) and using (2.21), the torsion tensor T of M with respect to the connection $\bar{D}$ is given by

$$T(U, W) = \bar{D}_U W - \bar{D}_W U - [U, W] = \eta(W)U - \eta(U)W. \tag{2.22}$$

The linear connection $\bar{D}$ satisfying (2.22) is a semi-symmetric connection. In view of (2.21), the metric g satisfies the relation

$$\begin{aligned} (\bar{D}_U g)(W, Z) &= \bar{D}_U g(W, Z) - g(\bar{D}_U W, Z) - g(W, \bar{D}_U Z) \\ &= 4\eta(U)g(W, Z) \neq 0, \end{aligned} \tag{2.23}$$

for all vector fields U, W, Z on M. The linear connection $\bar{D}$ satisfying (2.22) and (2.23) is called a semi-symmetric non-metric connection [28]. By making use of (2.13), (2.16) and (2.21), tensor field ϕ satisfies the relation

$$(\bar{D}_U \phi)(W) = \bar{D}_U \phi W - \phi(\bar{D}_U W) = 0. \tag{2.24}$$

The linear connection $\bar{D}$ defined by (2.21) satisfying (2.22), (2.23) and (2.24) is a special type of semi-symmetric non-metric ϕ-connection on Kenmotsu manifolds [5].

3. Proposed theorem on the complete lift of semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold to its tangent bundle

In this section, a study is done on the complete lift of semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold to its tangent bundle and we establish the relation between the Levi-Civita connection D^C and the semi-symmetric non-metric ϕ-connection $\bar{D}^C$ on TM.

Let M be a Kenmotsu manifold and TM its tangent bundle. Let $f^V, U^V, \eta^V, \phi^V, D^V$ and $f^C, U^C, \eta^C, \phi^C, D^C$ be the vertical and the complete lifts on the tangent bundle TM of a function f, a vector field U, a 1-form η, a tensor field ϕ of type (1,1) and an affine connection D, respectively, on the manifold M [14, 16].

Operating the complete lift on (1.1), (1.2), (2.13)-(2.12) by mathematical operators, the obtained equations are

$$\begin{aligned}
(\phi^C)^2 &= -I + \eta^V \otimes \xi^C - \eta^C \otimes \xi^V, && (3.1)\\
\eta^C \xi^C &= \eta^V \xi^V = 0, \quad \eta^C \xi^V = \eta^V \xi^C = 1, && (3.2)\\
\phi^C \xi^V &= \phi^V \xi^C = \phi^V \xi^V = \phi^C \xi^C = 0, && (3.3)\\
g(U^C, \xi^C) &= \eta^C(U^C), && (3.4)\\
\eta^V \circ \phi^C &= \eta^C \circ \phi^V = \eta^C \circ \phi^C = \eta^V \circ \phi^V = 0, && (3.5)\\
g^C((\phi U)^C, (\phi W)^C) &= g^C(U^C, W^C) - \eta^C(U^C)\eta^V(W^C)\\
&- \eta^V(U^C)\eta^C(W^C), && (3.6)
\end{aligned}$$

for arbitrary vector fields U^C and W^C on TM. Let D^C be the complete lift of the Levi-Civita connection D of a Riemannian metric g. The complete ϕ^C of a tensor field ϕ on TM satisfies the relation

$$\begin{aligned}
(D^C_{U^C}\phi^C)W^C &= g^C((\phi U)^C, W^C)\xi^V + g^C(((\phi U)^V, W^C)\xi^C\\
&- \eta^C(W^C)((\phi U)^V - \eta^V(W^C)((\phi U)^C. && (3.7)
\end{aligned}$$

Let R^C and S^C be the complete lifts on TM of the curvature tensor R and the Ricci tensor S of M, respectively. The following relations are given by [13]

$$\begin{aligned}
D^C_{U^C}\xi^C &= U^C - (\phi U)^C \xi^V - (\phi U)^V \xi^C, && (3.8)\\
(D^C_{U^C}\eta^C)(W^C) &= g^C(U^C, W^C) - \eta^C(U^C)\eta^V(W^C)\\
&- \eta^V(U^C)\eta^C(W^C), && (3.9)\\
R^C(U^C, W^C)\xi^C &= -\eta^C(W^C)X^V - \eta^V(W^C)U^C\\
&+ \eta^C(U^C)W^V + \eta^V(U^C)W^C, && (3.10)\\
R^C(\xi^C, U^C)W^C &= \eta^C(U^C)W^V + \eta^V(U^C)W^C\\
&- g^C(U^C, W^C)\xi^V - g^C(U^V, W^C)\xi^C, && (3.11)\\
S^C(U^C, \xi^C) &= -2n\eta^C(U^C), && (3.12)\\
\eta^C(R^C(U^C, W^C)Z^C &= g^C(U^C, W^C)\eta^V(W^C)\\
&+ g^C(U^V, W^C)\eta^C(W^C)\\
&- g^C(W^C, Z^C)\eta^V(U^C)\\
&- g^C(W^V, Z^C)\eta^C(U^C), && (3.13)
\end{aligned}$$

for any vector fields U^C, W^C on TM.

Let M be an almost contact metric manifold with a Riemannian connection D and TM its tangent bundle. The linear connection $\bar{D}$ and the tensor H of type (1,1) are related by [19]

$$\bar{D}_U W = D_U W + H(U, W), \qquad (3.14)$$

where

$$H(U, W) = \frac{1}{2}[T(U, W) + T'(U, W) + T'(W, U)]. \qquad (3.15)$$

and

$$g(T'(U,W),Z) = g(T(Z,U),W). \tag{3.16}$$

Using (1.2) and (3.16), the obtained equation is

$$T'(U,W) = \eta(U)W - g(W,U)\xi. \tag{3.17}$$

Making use of (1.2) and (3.17) in (3.15), then the equation (3.15) becomes

$$H(U,W) = -\eta(W)U - 2\eta(U)W + g(U,W)\xi. \tag{3.18}$$

Thus, a semi-symmetric non-metric ϕ-connection $\bar{D}$ on a Kenmotsu manifold is given by

$$\bar{D}_U W = D_U W - \eta(W)U - 2\eta(U)W + g(U,W)\xi. \tag{3.19}$$

Operating the complete lift on (1.1), (1.2), (1.4), (3.14), (3.15) and (3.16) by mathematical operators, the obtained equations are

$$\begin{aligned} T^C(U^C,W^C) &= \bar{D}^C_{U^C}W^C - \bar{D}^C_{W^C}U^C - [U^C,W^C] && (3.20)\\ &= \eta^C(W^C)U^V + \eta^V(W^C)U^C \\ &- \eta^C(U^C)W^V - \eta^V(U^C)W^C, && (3.21)\end{aligned}$$

$$\bar{D}^C_{U^C}W^C = D^C_{U^C}W^C + H^C(U^C,W^C), \tag{3.22}$$

where

$$H^C(U^C,W^C) = \frac{1}{2}[T^C(U^C,W^C) + T'^C(U^C,W^C) + T'^C(W^C,U^C)]. \tag{3.23}$$

and

$$g^C(T'^C(U^C,W^C),Z^C) = g^C(T^C(Z^C,U^C),W^C). \tag{3.24}$$

From (3.20) and (3.24), we infer

$$\begin{aligned} T'^C(U^C,W^C) &= -g^C(W^V,U^C)\xi^C - g^C(W^C,U^C)\xi^V \\ &+ \eta^C(U^C)W^V + \eta^V(U^C)W^C. \end{aligned} \tag{3.25}$$

Using (3.20) and (3.25) in (3.23), entails that

$$\begin{aligned} H^C(U^C,W^C) &= -\eta^C(W^C)U^V - \eta^V(W^C)U^C \\ &- 2\eta^C(U^C)W^V - 2\eta^V(U^C)W^C \\ &+ g^C(W^V,U^C)\xi^C + g^C(W^C,U^C)\xi^V, \end{aligned} \tag{3.26}$$

where H^C is the complete lift of H.

Thus a semi-symmetric non-metric ϕ-connection $\bar{D}^C$ in a Kenmotsu manifold with respect to D^C on TM is given by

$$\begin{aligned} \bar{D}^C_{U^C} W &= D^C_{U^C} W^C - \eta^C(W^C)U^V - \eta^V(W^C)U^C \\ &- 2\eta^C(U^C)W^V - 2\eta^V(U^C)W^C \\ &+ g^C(W^V, U^C)\xi^C + g^C(W^C, U^C)\xi^V. \end{aligned} \tag{3.27}$$

Thus equation (3.27) is the relation between the Levi-Civita connection D^C and the semi-symmetric non-metric ϕ-connection $\bar{D}^C$ on TM. Therefore, we have the following theorem:

Theorem 3.1. *Let $\bar{D}$ be the semi-symmetric non-metric ϕ-connection on a Kenmotsu manifold M and $\bar{D}^C$ its complete lift on the tangent bundle TM. Then the relation between the Levi-Civita connection D^C and the semi-symmetric non-metric ϕ-connection $\bar{D}^C$ on TM is given by equation (3.27).*

4. Some calculations for Curvature tensor of a Kenmotsu manifold to its tangent bundle

In this section, the complete lift of the curvature tensor, the scalar curvature and the Ricci tensor on the tangent bundle are constructed and it is shown that Ricci tensor is symmetric. A study of the complete lift of curvature tensor concerning the semi-symmetric non-metric ϕ-connection to its tangent bundle is done which shows that if the complete lift of the curvature tensor of $\bar{D}^C$ vanishes on TM, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(1)$ on TM.

Let $\bar{D}$ be the semi-symmetric non-metric ϕ-connection on M and TM its tangent bundle. The curvature tensor $\bar{R}$ of M with respect to the semi-symmetric non-metric ϕ-connection $\bar{D}$ is defined by [5]

$$\bar{R}(U, W)Z = \bar{D}_U \bar{D}_W Z - \bar{D}_W \bar{D}_U Z - \bar{D}_{[U,W]} Z, \tag{4.1}$$

for arbitrary vector fields U and W on M.

Operating complete lifts on (4.1), the obtained equation is

$$\bar{R}^C(U^C, W^C)Z^C = \bar{D}^C_{U^C} \bar{D}^C_{W^C} Z^C - \bar{D}^C_{W^C} \bar{D}^C_{U^C} Z^C - \bar{D}^C_{[U^C, W^C]} Z^C. \tag{4.2}$$

From (3.27) it follows that

$$\bar{D}^C_{U^C} \bar{D}^C_{W^C} Z^C = \bar{D}^C_{U^C} D^C_{W^C} Z^C - \bar{D}^C_{U^C} \eta^C(W^C)(\phi Z)^V - \bar{D}^C_{U^C} \eta^V(W^C)(\phi Z)^C. \tag{4.3}$$

In view of (3.27), (4.2), (4.3) and (3.7), the obtained formula for the curvature tensor $\bar{R}^C$ of the connection $\bar{D}^C$ on tangent bundle TM is

$$\begin{aligned}
&\bar{R}^C(U^C,W^C)Z^C \\
&= R^C(U^C,W^C)Z^C + (D_W\eta)^C(Z^C)U^V \\
&+ (D_W\eta)^C(Z^V)U^C + (D_W\eta)^V(Z^C)U^C \\
&- (D_U\eta)^C(Z^C)W^V - (D_U\eta)^C(Z^V)W^C - (D_U\eta)^V(Z^C)W^C \\
&+ \eta^C(W^C)\eta^C(Z^C)U^V + \eta^C(W^C)\eta^V(Z^C)U^C \\
&+ \eta^V(W^C)\eta^C(Z^C)U^C - \eta^C(U^C)\eta^C(Z^C)W^V \\
&- \eta^C(U^C)\eta^V(Z^C)W^C - \eta^V(U^C)\eta^C(Z^C)W^C, \qquad (4.4)
\end{aligned}$$

where $R^C(U^C,W^C)Z^C$ is the curvature tensor of the connection D^C.

A relation between the curvature tensor of TM concernig the semi-symmetric non-metric ϕ-connection $\bar{D}^C$ and the Levi-Civita connection D^C is given by the equation (4.4).

The following theorem is obtained:

Theorem 4.1. *Let M be the Kenmotsu manifold and TM its tangent bundle. Let $\bar{R}$ be the curvature tensor of a Kenmotsu manifold M concernig semi-symmetric non-metric ϕ-connection $\bar{D}$. Then the curvature tensor $\bar{R}^C$ concernig the semi-symmetric non-metric ϕ-connection $\bar{D}^C$ on TM is given by equation (4.4).*

Using (3.7) and (3.9) in (4.4), the obtained equation is

$$\begin{aligned}
\bar{R}^C(U^C,W^C)Z^C &= R^C(U^C,W^C)Z^C + g^C(W^V,Z^C)U^C + g^C(W^C,Z^C)U^V \\
&- g^C(U^V,Z^C)W^C - g^C(U^C,Z^C)W^V. \qquad (4.5)
\end{aligned}$$

From (4.5), we obtain

$$\bar{R}^C(U^C,W^C)Z^C = -\bar{R}^C(W^C,U^C)Z^C. \qquad (4.6)$$

and

$$\bar{R}^C(U^C,W^C)Z^C + \bar{R}^C(W^C,Z^C)U^C + \bar{R}^C(Z^C,U^C)W^C = 0. \qquad (4.7)$$

The equation (4.7) is the first Bianchi identity with respect to $\bar{D}^C$ on Kenmotsu manifolds to its tangent bundle TM.

Taking the inner product of (4.5) with X^C, provide

$$\begin{aligned}
&\tilde{\bar{R}}^C(U^C,W^C,Z^C,X^C) \\
&= \tilde{R}^C(U^C,W^C,Z^C,X^C) + g^C(W^C,Z^C)g^C(U^V,X^C) \\
&+ g^C(W^V,Z^C)g^C(U^C,X^C) - g^C(U^C,Z^C)g^C(W^V,X^C) \\
&- g^C(U^V,Z^C)g^C(W^C,X^C). \qquad (4.8)
\end{aligned}$$

where

$$\tilde{\bar{R}}^C(U^C,W^C,Z^C,X^C) = g^C(\bar{R}^C(U^C,W^C)Z^C,X^C)$$

and

$$\tilde{R}^C(U^C, W^C, Z^C, X^C) = g^C(R^C(U^C, W^C)Z^C, X^C).$$

Equation (4.8) turns into

$$\tilde{\bar{R}}^C(U^C, W^C, Z^C, X^C) = -\tilde{\bar{R}}^C(U^C, W^C, X^C, Z^C).$$

Setting $U^C = X^C = e_i^C, i = 1, 2, ..., 2n+1$ in (4.8), where $\{e_i^C, i = 1, 2, ..., 2n+1\}$ is a local orthonormal basis of TM and summing up for $i = 1, 2, ..., 2n+1$ and making use of (3.3), we get

$$\bar{S}^C(W^C, Z^C) = S^C(W^C, Z^C) + 2ng^C(W^C, Z^C), \tag{4.9}$$

where $\bar{S}^C$ and S^C represent the Ricci tensor of TM with respect to $\bar{D}^C$ and D^C, respectively.

Equation (4.9) becomes

$$\bar{S}^C(W^C, Z^C) = \bar{S}^C(Z^C, W^C).$$

Let r^C and $\bar{r}^C$ represent the scalar curvatures of connections D^C and $\bar{D}^C$, respectively, on TM , where r and $\bar{r}$ represent the scalar curvatures of connections D and $\bar{D}$, respectively, on M. Then $\bar{r} = \sum_{i=1}^{2n+1} \bar{S}^C(e_i^C, e_i^C)$ and $r = \sum_{i=1}^{2n+1} S^C(e_i^C, e_i^C)$ where $\bar{r}^C = \bar{r}$ =constant and $r^C = r$ =constant.

Setting $W^C = Z^C = e_i^C$ in (4.9), where $\{e_i^C, i = 1, 2, ..., 2n+1\}$ is a local orthonormal basis of TM and summing up for $i = 1, 2, ..., 2n+1$ and making use of (3.3), we get

$$\bar{r} = r + 2n(2n+1).$$

Therefore, the following theorem is obtained:

Theorem 4.2. *Let M be the Kenmotsu manifold and TM its tangent bundle. Let $\bar{D}^C$ be complete lift of the semi-symmetric non-metric ϕ-connection $\bar{D}$ on TM. Then*

(i) The curvature tensor $\bar{R}^C$ on TM is given by

$$\begin{aligned} \bar{R}^C(U^C, {}^C)Z^C &= R^C(U^C, W^C)Z^C + g^C(W^V, Z^C)U^C + g^C(W^C, Z^C)U^V \\ &- g^C(U^V, Z^C)W^C - g^C(U^C, Z^C)W^V. \end{aligned}$$

(ii) The Ricci tensor $\bar{S}^C$ is given by

$$\bar{S}^C(W^C, Z^C) = S^C(W^C, Z^C) + 2ng^C(W^C, Z^C).$$

(iii) The scalar curvature $\bar{r}$ is given by

$$\bar{r} = r + 2n(2n+1),$$

(iv) $\bar{R}^C(U^C, W^C)Z^C = -\bar{R}^C(W^C, U^C)Z^C$.
(v) $\bar{R}^C(U^C, W^C)Z^C + \bar{R}^C(W^C, Z^C)U^C + \bar{R}^C(Z^C, U^C)W^C = 0$.
(vi)The Ricci tensor $\bar{S}^C$ is symmetric.
(vii) $\tilde{\bar{R}}^C(U^C, W^C, Z^C, X^C) = -\tilde{\bar{R}}^C(U^C, W^C, X^C, Z^C)$.

Definition 4.1. [5] A Kenmotsu manifold with respect to the Levi-Civita connection is of constant curvature k if its curvature tensor R is of the form

$$\begin{aligned} g(R(U,W)Z,X) &= k[g(W,Z)g(U,X) \\ &- g(U,Z)g(W,X)]. \end{aligned} \tag{4.10}$$

Operating complete lifts on the above equation, the obtained equation is

$$\begin{aligned} g^C(R^C(U^C,W^C)Z^C,X^C) &= k[g^C(W^C,Z^C)g^C(U^V,X^C) \\ &+ g^C(W^V,Z^C)g^C(U^C,X^C) \\ &- g^C(U^C,Z^C)g^C(W^V,X^C) \\ &- g^C(U^V,Z^C)g^C(W^C,X^C)], \end{aligned} \tag{4.11}$$

where k is a constant.

If $\tilde{\bar{R}}^C = 0$, then (4.8) becomes

$$\begin{aligned} \tilde{R}^C(U^C,W^C,Z^C,X^C) &= g^C(U^C,Z^C)g^C(W^V,X^C) \\ &+ g^C(U^V,Z^C)g^C(W^C,X^C) \\ &- g^C(W^C,Z^C)g^C(U^V,X^C) \\ &- g^C(W^V,Z^C)g^C(U^C,X^C). \end{aligned} \tag{4.12}$$

In view of (4.11) and (4.12), we have

$$\begin{aligned} g^C(R^C(U^C,W^C)Z^C,X^C) &= k[g^C(W^C,Z^C)g^C(U^V,X^C) \\ &+ g^C(W^V,Z^C)g^C(U^C,X^C) \\ &- g^C(U^C,Z^C)g^C(W^V,X^C) \\ &- g^C(U^V,Z^C)g^C(W^C,X^C)], \end{aligned}$$

such that $k = -1$. This shows that the Kenmotsu manifold with respect to D^C on TM is of constant curvature $k-1$. Therefore, the following theorem is obtained:

Theorem 4.3. *Let M be the Kenmotsu manifold and TM its tangent bundle. If the complete lift of the curvature tensor of $\bar{D}^C$ vanishes on TM, then the Kenmotsu manifold is locally isometric to the hyperbolic space $H^n(1)$ on TM.*

Definition 4.2. [5, 3] Let (M,g) be Kenmotsu manifold and TM its tangent bundle. For each plane p in the tangent space $T_x(M)$, the sectional curvature $K(p)$ is defined by

$$K(p) = \frac{\tilde{R}(U,W,U,W)}{g(U,U)g(W,W) - g(U,W)^2}, \tag{4.13}$$

where $\{U,W\}$ is orthonormal basis for p. Clearly $K(p)$ is independent of the choice of the orthonormal basis $\{U,W\}$.

Putting $Z^C = X^C, X^C = W^C$ in (4.12), the obtained equation is

$$\begin{aligned} \tilde{R}(U^C, W^C, U^C, W^C) &= g^C(U^C, U^C)g^C(W^V, W^C) \\ &+ g^C(U^V, U^C)g^C(W^C, W^C) \\ &- g^C(W^C, U^C)g^C(U^V, W^C) \\ &- g^C(W^V, U^C)g^C(U^C, W^C). \end{aligned} \tag{4.14}$$

Operating complete lift on (4.13), reveals that

$$K^C(p) = \frac{\tilde{R}^C(U^C, W^C, U^C, W^C)}{g(U^C, U^C)g(W^C, W^C) - g(U^C, W^C)^2}, \tag{4.15}$$

where $\tilde{R}^C(U^C, W^C, U^C, W^C)$ is given in (4.14). Therefore, the following theorem is obtained:

Theorem 4.4. *Let M be the Kenmotsu manifold and TM its tangent bundle. If the complete lift of the curvature tensor of $\bar{D}^C$ vanishes on TM, then the sectional curvature on TM is given by (4.15).*

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