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Analysis and computational modelling of a coupled epidemic reaction-diffusion

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Abstract. This paper mainly focuses on the dynamics of an epidemiologically emerging reaction-diffusion system. We establish the global existence and the local and global asymptotic stability results for solution of proposed system for a rather broad class of nonlinearities, and the numerical simulations are conducted by using MATLAB.

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1. Introduction

There have been some recent theoretical studies on SI (susceptible-infected)-type reaction-diffusion models. For references to other spatial deterministic epidemic models, we refer the reader to excellent surveys in a [14, 16, 6, 18, 17]. In the study by de Jong et al.[7], the standard incidence transmission term $\beta SI/(S+I)$ was suggested as an alternative to mass action. For such purpose, Allen et al. proposed in [3] a frequency-dependent SIS reaction-diffusion model for a population living in a continuous spatial habitat.

We consider the following reaction–diffusion system:

$$(1.1) \qquad \begin{cases} \frac{\partial s}{\partial t} - d_s \triangle s = \Lambda - \beta \frac{su}{s+u} - \mu s & \text{in } R^+ \times \Omega \\ \frac{\partial u}{\partial t} - d_u \triangle u = \beta \frac{su}{s+u} - (\mu + \sigma) u. & \text{in } R^+ \times \Omega \end{cases}$$

The system proposed describes the transmission of HIV in a population. The studied population contains individuals susceptible s and individuals infected u. The study of epidemiology has attracted the attention of a vast number of researchers through planning and predictions of the spread of the disease thereby reducing mortality rates. What we understand of the dynamics of HIV in the context of mathematical models for multiple groups is critical to our understanding of the dynamics of HIV in a highly heterogeneous population.

With the initial data s, u on R^+. Several infectious diseases are still targeting huge populations. They are considered amongt the principal causes of mortality. The constant Λ represents the flow rate of newly exposed individuals, μ is the death rate, the parameter β describes the rate of disease prevalence among individuals per unit time, the parameter σ given by $\sigma+\eta$, where $\frac{1}{\eta}$ mean period of sexual activity of affected individuals.

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The notation $\triangle$ denotes the Laplacian operators on Ω, where Ω is an open bounded subset of R^n with smooth boundary $\partial\Omega$. The constant parameters d_s and $d_u > 0$ are the diffusion coefficients. We assume the initial conditions

$$s_0(x) = s(x,0) > 0, \quad u_0(x) = u(x,0) \geq 0, \qquad \text{in } \Omega \tag{1.2}$$

and Neumann boundary conditions

$$\frac{\partial s}{\partial \upsilon} = \frac{\partial u}{\partial \upsilon} = 0. \qquad \text{in } R^+ \times \Omega \tag{1.3}$$

In [4] the authors presented the global stability of an epidemiological model with the strength of the infection under intervention strategies in a spatially heterogeneous environment. In [18], the stability of the infected equilibrium has been analyzed locally. Nevertheless, the question of global stability for this type of viral infections dynamics is intriguing. In the present study, we study the existence of equilibria and their asymptotic stability conditions for the model considered in [18], in Section 2, we present the local existence of solutions to problem (1.1)-(1.3), we define the basic reproduction number R_0 of the proposed model and establish the existence of two equilibria. The local asymptotic stability and instability of the disease-free equilibrium and the endemic equilibrium are investigated. Section 3 proves that the two steady states of the model are globally asymptotically stable using an appropriate Lyapunov functional. Finally, Section 4 presents a numerical test to validate the theoretical analysis presented.

2. Properties of the Model

In the following section, we define the system's equilibria and their relation to the basic reproduction number R_0, investigate the local stability of the system in the ODE and PDE

2.1. Existence and positivity of solutions

Throughout this study, we denote by

$$\begin{aligned} \|f\|_p^p &= \int_\Omega |f(x)|^p \, dx, 1 \leq p < \infty, \\ \|f\|_\infty &= \operatorname{ess\,sup}_{x\in\Omega} |f(x)|, \\ \|f\|_{C(\overline{\Omega})} &= \max_{x\in\overline{\Omega}} |f(x)|, \end{aligned}$$

the usual norms in spaces $L^p(\Omega), L^\infty(\Omega)$ and $C(\overline{\Omega})$, respectively.

According to the classical results for Abstract parabolic equations (see Theorem 3.1 in Chapter 7 [15] and Section 2 in [13]), when $s_0, u_0 \in C(\overline{\Omega})$, we can deduce that there exists a unique local (i.e. in some interval $(0, T_{max})$, $0 < T_{max} \leq +\infty$) classical solution of the system (1.1)-(1.3).

From the maximum principle and the assumption (1.3), it follows that the solution. (s, u) of the system (1.1)–(1.3) remains nonnegative on $(0, T^*)$ and

$$u(t, x) \geq 0. \qquad \forall (t, x) \in (0, T^*) \times \Omega$$

Again, the maximum principle applied to the first and the third equation of (1.3) permits us to deduce that the components s are bounded on $(0, T^*) \times \Omega$

$$0 < s(t, x) \leq \max \left\{ \frac{\Lambda}{\mu}, \|s_0\|_\infty \right\},$$

Proposition 2.1. *The solution (s, u) of the system (1.1)-(1.3) exists uniquely and globally in time. Moreover, there exists a positive constant A depending on initial data, such that*

$$\|s(., t)\|_{L^\infty(\Omega)} + \|u(., t)\|_{L^\infty(\Omega)} \leq A. \ \text{for all } t > 0 \tag{2.1}$$

Furthermore, there exists a positive constant $\tilde{A}$ such that for a large $T > 0$,

$$\|s(., t)\|_{L^\infty(\Omega)} + \|u(., t)\|_{L^\infty(\Omega)} \leq \tilde{A}. \ \text{for all } t > T \tag{2.2}$$

Proof. Let $s(t, x) \in (0, T_{max}) \times \Omega$ be the first component of the local solution for system (1.1), which can be formulated as follows

$$\begin{cases} \dfrac{\partial s}{\partial t} - d_s \Delta s = \Lambda - \beta \dfrac{su}{s+u} - \mu s, & \text{in } (0, T_{max}) \times \Omega, \\ s(0, x) = s_0(x), & \text{on } \Omega, \\ \dfrac{\partial s}{\partial \nu} = 0, & \text{on } (0, T_{max}) \times \partial\Omega. \end{cases} \tag{2.3}$$

We notice that an upper solution exists for (2.3) for any positive function $u(t, x) \in (0, T_{max}) \times \Omega$. This upper solution is provided by

$$C_1 := \max \left\{ \frac{\Lambda}{\mu}, \|s_0\|_{C(\overline{\Omega})} \right\}.$$

By using the comparison principle, we obtain $s(t, x) \leq C_1$ in $[0, T_{max}) \times \overline{\Omega}$, thus, $s(t, x)$ is uniformly bounded in $[0, T_{max}) \times \overline{\Omega}$. On the other hand, we consider $\tilde{\chi} = \int_\Omega (s(x, t) + u(x, t))dx$ and from (1.1)-(1.3), we have

$$\begin{aligned} \frac{d}{dt}\tilde{\chi}(t) &= \Lambda |\Omega| - \int_\Omega (\mu s(t, x) + (\mu + \sigma) u(t, x))dx \\ &\leq \Lambda |\Omega| - \mu \tilde{\chi}(t). \end{aligned} \tag{2.4}$$

Thanks to the Gronwall's inequality, we have for $t \in (0, T_{max})$,

$$\tilde{\chi}(t) \leq C_2, \tag{2.5}$$

where $C_2 > 0$. Hence, for $t \in (0, T_{max})$,

$$u(t,.) \in L^1(\Omega). \tag{2.6}$$

By using the second equation of (1.1), we conclude that there exists $C_3 > 0$ depending on C_2 such that $u(t,x) \leq C_3$ in $[0, T_{max}) \times \overline{\Omega}$. By using the standard theory of semilinear parabolic systems described in [11], we deduce $T_{max} = \infty$.

When $T_{max} = +\infty$, the problem (2.3) becomes (for any positive function u)

$$\begin{cases} \dfrac{\partial s}{\partial t} - d_1 \Delta s \leq \Lambda - \mu s, & \text{in } (0, +\infty) \times \Omega, \\ s(0,x) = s_0(x) \leq \|s_0\|_{C(\overline{\Omega})}, & \text{on } \Omega, \\ \dfrac{\partial s}{\partial \nu} = 0, & \text{on } (0, \infty) \times \partial\Omega. \end{cases} \tag{2.7}$$

Using the comparison principle, we get $s(t,x) \leq \omega(t)$ for $t \in [0, +\infty)$, where $\omega(t) = \|s_0\|_{C(\overline{\Omega})} e^{-\mu t} + \left(\frac{\Lambda}{\mu}\right)(1 - e^{-\mu t})$ is the unique solution of the initial value problem

$$\begin{cases} \frac{d\omega}{dt} = \Lambda - \mu \omega, & t > 0, \\ \omega(0) = \|s_0\|_{C(\overline{\Omega})}. \end{cases} \tag{2.8}$$

Then, for $x \in \overline{\Omega}$, we have

$$s(t,x) \leq \omega(t) \overset{t \to \infty}{\longrightarrow} \frac{\Lambda}{\mu}.$$

Thus, we have an upper bound for $\|s(t,.)\|_{L^\infty(\Omega)}$ independent of the initial data for a given sufficiently large t. Thanks to [16, Lemma 3.1] we find that $\|u(t,.)\|_{L^\infty(\Omega)}$ is also bounded by a positive constant independent of the initial data for a large enough t. □

2.2. Equilibrium Points and Basic Reproduction Number

There are two equilibrium points, the disease free equilibrium point (DFE) $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ and endemic equilibrium point (EE) $E^* = (s^*, u^*)$.

Moreover, basic reproduction number R_0 is defined as the spectral radius of the matrix FV^{-1} [8, 19]. Where F and V respectively denote the matrix of transmission terms of the system (1.1) at E_0 such as

$$\begin{aligned} \begin{pmatrix} u_t \\ s_t \end{pmatrix} &= \begin{pmatrix} \beta \frac{su}{s+u} - (\mu + \sigma) u \\ \Lambda - \beta \frac{su}{s+u} - \mu s \end{pmatrix} \\ &= \begin{pmatrix} \beta \frac{su}{s+u} \\ 0 \end{pmatrix} - \begin{pmatrix} (\mu + \sigma) u \\ -\Lambda + \beta \frac{su}{s+u} + \mu s \end{pmatrix}. \end{aligned}$$

The Jacobian matrices corresponding to vectors $\begin{pmatrix} \beta \frac{su}{s+u} \\ 0 \end{pmatrix}$ and $\begin{pmatrix} (\mu+\sigma)\,u \\ -\Lambda + \beta\frac{su}{s+u} + \mu s \end{pmatrix}$ at the disease-free equilibrium $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ are given, respectively, by

$$\begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} \mu+\sigma & 0 \\ \beta & \mu \end{pmatrix} = \begin{pmatrix} V & 0 \\ V_1 & V_2 \end{pmatrix}.$$

The basic reproduction number R_0 is simply the spectral radius of the next generation matrix $R_0 = \rho\left(FV^{-1}\right)$, which is given by

$$R_0 = \frac{\beta}{\sigma+\mu}. \tag{2.9}$$

The endemic equilibrium point $E^* = (s^*, u^*)$,

where

$$s^* = \frac{\Lambda}{\beta-\sigma},$$
$$u^* = \Lambda\frac{\sigma-\beta+\mu}{(\sigma+\mu)(\sigma-\beta)},$$

Proposition 2.2. *i) If* $R_0 \leq 1$ *the system* (1.1) *accepts one equilibrium point* E_0.

ii) If $R_0 > 1$ *the system* (1.1) *has a two equilibriums points* E_0 *and* E^*.

Proof. We put

$$\begin{cases} \Lambda - \beta\frac{su}{s+u} - \mu s = 0 \\ \beta\frac{su}{s+u} - (\mu+\sigma)\,u = 0, \end{cases} \tag{2.10}$$

we obtain

$$\begin{cases} -\mu s^2 + (\Lambda - \beta u - \mu u)s + \Lambda u = 0 \\ \Lambda - (\mu+\sigma)\,u = \mu s. \end{cases} \tag{2.11}$$

1) If $\mu \neq 0$, we find s and then substitute in the first equation to find u, we obtain

$$-\mu s^2 + (\Lambda - \beta u - \mu u)s + \Lambda u = 0 \tag{2.12}$$
$$\frac{\Lambda}{\mu} - \frac{\mu+\sigma}{\mu}u = s, \tag{2.13}$$

we substitute (2.13) in (2.12), we get

$$-\mu\left(\frac{\Lambda}{\mu} - \frac{\mu+\sigma}{\mu}u\right)^2 + (\Lambda - \beta u - \mu u)\left(\frac{\Lambda}{\mu} - \frac{\mu+\sigma}{\mu}u\right) + \Lambda u = 0,$$

after simplification we find

$$u(-\frac{\sigma^2}{\mu}u - \sigma u + \beta u + \frac{\beta\sigma}{\mu}u + \Lambda + \frac{\Lambda\sigma}{\mu} - \frac{\beta\Lambda}{\mu}) = 0.$$

And from that

$$u = 0 \text{ or } u\left(-\frac{\sigma^2}{\mu} - \sigma + \beta + \frac{\beta\sigma}{\mu}\right) = -\Lambda\left(1 + \frac{\sigma}{\mu} - \frac{\beta}{\mu}\right).$$

From it the result

$$\left\{\begin{array}{c} u = 0, \\ \text{or} \\ u(\sigma+\mu)(\sigma-\beta) = \Lambda(\sigma-\beta+\mu). \end{array}\right. \tag{2.14}$$

∘ The first case of $u = 0$, then $s = \frac{\Lambda}{\mu}$, hence the equilibrium point

$$E_0 = \left(\frac{\Lambda}{\mu}, 0\right). \tag{2.15}$$

∘ In the second case:
If $\sigma \neq \beta$, then $u = \Lambda\dfrac{\sigma-\beta+\mu}{(\sigma+\mu)(\sigma-\beta)}$, we get the second equilibrium point

$$E^* = \left(\frac{\Lambda}{\beta-\sigma}, \Lambda\frac{\sigma-\beta+\mu}{(\sigma+\mu)(\sigma-\beta)}\right) \tag{2.16}$$

2) If $\mu = 0$, we substitute it into the (2.11), we obtain a solution that satisfies E^*.

We now discuss the existence of equilibriums points obtained in (2.15)-(2.16).

∘ Through the form of E_0 the equilibrium point exists regardless of the value of R_0.

∘ If $R_0 > 1$, then $\sigma-\beta+\mu < 0$, and $\sigma-\beta < 0$, the solution remains positive, and from it there is a equilibrium point E^*.

∘ If $R_0 < 1$, then $\sigma - \beta + \mu > 0$, and $\sigma - \beta < 0$, we get negative u, from which the equilibrium point E^* does not exist. □

2.3. The Local Stability of ODE and properties of R_0

We now move to study the local asymptotic stability to the two equilibrium points E_0 and E^* as shown in the following theorem.

Proposition 2.3. *a) if $R_0 < 1$ the equilibrium point E_0 is locally asymptotically stable.*

b) if $R_0 > 1$, E_0 is unstable and the second equilibrium point E^* is locally asymptotically stable.

Proof. To prove the local asymptotic stability, we make advantage of the Jacobian matrix, which may be given by

$$J(s,u) = \begin{pmatrix} -\mu - \beta\frac{u^2}{(s+u)^2} & -\beta\frac{s^2}{(s+u)^2} \\ \beta\frac{u^2}{(s+u)^2} & \beta\frac{s^2}{(s+u)^2} - (\sigma+\mu) \end{pmatrix}. \tag{2.17}$$

Evaluating $J(s,u)$ at E_0, we obtain

$$J(E_0) = \begin{pmatrix} -\mu & -\beta \\ 0 & \beta - (\sigma+\mu) \end{pmatrix}.$$

The eigenvalues can be easily shown to be

$$\lambda_1 = -\mu < 0 \text{ and } \lambda_2 = \beta - (\sigma+\mu). \tag{2.18}$$

It is easy to see that $\lambda_2 < 0$ if $R_0 < 1$, the real parts of the eigenvalues of $J(E_0)$ are negative, we depend on [2], leading to the asymptotic stability of E_0.

The second case if where $R_0 > 1$. The equilibrium E_0 is clearly unstable but the system possesses an equilibrium point E^*.

Evaluating the Jacobian matrix (2.17) at E^* yields

$$J(E^*) = \begin{pmatrix} -\mu - \frac{(\sigma-\beta+\mu)^2}{\beta} & -\frac{(\sigma+\mu)^2}{\beta} \\ \frac{(\sigma-\beta+\mu)^2}{\beta} & \frac{(\sigma+\mu)^2}{\beta} - (\sigma+\mu) \end{pmatrix}. \tag{2.19}$$

We solve $\det(J(E^*) - \lambda I) = 0$. Hence,

$$\det\left(J(E^*) - \lambda I\right) = \lambda^2 + (\beta - \sigma)\lambda + \det J(E^*) = 0, \tag{2.20}$$

where

$$\det J(E^*) = \frac{1}{\beta}\left(\sigma - \beta\right)\left(\sigma+\mu\right)\left(\sigma - \beta + \mu\right). \tag{2.21}$$

By using the basic reproduction number R_0, we obtain

$$\begin{cases} \sigma - \beta < 0 \\ \sigma - \beta + \mu < 0. \end{cases}$$

The previous (2.21) has two eigenvalues negative real parts solutions λ_1, λ_2 becouse $\lambda_1 + \lambda_2 = \sigma - \beta < 0$ and $\lambda_1\lambda_2 = \det J(E^*) > 0$.

Hence, the equilibrium E^* is locally asymptotically stable. □

2.4. The Local Stability of PDE

We have already established sufficient conditions for the local asymptotic stability in the ODE scenario. Let us now examine the more general PDE case (1.1)-(1.3).

Theorem 2.4. *For system (1.1):*
(i) If $R_0 < 1$, the DFE point E_0 is locally asymptotically stable.
(ii) If $R_0 > 1$, the EE point E^* is locally asymptotically stable.

Proof. (i) In the presence of diffusion, the equilibrium point $E_0 = \left(\frac{\Lambda}{\mu}, 0\right)$ satisfes

$$(2.22) \qquad \begin{cases} d_s \triangle s^* + \Lambda - \beta \frac{s^* u^*}{s^* + u^*} - \mu s^* = 0 & \text{in } R^+ \times \Omega \\ d_u \triangle u^* + \beta \frac{s^* u^*}{s^* + u^*} - (\mu + \sigma) u^* = 0. & \text{in } R^+ \times \Omega \end{cases}$$

With

$$L(E_0) = \begin{pmatrix} d_s \triangle - \mu & -\beta \\ 0 & d_u \triangle + \beta - (\sigma + \mu) \end{pmatrix}.$$

Using the same method from [1], the stability of E_0 reduces to examining the eigenvalues of the matrices

$$(2.23) \qquad J_i(E_0) = \begin{pmatrix} -d_s \lambda_i - \mu & -\beta \\ 0 & -d_u \lambda_i + \beta - (\sigma + \mu) \end{pmatrix}, \quad \text{for all } i,$$

which are given for all $i \geq 0$ by

$$\begin{cases} k_{1i} = -d_s \lambda_i - \mu \\ k_{2i} = -d_u \lambda_i + \beta - (\sigma + \mu). \end{cases}$$

Since the Laplacian eigenvalues are positive and in ascending order, both k_{1i} and k_{2i} clearly have negative real parts for $R_0 < 1$ leading to the local stability of E_0.

The second equilibrium E^* satisfes (2.22)-(1.3). The corresponding linearization operator is

$$L(E^*) = \begin{pmatrix} d_s \triangle - \mu - \frac{(\sigma - \beta + \mu)^2}{\beta} & -\frac{(\sigma + \mu)^2}{\beta} \\ \frac{(\sigma - \beta + \mu)^2}{\beta} & d_u \triangle + \frac{(\sigma + \mu)^2}{\beta} - (\sigma + \mu) \end{pmatrix}.$$

Hence, the stability of E^* rests on the negativity of the real parts of the eigenvalues of matrices

$$J_i(E^*) = \begin{pmatrix} -d_s \lambda_i - \mu - \frac{(\sigma - \beta + \mu)^2}{\beta} & -\frac{(\sigma + \mu)^2}{\beta} \\ \frac{(\sigma - \beta + \mu)^2}{\beta} & -d_u \lambda_i + \frac{(\sigma + \mu)^2}{\beta} - (\sigma + \mu) \end{pmatrix}, \quad \text{for all } i$$

which has trace

$$tr J_i(E^*) = -(d_s + d_u)\lambda_i + \sigma - \beta.$$

For $R_0 > 1$ we obtain $tr J_i(E^*) < 0$.

The determinant of the Jacobian may be given by

$$\det J_i(E^*) = d_s d_u \lambda_i^2 + H_0 \lambda_i + \det J(E^*),$$

where

$$H_0 = d_s \frac{(\sigma+\mu)(-\sigma+\beta-\mu)}{\beta} + d_u(\mu + \frac{(\sigma-\beta+\mu)^2}{\beta}) > 0,$$

which leads to $\det J_i(E^*) > 0$. Hence, E^* is locally asymptotically stable. □

3. Global stability

In this section, we study the global asymptotic stability of the two steady states E_0 and E^*.

3.1. Global stability of DFE point E_0 with $R_0 < 1$

Theorem 3.1. *Let:*

$$F_\theta(t) = \int_\Omega \left[su + \frac{\theta}{2}\left(s - \frac{\Lambda}{\mu}\right)^2 + \frac{1}{2}u^2 + 2\frac{\Lambda}{\mu+\sigma}u \right] dx,$$

where

$$\beta\left(\theta\frac{1}{\mu} + \frac{2}{\mu+\sigma}\right) \leq 1, \tag{3.1}$$

with

$$\theta > \frac{(d_s+d_u)^2}{4d_s d_u}. \tag{3.2}$$

Then, $F_\theta(t)$ is a Lyapunov functional.

Proof. We must show that $F_\theta(t)$ is a Lyapunov function.

At $E_0 = (\frac{\Lambda}{\mu}, 0)$, $F_\theta(t) = 0$.

At first we have to show that $F_\theta(t) > 0$ for all $(\frac{\Lambda}{\mu}, 0) \neq (0,0)$.

The evaluation of the derivative is given as follows

$$\frac{d}{dt}F_\theta(t) = \int_\Omega \left(\frac{\partial s}{\partial t}u + \frac{\partial u}{\partial t}s\right)dx + \theta\int_\Omega \left(s - \frac{\Lambda}{\mu}\right)\frac{\partial s}{\partial t}dx + \int_\Omega u\frac{\partial u}{\partial t}dx + 2\frac{\Lambda}{\mu+\sigma}\int_\Omega \frac{\partial u}{\partial t}dx.$$

Substituting the partial derivatives $\frac{\partial s}{\partial t}$ and $\frac{\partial u}{\partial t}$ with their respective values from (1.1)

$$\begin{aligned} \frac{d}{dt}F_\theta(t) &= \int_\Omega \left(u + \theta\left(s - \frac{\Lambda}{\mu}\right)\right)\frac{\partial s}{\partial t}dx + \int_\Omega \left(s + u + 2\frac{\Lambda}{\mu+\sigma}\right)\frac{\partial u}{\partial t}dx \\ &= \int_\Omega \left(u + \theta\left(s - \frac{\Lambda}{\mu}\right)\right)\left(d_s \triangle s + \Lambda - \beta\frac{su}{s+u} - \mu s\right)dx \\ &\quad + \int_\Omega \left(s + u + 2\frac{\Lambda}{\mu+\sigma}\right)\left(d_u \triangle u + \beta\frac{su}{s+u} - (\mu+\sigma)u\right)dx. \\ &= I_1 + I_2. \end{aligned} \tag{3.3}$$

We start by looking at I_1. Using Green's formula and assuming the Neumann boundary conditions in (1.3), we obtain

$$\begin{aligned} I_1 &= \int_\Omega \left(u+\theta\left(s-\frac{\Lambda}{\mu}\right)\right) d_s \triangle s dx + \int_\Omega \left(s+u+2\frac{\Lambda}{\mu+\sigma}\right) d_u \triangle u dx \\ &= -d_s \int_\Omega \left(\nabla u + \theta \nabla s\right) \nabla s dx - d_u \int_\Omega \left(\nabla s + \nabla u\right) \nabla u dx \\ &= -d_s \int_\Omega \nabla u \nabla s dx - \theta d_s \int_\Omega \left|\nabla s\right|^2 dx - d_u \int_\Omega \nabla s \nabla u dx - d_u \int_\Omega \left|\nabla u\right|^2 dx. \end{aligned}$$

We start with the term I_1, we can rewrite it as

$$I_1 = -\int_\Omega \theta d_s \left|\nabla s\right|^2 + \left(d_s + d_u\right) \nabla u \nabla s + d_u \left|\nabla u\right|^2 dx.$$

We know Q is positive as θ, d_s and d_u satisfy the conditions $\theta d_s > 0$ and $\theta > \frac{(d_s+d_u)^2}{4d_s d_u}$, from which we obtain

$$I_1 \leq 0. \tag{3.4}$$

The second part of the derivative is

$$\begin{aligned} I_2 &= -\mu \int_\Omega su dx + \Lambda \int_\Omega u dx - \beta \int_\Omega \frac{su^2}{s+u} dx - \theta\mu \int_\Omega \left(s-\frac{\Lambda}{\mu}\right)^2 dx \\ &\quad -\theta\beta \int_\Omega \frac{s^2 u}{s+u} dx + \theta\beta \frac{\Lambda}{\mu} \int_\Omega \frac{su}{s+u} dx \\ &\quad +\beta \int_\Omega \frac{s^2 u}{s+u} dx - (\mu+\sigma) \int_\Omega su dx + \beta \int_\Omega \frac{su^2}{s+u} dx \\ &\quad -(\mu+\sigma) \int_\Omega u^2 dx + 2\frac{\Lambda\beta}{\mu+\sigma} \int_\Omega \frac{su}{s+u} dx - 2\Lambda \int_\Omega u dx \\ &= I_{21} + I_{22} + I_{23} + I_{24}, \end{aligned}$$

with

$$I_{21} = \beta \int_\Omega \frac{s^2 u}{s+u} dx - \theta\beta \int_\Omega \frac{s^2 u}{s+u} dx - \beta \int_\Omega \frac{su^2}{s+u} dx \tag{3.5}$$

$$I_{22} = \beta \int_\Omega \frac{su^2}{s+u} dx \tag{3.6}$$

$$I_{23} = -\Lambda \int_\Omega u dx - (2\mu+\sigma) \int_\Omega su dx$$

(3.7)

$$I_{24} = \theta\beta \frac{\Lambda}{\mu} \int_\Omega \frac{su}{s+u} dx + 2\frac{\Lambda\beta}{\mu+\sigma} \int_\Omega \frac{su}{s+u} dx - \theta\mu \int_\Omega \left(s-\frac{\Lambda}{\mu}\right)^2 dx - (\mu+\sigma) \int_\Omega u^2 dx.$$

We have

$$\frac{s}{s+u} \leq 1. \tag{3.8}$$

Using the inequality (3.8) in I_{24} yields

$$I_{24} \leq \left(\theta\beta\frac{\Lambda}{\mu} + 2\frac{\Lambda\beta}{\mu+\sigma}\right)\int_\Omega udx - \theta\mu\int_\Omega\left(s-\frac{\Lambda}{\mu}\right)^2 dx - (\mu+\sigma)\int_\Omega u^2dx. \tag{3.9}$$

Beside that, we can write

$$I_{23} \leq -\Lambda\int_\Omega udx. \tag{3.10}$$

Using (3.5), (3.6), (3.9) and (3.10), we get

$$\begin{aligned} I_2 &= I_{21} + I_{22} + I_{23} + I_{24} \\ &\leq \beta\int_\Omega \frac{s^2u}{s+u}dx - \theta\beta\int_\Omega \frac{s^2u}{s+u}dx \\ &\quad + \left[\left(\theta\beta\frac{\Lambda}{\mu} + 2\frac{\Lambda\beta}{\mu+\sigma}\right) - \Lambda\right]\int_\Omega udx - \theta\mu\int_\Omega\left(s-\frac{\Lambda}{\mu}\right)^2 dx - (\mu+\sigma)\int_\Omega u^2dx. \end{aligned}$$

Since θ verifies the estimates (3.2), then

$$I_2 \leq \beta(1-\theta)\int_\Omega \frac{s^2u}{s+u}dx - \theta\mu\int_\Omega\left(s-\frac{\Lambda}{\mu}\right)^2 dx - (\mu+\sigma)\int_\Omega u^2dx. \tag{3.11}$$

Then, by (3.4) and (3.11)

$$\begin{aligned} \frac{d}{dt}F_\theta(t) &\leq -\theta d_s\int_\Omega |\nabla s|^2 dx - d_u\int_\Omega |\nabla u|^2 dx - \theta\mu\int_\Omega\left(s-\frac{\Lambda}{\mu}\right)^2 dx - (\mu+\sigma)\int_\Omega u^2dx \\ &\quad (3.12) \\ &\leq 0. \end{aligned}$$

Finally, $F_\theta(t)$ is a Lyapunov functional. □

Theorem 3.2. *Let* $E_0 = (\frac{\Lambda}{\mu}, 0)$ *be the solution of (1.1)-(1.3) in* $(0,+\infty)$, *with hypotheses (3.1) and (3.2) then*

$$\lim_{t\to+\infty}\left\|s(t,.) - \frac{\Lambda}{\mu}\right\|_\infty = 0, \tag{3.13}$$

and

$$\lim_{t\to+\infty}\|u(t,.)\|_\infty = 0. \tag{3.14}$$

Proof. Using inequality (3.12), and integrating over $(0,t)$ yields

$$\begin{aligned} & F_\theta(t)+\theta d_s\int_0^t\left[\int_\Omega |\nabla s|^2\,dx\right]dS+d_u\int_0^t\left[\int_\Omega |\nabla u|^2\,dx\right]dS \\ & +\theta\mu\int_0^t\left[\int_\Omega\left(s-\frac{\Lambda}{\mu}\right)^2dx\right]dS+(\mu+\sigma)\int_0^t\left[\int_\Omega u^2dx\right]dS \\ (3.15)\qquad \le\ & F_\theta(0). \end{aligned}$$

Since $F_\theta(t)\ge 0$, we have from (3.15) that

$$\int_0^t\left[\int_\Omega\left(s-\frac{\Lambda}{\mu}\right)^2dx\right]dS\le\frac{F_\theta(0)}{\theta\mu}, \tag{3.16}$$

and

$$\int_0^t\left[\int_\Omega u^2dx\right]dS\le\frac{F_\theta(0)}{\mu+\sigma}. \tag{3.17}$$

Thus, we conclude from (3.15), (3.16) and (3.17) that $F_\theta(t)\in L^1(0,+\infty)$ and $\frac{d}{dt}F_\theta(t)\in L^1(0,+\infty)$.
By Barbalate's lemma ([9] Lemma (1.2.2)), we obtain $F_\theta(t)\to 0$, that is

$$\lim_{t\to+\infty}\left\|s(t,.)-\frac{\Lambda}{\mu}\right\|_2=0, \tag{3.18}$$

and

$$\lim_{t\to+\infty}\|u(t,.)\|_2=0. \tag{3.19}$$

Combining (3.18) and (3.19) and the fact that the orbits $\{s(t,x),t>0\}$ and $\{u(t,x),t>0\}$ are relatively compact on $C(\bar{\Omega})$[10]. Using this result, the limits (3.13) and (3.14) are evident. And the theorem is completely proved. □

3.2. Global stability of EE point E^* with $R_0>1$

Henshaw and McCluskey [12] established the global asymptotic stability of the unique equilibrium using an appropriate Lyapunov function. We consider

$$L(x)=x-1-\ln x,\quad \text{for all } x>0 \tag{3.20}$$

Lemma 3.3. *The second equilibrium point E^*, satisfies the inequality*

$$L\left(\frac{\frac{u}{s+u}}{\frac{u^*}{s^*+u^*}}\right)\le L\left(\frac{u}{u^*}\right). \tag{3.21}$$

Proof. We put $g(u)=\frac{u}{s+u}$, for all $u>0$. Since $\frac{g(u)}{u}$ is a decreasing function, we may separate the proof into two regions:

1. Suppose $u \geq u^*$. Since $\frac{g(u)}{u}$ is a decreasing function, we have

$$\frac{g(u)}{g(u^*)} \leq \frac{u}{u^*}.$$

And from it the result

$$\frac{\frac{u}{s+u}}{\frac{u^*}{s^*+u^*}} \leq \frac{u}{u^*}.$$

Since g is non-decreasing, which leads to

$$g(u) \geq g(u^*),$$

and, consequently,

$$1 \leq \frac{\frac{u}{s+u}}{\frac{u^*}{s^*+u^*}} \leq \frac{u}{u^*}.$$

Since L is increasing for all $x > 1$, (3.21) holds.

2.The second region is $0 < u < u^*$. Again, Since $\frac{g(u)}{u}$ is a decreasing function, we have

$$\frac{g(u)}{g(u^*)} > \frac{u}{u^*}.$$

This gives us

$$\frac{\frac{u}{s+u}}{\frac{u^*}{s^*+u^*}} > \frac{u}{u^*},$$

and given the non-decreasing nature of g we end up with

$$g(u) < g(u^*),$$

we get

$$1 > \frac{\frac{u}{s+u}}{\frac{u^*}{s^*+u^*}} > \frac{u}{u^*} > 0.$$

Hince L is decreasing for $0 < x < 1$, (3.21) holds. □

Theorem 3.4. *Let*

$$W(t) = \int_\Omega \left[s^* L\left(\frac{s}{s^*}\right) + u^* L\left(\frac{u}{u^*}\right) \right] dx. \tag{3.22}$$

Then, $W(t)$ is non-negative and is strictly minimized at the unique equilibrium E^*. Hence, $W(t)$ is a Lyapunov functional.

Proof. The derivative of $W(t)$ is evaluated as follows

$$\begin{aligned} \frac{d}{dt}W(t) &= \int_\Omega \left[\left(1 - \frac{s^*}{s}\right)\frac{ds}{dt} + \left(1 - \frac{u^*}{u}\right)\frac{du}{dt} \right] dx \\ &= \int_\Omega \left(1 - \frac{s^*}{s}\right)\left[d_s \triangle s + \Lambda - \beta\frac{su}{s+u} - \mu s \right] dx \\ &\quad + \int_\Omega \left(1 - \frac{u^*}{u}\right)\left[d_u \triangle u + \beta\frac{su}{s+u} - (\mu+\sigma)u \right] dx. \end{aligned} \tag{3.23}$$

Using Green's formula and Neuman boundary conditions we get

$$\begin{aligned}\frac{d}{dt}W(t) &= -d_s\int_\Omega \nabla\left(1-\frac{s^*}{s}\right)\nabla s dx + \int_\Omega\left(1-\frac{s^*}{s}\right)\left(\Lambda-\beta\frac{su}{s+u}-\mu s\right)dx \\ &- d_u\int_\Omega \nabla\left(1-\frac{u^*}{u}\right)d_2\nabla u dx + \int_\Omega\left(1-\frac{u^*}{u}\right)\left(\beta\frac{su}{s+u}-(\mu+\sigma)u\right)dx \\ &= -d_s\int_\Omega \frac{s^*}{s^2}|\nabla s|^2 dx + \int_\Omega\left(1-\frac{s^*}{s}\right)\left(\Lambda-\beta\frac{su}{s+u}-\mu s\right)dx \\ &- d_u\int_\Omega \frac{u^*}{u^2}|\nabla u|^2 dx + \int_\Omega\left(1-\frac{u^*}{u}\right)\left(\beta\frac{su}{s+u}-(\mu+\sigma)u\right)dx \\ &= M+N,\end{aligned}$$

where

$$M = -d_s\int_\Omega \frac{s^*}{s^2}|\nabla s|^2 dx - d_u\int_\Omega \frac{u^*}{u^2}|\nabla u|^2 dx \le 0, \tag{3.24}$$

and

$$N = \int_\Omega\left(1-\frac{s^*}{s}\right)\left(\Lambda-\beta\frac{su}{s+u}-\mu s\right)dx + \int_\Omega\left(1-\frac{u^*}{u}\right)\left(\beta\frac{su}{s+u}-(\mu+\sigma)u\right)dx. \tag{3.25}$$

Considering that (s^*, u^*) are solutions to system (1.1) we find

$$\begin{cases} \Lambda = \beta\frac{S^*U^*}{S^*+U^*}+\mu s^* \\ (\mu+\sigma) = \beta\frac{S^*}{S^*+U^*}. \end{cases} \tag{3.26}$$

We substitute in (3.25) we get

$$\begin{aligned} N &= \int_\Omega\left(1-\frac{s^*}{s}\right)\left(\beta\frac{s^*u^*}{s^*+u^*}+\mu s^*-\beta\frac{su}{s+u}-\mu s\right)dx \\ &+ \int_\Omega\left(1-\frac{u^*}{u}\right)\left[\beta\frac{su}{s+u}-\beta\frac{s^*u}{s^*+u^*}\right]dx \\ &= \int_\Omega\left[\left(1-\frac{s^*}{s}\right)(\mu s^*-\mu s)+\left(1-\frac{s^*}{s}\right)\left(\beta\frac{s^*u^*}{s^*+u^*}-\beta\frac{su}{s+u}\right)\right]dx \\ &+ \int_\Omega\left(1-\frac{u^*}{u}\right)\left[\beta\frac{su}{s+u}-\beta\frac{s^*u}{s^*+u^*}\right]dx \\ &= \int_\Omega \mu s^*\left(1-\frac{s^*}{s}\right)\left(1-\frac{s}{s^*}\right)dx \\ &+\beta\frac{s^*u^*}{s^*+u^*}\int_\Omega\left[\left(1-\frac{s^*}{s}\right)\left(1-\frac{\frac{su}{s+u}}{\frac{s^*u^*}{s^*+u^*}}\right)+\left(1-\frac{u^*}{u}\right)\left(\frac{\frac{su}{s+u}}{\frac{s^*u^*}{s^*+u^*}}-\frac{u}{u^*}\right)\right]dx \\ &= \mu s^*\int_\Omega\left(1-\frac{s^*}{s}\right)\left(1-\frac{s}{s^*}\right)dx \\ &+\beta\frac{s^*u^*}{s^*+u^*}\int_\Omega\left[1-\frac{s^*}{s}+\frac{\frac{u}{s+u}}{\frac{u^*}{s^*+u^*}}+1-\frac{u}{u^*}-\frac{\frac{u^*su}{s+u}}{\frac{us^*u^*}{s^*+u^*}}\right]dx. \end{aligned}$$

After simplification, we find

$$\begin{aligned} N &= -\mu s^* \int_\Omega \left[L\left(\frac{s^*}{s}\right) + L\left(\frac{s}{s^*}\right)\right] dx \\ &\quad -\beta \frac{s^* u^*}{s^* + u^*} \int_\Omega \left[L\left(\frac{s^*}{s}\right) + L\left(\frac{\frac{u^* su}{s+u}}{\frac{us^*u^*}{s^*+u^*}}\right)\right] dx \\ &\quad +\beta \frac{s^* u^*}{s^* + u^*} \int_\Omega \left[-L\left(\frac{u}{u^*}\right) + L\left(\frac{\frac{u}{s+u}}{\frac{u^*}{s^*+u^*}}\right)\right] dx. \end{aligned}$$

Using (3.21), we get $N \leq 0$. Back to (3.24), which leads to $\frac{d}{dt} W(t) \leq 0$. Hence, $W(t)$ is a Lyapunov function. □

Theorem 3.5. *Let* $E^* = (s^*, u^*)$ *be the solution of (1.1)-(1.3) in* $(0, +\infty)$, *Then*

$$\lim_{t \to +\infty} \|s(t,.) - s^*\|_\infty = 0, \tag{3.27}$$

and

$$\lim_{t \to +\infty} \|u(t,.) - u^*\|_\infty = 0. \tag{3.28}$$

Proof. In order to prove this theorem, we need the following corollary in ([5] pp. 386-387). As there is complete proof of this result in [5], we omit the proof of this for simplicity. □

4. Numerical experiments

In order to demonstrate the changes in solution behaviour that arise when the parameters are varied. The computer algorithm for numerical simulation was written in MATLAB.

The resulting problem is given by

$$\begin{cases} \frac{\partial s}{\partial t} - d_s \triangle s = \Lambda - \beta \frac{su}{s+u} - \mu s & \text{in } R^+ \times \Omega \\ \frac{\partial u}{\partial t} - d_u \triangle u = \beta \frac{su}{s+u} - (\mu + \sigma) u. & \text{in } R^+ \times \Omega \\ s_0(x) = s(x,0) > 0, u_0(x) = u(x,0) \geq 0 & \text{in } \Omega, \\ \frac{\partial s}{\partial v} = \frac{\partial u}{\partial v} = 0. & \text{in } R^+ \times \Omega \end{cases} \tag{4.1}$$

System (4.1) possesses two constant steady states

$$E_0 = \left(\frac{\Lambda}{\mu}, 0\right), \tag{4.2}$$

and

$$E^* = \left(\frac{\Lambda}{\beta - \sigma}, \Lambda \frac{\sigma - \beta + \mu}{(\sigma + \mu)(\sigma - \beta)}\right). \tag{4.3}$$

Table 1: Simulation parameters for system (4.1).

	Set	Figure	s_0	u_0	d_s	d_u	Λ	β	μ	σ
ODE	1	1(a)	2	8	–	–	5	$\frac{8}{11}$	$\frac{1}{2}$	$\frac{1}{3}$
	2	1(b)	6	1.5	–	–	8	$\frac{9}{10}$	$\frac{1}{5}$	$\frac{1}{2}$
EDP	1	2	$2+\cos(x)$	$8+2\sin(x)$	2	3	5	$\frac{8}{11}$	$\frac{1}{2}$	$\frac{1}{3}$
	2	3	$2+\cos(x)$	$8+2\sin(x)$	2	3	8	$\frac{9}{10}$	$\frac{1}{5}$	$\frac{1}{2}$

Note that the second steady state E^* exists only when the reproduction number $R_0 > 1$.

As detailed in Table 1, we use different sets of parameters to obtain numerical solutions in the ODE and PDE. Note that throughout the PDE simulations, we assume a single spatial dimension with $\Omega = (0, 10)$.

The following is a description of the results:

∘ Figure 1 shows the solutions in the ODE case subject to sets 1 and 2, with $R_0 = 0.87$ and $R_0 = 1.28$, respectively. In the first case, as $R_0 < 1$, $E_0 = (10, 0)$ is globally asymptotically stable. In the second case, $R_0 > 1$ and $E^* = (20, 40/7)$ is globally asymptotically stable.

∘ Figure 2 depicts the solution in the PDE case subject to parameter set 1, where $R_0 = 1.28 > 1$, which by Theorem 5 means that $E^* = (20, 40/7)$ is globally asymptotically stable.

∘ Figure 3 depicts the solution in the PDE case subject to parameter set 2, where $R_0 = 0.87 < 1$. By Theorem 4, $E_0 = (10, 0)$ is globally asymptotically stable.

Throughout the simulations we considered the following initial conditions:

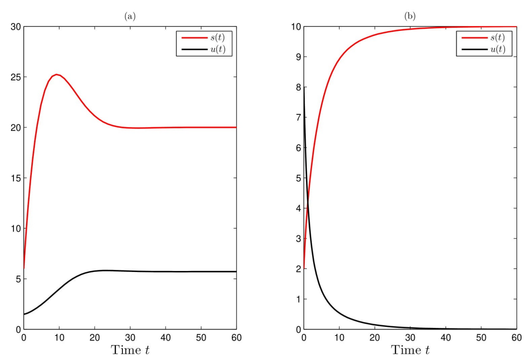

Figure 1: Numerical solutions of system (4.1) (ODE case) subject to the first and second sets of parameters.

Remark 4.1. The approximate solution depicted in Figure 1, 2 and 3 agree with the theoretical results obtained, regarding the dynamics of system (4.1).

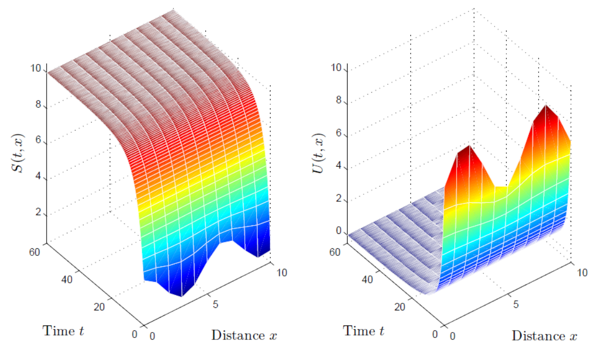

Figure 2: Numerical solutions of system (4.1) subject to the first set of parameters.

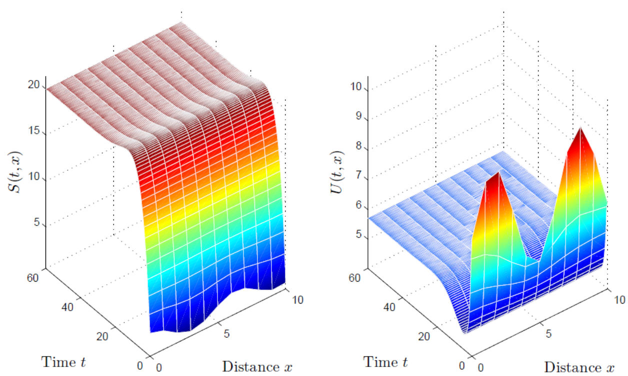

Figure 3: Numerical solutions of system (4.1) subject to the second set of parameters.

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