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Applications of the (p,q)-derivative involving a certain family of analytic and te-univalent functions

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Abstract. In this paper, we introduce a new subclass of analytic and te-univalent functions in the open unit disc associated with the operator $\Re^{\alpha,\beta,\gamma}_{\lambda,p,q}$, which is defined by using the (p,q)-derivative. We find estimates for the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this subclass, and we obtain an estimation for the Fekete-Szegő problem for this function class. Our results generalize some previously published results.

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1. Introduction

Let A denote the class of all functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Also let S denote the class of all functions in A which are univalent in U.

For the function f, given by (1.1), and $\zeta \in A$, given by

$$\zeta(z) = z + \sum_{k=2}^{\infty} b_k z^k,$$

the Hadamard product (or convolution) of f and ζ is defined by

$$(f * \zeta)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (\zeta * f)(z).$$

The theory of $q-$calculus plays an important role in many fields of mathematical, physical, and engineering sciences. The first application of the $q-$calculus was introduced by Jackson in [15, 16]. Recently, there is an extension of $q-$calculus, denoted by $(p, q)-$calculus which is obtained by substituting q by q/p in q-calculus. The $(p, q)-$integer was introduced by Chakrabarti and Jagannathan in [11]. For definitions and properties of the $(p, q)-$calculus, one may refer to [9, 24].

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For $0 < q < p \leq 1$, the $(p;q)-$derivative operator for f is defined as in [2]:

(1.2)
$$D_{pq}f(z) = \begin{cases} \frac{f(pz)-f(qz)}{(p-q)z}, & if \quad z \in \mathbb{U}^* := \mathbb{U} - \{0\}; \\ f'(0), & if \quad z = 0. \end{cases}$$

From (1.2) we deduce that

$$D_{pq}f(z) = 1 + \sum_{k=2}^{\infty} [k,p,q]\, a_k z^{k-1} \qquad (z \in \mathbb{U}),$$

where the $(p,q)-$bracket number is given by

(1.3)
$$\begin{aligned}[k,p,q] &= \frac{p^k - q^k}{p-q} = \sum_{j=0}^{k-1} p^{k-(j+1)} q^j \\ &= p^{k-1} + p^{k-2}q + p^{k-3}q^2 + ... + q^{k-1} \qquad (0 < q < p \leq 1),\end{aligned}$$

which is a natural generalization of the $q-$number. Clearly, we note that $[k,1,q] = [k]_q = \frac{1-q^k}{1-q}$ and $\lim_{q \longrightarrow 1^-} [k,1,q] = k$. Also, we note that $D_{pq}f(z) \longrightarrow f'(z)$ as $p = 1$ and $q \longrightarrow 1^-$, where f' is the ordinary derivative of the function f.

By using (1.3) the $(p,q)-$shifted factorial is given by

$$[k,p,q]! = \begin{cases} 1, & if \quad k = 0; \\ \prod_{i=1}^{k} [i,p,q], & if \quad k \in \mathbb{N} := \{1,2,3,...\}, \end{cases}$$

and for any positive number δ, the $(p,q)-$generalized Pochhammer symbol is defined by

$$[\delta,p,q]_k = \begin{cases} 1, & if \quad k = 0; \\ \prod_{i=1}^{k} [\delta + i - 1, p, q], & if \quad k \in \mathbb{N}. \end{cases}$$

For the function $f \in A$, we define the operator $\Re_{\lambda,p,q}^{\alpha,\beta,\gamma} : A \to A$ as follows:

$$\Re_{\lambda,p,q}^{\alpha,\beta,\gamma} f(z) * \mathcal{M}_{\lambda,p,q}^{\alpha,\beta,\gamma}(z) = zD_{pq}f(z),$$

where the function $\mathcal{M}_{\lambda,p,q}^{\alpha,\beta,\gamma}$ is defined, in terms of the Gamma function Γ, by

$$\mathcal{M}_{\lambda,p,q}^{\alpha,\beta,\gamma}(z) = z + \sum_{k=2}^{\infty} \left[\frac{\Gamma(\beta+1)\,\Gamma(\alpha+\beta+k-\gamma+2)\,[\lambda+1,p,q]_{k-1}}{\Gamma(\alpha+\beta-\gamma+2)\,\Gamma(\beta+k+1)\,[k-1,p,q]!} \right] z^k$$
$$(\beta > -1;\ \alpha \geq \gamma - 1; \gamma > 0; \lambda > -1; 0 < q < p \leq 1; z \in U).$$

A simple computation shows that

$$\Re_{\lambda,p,q}^{\alpha,\beta,\gamma} f(z) = z + \sum_{k=2}^{\infty} \psi_k a_k z^k, \tag{1.4}$$

where

$$\psi_k = \frac{\Gamma(\alpha+\beta-\gamma+2)\,\Gamma(\beta+k+1)\,[k,p,q]!}{\Gamma(\beta+1)\,\Gamma(\alpha+\beta+k-\gamma+2)\,[\lambda+1,p,q]_{k-1}} \tag{1.5}$$
$$(\beta > -1;\ \alpha \geq \gamma - 1; \gamma > 0; \lambda > -1; 0 < q < p \leq 1).$$

We note that $\Re_{0,1,q}^{0,\beta,1} f(z) \longrightarrow zf'(z)$ as $\gamma = p = 1$, $\alpha = \lambda = 0$ and $q \longrightarrow 1^-$. Also, for $\gamma = \lambda = 1$ and $\alpha = 0$ we have $\Re_{1,p,q}^{0,\beta,1} f(z) = f(z)$.

Remark 1.1.

(i) For $\alpha = 0$ and $\gamma = p = 1$,

$$\begin{aligned}\Re_{\lambda,1,q}^{0,\beta,1} f(z) &= \Im_q^{\lambda} f(z) \\ &:= z + \sum_{k=2}^{\infty} \frac{[k,q]!}{[\lambda+1,q]_{k-1}} a_k z^k \quad (\lambda > -1; 0 < q < 1; z \in U),\end{aligned}$$

where the operator J_q^{λ} was studied by Arif et al. [5];

(ii) For $\lambda = 1$, $\frac{[k,p,q]!}{[\lambda+1,p,q]_{k-1}} = 1$, and we obtain

$$\begin{aligned}&\Re_{1,p,q}^{\alpha,\beta,\gamma} f(z) \\ &= \Re_{\beta}^{\alpha,\gamma} f(z) \\ &:= \binom{\alpha+\beta-\gamma+1}{\beta} \frac{(\alpha-\gamma+1)}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-\gamma} t^{\beta-1} f(t)\,dt \\ &= \frac{\Gamma(\alpha+\beta-\gamma+2)}{\Gamma(\beta+1)\,\Gamma(\alpha-\gamma+1)} \frac{1}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-\gamma} t^{\beta-1} f(t)\,dt \qquad (1.6) \\ &= z + \sum_{k=2}^{\infty} \left[\frac{\Gamma(\alpha+\beta-\gamma+2)\,\Gamma(\beta+k+1)}{\Gamma(\beta+1)\,\Gamma(\alpha+\beta+k-\gamma+2)}\right] a_k z^k \\ &\quad (\beta > -1;\ \alpha \geq \gamma - 1; \gamma > 0; z \in U),\end{aligned}$$

where the operator $\Re_{\beta}^{\alpha,\gamma}$ was introduced and studied by Aouf et al. [4];

(iii) For $\gamma = \lambda = 1$,

$$\begin{aligned}\Re^{\alpha,\beta,1}_{1,p,q} f(z) &= Q^{\alpha}_{\beta} f(z) \\ &:= \binom{\alpha+\beta}{\beta} \frac{\alpha}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\, dt \\ &= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(\alpha)} \frac{1}{z^{\beta}} \int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\, dt \\ &= z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{k=2}^{\infty} \left[\frac{\Gamma(\beta+k+1)}{\Gamma(\alpha+\beta+k+1)}\right] a_k z^k \\ &\quad (\beta > -1;\ \alpha \geq 0; z \in U),\end{aligned}$$

where the operator Q^{α}_{β} was introduced and studied by Jung et al. [17];

(iv) For $\alpha = \gamma = \lambda = 1$ and $\beta = c$,

$$\begin{aligned}\Re^{1,c,1}_{1,p,q} f(z) &= J_c f(z) \\ &:= \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\, dt \\ &= z + \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k+1}\right) a_k z^k \\ &\quad (c > -1; z \in U),\end{aligned}$$

where the operator $J_c f(z)$ was introduced by Bernardi [6] and we note that $J_1 f(z) = J f(z)$ was introduced and studied by Libera [19] and Livingston [20].

According to the Koebe one-quarter theorem, see Duren [12], the images of U under every univalent functions $f \in S$ contain a disc of radius $\frac{1}{4}$ centered at 0. Thus, every univalent function f on U has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where $r_0(f)$ is the radius of the image of U under f, and

$$h(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots. \tag{1.7}$$

A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of all bi-univalent functions in U given by (1.1). Some examples of functions in the class Σ are $\frac{z}{1-z}$, $-\log(1-z)$, and $\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$.

Abd-Eltawab [1] introduced the concept of te-univalency associated with an operator, which is a generalization and extension of the concept of bi-univalency. For $f \in A$, let $S^{\alpha,\beta,\gamma}_{\lambda,p,q}$ denote the class of all functions given by (1.4), which are univalent in U. It is well known that every function $\Re^{\alpha,\beta,\gamma}_{\lambda,p,q} f \in S^{\alpha,\beta,\gamma}_{\lambda,p,q}$ has an inverse $\left(\Re^{\alpha,\beta,\gamma}_{\lambda,p,q} f\right)^{-1}$, defined by

$$g(\Re^{\alpha,\beta,\gamma}_{\lambda,p,q} f(z)) = z \quad (z \in U),$$

and

$$\Re^{\alpha,\beta,\gamma}_{\lambda,p,q} f(g(w)) = w \quad \left(|w| < r_0(\Re^{\alpha,\beta,\gamma}_{\lambda,p,q} f); r_0(\Re^{\alpha,\beta,\gamma}_{\lambda,p,q} f) \geq \frac{1}{4}\right),$$

where

$$\begin{aligned} g(w) &= \left(\Re^{\alpha,\beta,\gamma}_{\lambda,p,q} f\right)^{-1}(w) \\ &= w - \psi_2 a_2 w^2 + \left[2\psi_2^2 a_2^2 - \psi_3 a_3\right] w^3 \\ &- \left[5\psi_2^3 a_2^3 - 5\psi_2\psi_3 a_2 a_3 + \psi_4 a_4\right] w^4 + ..., \end{aligned} \tag{1.8}$$

and ψ_k is given by (1.5).

A function f given by (1.1) is said to be te-univalent in U associated with the operator $\Re^{\alpha,\beta,\gamma}_{\lambda,p,q}$, if both $\Re^{\alpha,\beta,\gamma}_{\lambda,p,q} f$ and $\left(\Re^{\alpha,\beta,\gamma}_{\lambda,p,q} f\right)^{-1}$ are univalent in U. Let $\Sigma^{\alpha,\beta,\gamma}_{\lambda,p,q}$ denote the class of all functions given by (1.1), which are te-univalent in U associated with $\Re^{\alpha,\beta,\gamma}_{\lambda,p,q}$.

Remark 1.2.

(i) For $\lambda = 1$, we have $\Sigma^{\alpha,\beta,\gamma}_{1,p,q} = \Sigma^{\alpha,\beta,\gamma}$ denote the class of all functions given by (1.1), which are te-univalent in U associated with the operator $\Re^{\alpha,\gamma}_{\beta} f(z)$, where the operator $\Re^{\alpha,\gamma}_{\beta} f(z)$ is given by (1.6);

(ii) For $\lambda = \gamma = 1$ and $\alpha = 0$, we have $\Sigma^{0,\beta,1}_{1,p,q} = \Sigma$.

For two functions f and g, which are analytic in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function s, which (by definition) is analytic in U with $s(0) = 0$ and $|s(z)| < 1$ for all $z \in U$, such that $f(z) = g(s(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g., [10] , and [22]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let φ is an analytic function with positive real part in the unit disc U, which satisfies the following conditions:

$$\varphi(0) = 1 \text{ and } \varphi'(0) > 0$$

and is so constrained that $\varphi(U)$ is symmetric with respect to the real axis. Ma and Minda [21] unified various subclasses of starlike and convex functions consisting of functions $f \in A$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \varphi(z)$ and $1+\frac{zf''(z)}{f'(z)} \prec \varphi(z)$, respectively. A function f is bi-starlike of Ma-Minda type or bi-convex of Ma-Minda type if both f and f^{-1} are, respectively, Ma-Minda starlike or convex (see [3]). Many interesting examples of the functions of the class Σ, together with various other properties and characteristics associated with bi-univalent functions can be found in the earlier work (cf., e.g., [7, 18, 23]). Brannan and Taha [8] introduced certain subclasses of bi-univalent functions similar to the familiar subclasses of univalent functions consisting of starlike, convex and strongly starlike functions. They investigated the bound on the initial coefficients of the classes bi-starlike and bi-convex functions. Recently, many researchers (see [1, 3, 14]) introduced and investigated some new subclasses of Σ and obtained bounds for the initial coefficients of the function given by (1.1). For a brief history and interesting examples in the class Σ, (see [29]).

Earlier in 1933, Fekete and Szegő [13] made use of Lowner's parametric method in order to prove that, if $f \in S$ and is given by (1.1),

$$\left|a_3-\mu a_2^2\right| \leq 1+2\exp\left(-\frac{2\lambda}{1-\lambda}\right) \quad (0 \leq \lambda \leq 1).$$

For some history of Feketo-Szegő problem for class of starlike, convex and close-to-convex functions, refer to work produced by Srivastava et al. [28]. Besides that, some authors [1, 26, 31] have studied the Feketo-Szegő inequalities for certain subclasses of bi-univalent functions.

The object of the present paper is to introduce a new subclass of analytic and te-univalent functions in the open unit disc associated with the operator $\Re_{\lambda,p,q}^{\alpha,\beta,\gamma}$ based on the Ma-Minda concept, and the bound for second and third coefficients of functions in this class are obtained. Also the Fekete-Szegő inequality is determined for this function class. Our results generalize several well-known results in [3, 14, 25, 29, 31] and these are pointed out.

In order to derive our main results, we have to recall here the following lemmas.

Lemma 1.3. *[27] If $p \in \mathcal{P}$ then $|c_k| \leq 2$ for each k, where $\mathcal{P}$ is the family of all functions p analytic in U for which $Re\{p(z)\} > 0$, $p(z) = 1+c_1z+c_2z^2+c_3z^3+\cdots$ for $z \in U$.*

Lemma 1.4. *[30] Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < \chi$ and $|z_2| < \chi$ then*

$$|(k+l)z_1+(k-l)z_2| \leq \begin{cases} 2|k|\chi & for \quad |k| \geq |l|; \\ 2|l|\chi & for \quad |k| \leq |l|. \end{cases}$$

2. Coefficient Bounds for the Function Class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$

We begin this section by assuming that φ is an analytic function with positive real part in U with $\varphi(0) = 1$ and $\varphi'(0) > 0$ and is so constrained that

$\varphi(U)$ is symmetric with respect to the real axis. Such a function has a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \qquad (z \in U) \tag{2.1}$$

where $B_1, B_2, B_3, ...$ are real and $B_1 > 0$.

Unless otherwise mentioned, we assume throughout this paper that the function φ satisfies the above conditions, $\beta > -1$, $\alpha \geq \gamma - 1$, $\gamma > 0$, $\eta \geq 1$, $\lambda > -1$, $0 < q < p \leq 1$ and $z \in U$.

Definition 2.1. A function f, given by (1.1), is said to be in the class $\Sigma^{\alpha,\beta,\gamma}_{\lambda,p,q}[\varphi,\eta]$, if the following subordination conditions hold true:

$$f \in \Sigma^{\alpha,\beta,\gamma}_{\lambda,p,q} \text{ with } (1-\eta)\frac{\Re^{\alpha,\beta,\gamma}_{\lambda,p,q}f(z)}{z} + \eta D_{pq}\Re^{\alpha,\beta,\gamma}_{\lambda,p,q}f(z) \prec \varphi(z), \tag{2.2}$$

and

$$(1-\eta)\frac{g(w)}{w} + \eta D_{pq}g(w) \prec \varphi(w), \tag{2.3}$$

where the functions $\Re^{\alpha,\beta,\gamma}_{\lambda,p,q}f$, and g are given by (1.4), and (1.8), respectively.

It is interesting to note that for appropriate values of parameters involved in the class $\Sigma^{\alpha,\beta,\gamma}_{\lambda,p,q}[\varphi,\eta]$, we have the following known and new classes:

(i) For $\eta = 1$, the class $\Sigma^{\alpha,\beta,\gamma}_{\lambda,p,q}[\varphi,\eta]$ reduces to the class $\Sigma^{\alpha,\beta,\gamma}_{\lambda,p,q}[\varphi]$, that represents the functions $f \in \Sigma^{\alpha,\beta,\gamma}_{\lambda,p,q}$ that satisfy the following subordination conditions:

$$D_{pq}\Re^{\alpha,\beta,\gamma}_{\lambda,p,q}f(z) \prec \varphi(z),$$

and

$$D_{pq}g(w) \prec \varphi(w),$$

where the functions $\Re^{\alpha,\beta,\gamma}_{\lambda,p,q}f$ and g are given by (1.4) and (1.8), respectively;

(ii) For $\lambda = p = 1$ and $q \longrightarrow 1^-$, we have $\Re^{\alpha,\beta,\gamma}_{1,1,q}f(z) = \Re^{\alpha,\gamma}_{\beta}f(z)$,

$$\lim_{q\longrightarrow 1^-} D_{pq}\Re^{\alpha,\beta,\gamma}_{1,1,q}f(z) \longrightarrow \left(\Re^{\alpha,\gamma}_{\beta}f\right)'(z),$$

and

$$\lim_{q\longrightarrow 1^-} D_{pq}g(w) \longrightarrow g'(w),$$

therefore the class $\Sigma^{\alpha,\beta,\gamma}_{\lambda,p,q}[\varphi,\eta]$ reduces to the class $\Sigma^{\alpha,\beta,\gamma}[\varphi,\eta]$, that represents the functions $f \in \Sigma^{\alpha,\beta,\gamma}$ that satisfy (2.2) and (2.3) for $\Re^{\alpha,\beta,\gamma}_{\lambda,p,q}f(z)$, $D_{pq}\Re^{\alpha,\beta,\gamma}_{\lambda,p,q}f(z)$ and $D_{pq}g(w)$ replaced with $\Re^{\alpha,\gamma}_{\beta}f(z)$, $\left(\Re^{\alpha,\gamma}_{\beta}f\right)'(z)$ and $g'(w)$, respectively, where the operator $\Re^{\alpha,\gamma}_{\beta}$ is given by (1.6) and $g = \left(\Re^{\alpha,\gamma}_{\beta}f\right)^{-1}$;

(iii) For $\lambda=\gamma=p=1$, $\alpha=0$ and $q\longrightarrow 1^-$, we have $\Re_{1,1,q}^{0,\beta,1}f(z)=f(z)$, $g(w)=h(w)$, $\lim_{q\longrightarrow 1^-}D_{pq}\Re_{1,1,q}^{0,\beta,1}f(z)\longrightarrow f^{'}(z)$ and $\lim_{q\longrightarrow 1^-}D_{pq}g(w)\longrightarrow h^{'}(w)$, therefore the class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$ reduces to the class $B_\Sigma(\varphi,\eta)$, that represents the functions $f\in\Sigma$ that satisfy (2.2) and (2.3) for $\Re_{\lambda,p,q}^{\alpha,\beta,\gamma}f(z)$, $D_{pq}\Re_{\lambda,p,q}^{\alpha,\beta,\gamma}f(z)$, $g(w)$ and $D_{pq}g(w)$ replaced with $f(z)$, $f^{'}(z)$, $h(w)$ and $h^{'}(w)$, respectively, where the functions f and h are given by (1.1) and (1.7), respectively. The class $B_\Sigma(\varphi,\eta)$ was introduced and studied by Omar et al.[25];

(iv) For $\lambda=\gamma=\eta=p=1$, $\alpha=0$, and $q\longrightarrow 1^-$ we have $\lim_{q\longrightarrow 1^-}D_{pq}\Re_{1,1,q}^{0,\beta,1}f(z)\longrightarrow f^{'}(z)$, $\lim_{q\longrightarrow 1^-}D_{pq}g(w)\longrightarrow h^{'}(w)$, where the functions f and h are given by (1.1) and (1.7), respectively. Therefore the class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$ reduces to the class $\mathcal{H}_\Sigma(\varphi)$, which was introduced and studied by Ali et al.[3];

(v) For $\lambda=\gamma=p=1$, $\alpha=0$, $\varphi=\left(\frac{1+z}{1-z}\right)^\zeta$ $(0<\zeta\le 1)$, and $q\longrightarrow 1^-$ we have $\Re_{1,1,q}^{0,\beta,1}f(z)=f(z)$, $g(w)=h(w)$, $\lim_{q\longrightarrow 1^-}D_{pq}\Re_{1,1,q}^{0,\beta,1}f(z)\longrightarrow f^{'}(z)$ and $\lim_{q\longrightarrow 1^-}D_{pq}g(w)\longrightarrow h^{'}(w)$, where the functions f and h are given by (1.1) and (1.7), respectively. Therefore the class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$ reduces to the class $B_\Sigma(\zeta,\eta)$ $(0<\zeta\le 1)$, which was introduced and studied by Frasin and Aouf [14];

(vi) For $\lambda=\gamma=p=1$, $\alpha=0$, $\varphi=\frac{1+(1-2\sigma)z}{1-z}$ $(0\le\sigma<1)$, and $q\longrightarrow 1^-$ we have $\Re_{1,1,q}^{0,\beta,1}f(z)=f(z)$, $g(w)=h(w)$, $\lim_{q\longrightarrow 1^-}D_{pq}\Re_{1,1,q}^{0,\beta,1}f(z)\longrightarrow f^{'}(z)$ and $\lim_{q\longrightarrow 1^-}D_{pq}g(w)\longrightarrow h^{'}(w)$, where the functions f and h are given by (1.1) and (1.7), respectively. Therefore the class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$ reduces to the class $B_\Sigma(\sigma,\eta)$ $(0\le\sigma<1)$, which was introduced and studied by Frasin and Aouf [14];

(vii) For$\lambda=\gamma=\eta=p=1$, $\alpha=0$, $\varphi=\left(\frac{1+z}{1-z}\right)^\zeta$ $(0<\zeta\le 1)$, and $q\longrightarrow 1^-$ we have $\lim_{q\longrightarrow 1^-}D_{pq}\Re_{1,1,q}^{0,\beta,1}f(z)\longrightarrow f^{'}(z)$, $\lim_{q\longrightarrow 1^-}D_{pq}g(w)\longrightarrow h^{'}(w)$, where the functions f and h are given by (1.1) and (1.7), respectively. Therefore the class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$ reduces to the class $\mathcal{H}_\Sigma^\zeta$ $(0<\zeta\le 1)$, which was introduced and studied by Srivastava et al.[29];

(viii) For $\lambda=\gamma=\eta=p=1$, $\alpha=0$, $\varphi=\frac{1+(1-2\sigma)z}{1-z}$ $(0\le\sigma<1)$, and $q\longrightarrow 1^-$ we have $\lim_{q\longrightarrow 1^-}D_{pq}\Re_{1,1,q}^{0,\beta,1}f(z)\longrightarrow f^{'}(z)$, $\lim_{q\longrightarrow 1^-}D_{pq}g(w)\longrightarrow h^{'}(w)$, where the functions f and h are given by (1.1) and (1.7), respectively. Therefore the class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$ reduces to the class $\mathcal{H}_\Sigma(\sigma)$ $(0\le\sigma<1)$, which was introduced and studied by Srivastava et al.[29].

Theorem 2.2.
If f, given by (1.1), is in the class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$, then

$$(2.4)\quad |a_2| \leq \frac{B_1\sqrt{B_1}}{\psi_2\sqrt{\left|\left[1+\eta\left([3,p,q]-1\right)\right]B_1^2+\left[1+\eta\left([2,p,q]-1\right)\right]^2\left(B_1-B_2\right)\right|}},$$

and

$$(2.5)\qquad |a_3| \leq \frac{B_1}{\left[1+\eta\left([3,p,q]-1\right)\right]\psi_3}+\frac{B_1^2}{\left[1+\eta\left([2,p,q]-1\right)\right]^2\psi_3},$$

where ψ_k, $k\in\{2,3\}$, is given by (1.5).

Proof. If $f\in\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$, from (2.2), (2.3), and the definition of subordination it follows that there exist two analytic functions $u,v:U\longrightarrow U$ with $u(0)=v(0)=0$, such that

$$(2.6)\qquad (1-\eta)\frac{\Re_{\lambda,p,q}^{\alpha,\beta,\gamma}f(z)}{z}+\eta D_{pq}\Re_{\lambda,p,q}^{\alpha,\beta,\gamma}f(z)=\varphi\left(u\left(z\right)\right),$$

and

$$(2.7)\qquad (1-\eta)\frac{g(w)}{w}+\eta D_{pq}g(w)=\varphi\left(v\left(w\right)\right).$$

We define the functions p and q in $\mathcal{P}$ given by

$$(2.8)\qquad p\left(z\right)=\frac{1+u\left(z\right)}{1-u\left(z\right)}=1+p_1z+p_2z^2+p_3z^3+...,$$

and

$$(2.9)\qquad q\left(z\right)=\frac{1+v\left(z\right)}{1-v\left(z\right)}=1+q_1z+q_2z^2+q_3z^3+....$$

It follows from (2.8) and (2.9) that

$$(2.10)\qquad u\left(z\right)=\frac{p\left(z\right)-1}{p\left(z\right)+1}=\frac{p_1}{2}z+\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)z^2+...,$$

and

$$(2.11)\qquad v\left(z\right)=\frac{q\left(z\right)-1}{q\left(z\right)+1}=\frac{q_1}{2}z+\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)z^2+....$$

Using (2.10) and (2.11) with (2.1) lead us to

$$\varphi\left(u\left(z\right)\right)=\varphi\left(\tfrac{p(z)-1}{p(z)+1}\right)=1+\tfrac{B_1p_1}{2}z+\left[\tfrac{1}{2}\left(p_2-\tfrac{p_1^2}{2}\right)B_1+\tfrac{1}{4}p_1^2B_2\right]z^2+...,$$

and

$$\varphi\left(v\left(z\right)\right)=\varphi\left(\tfrac{q(z)-1}{q(z)+1}\right)=1+\tfrac{B_1 q_1}{2}z+\left[\tfrac{1}{2}\left(q_2-\tfrac{q_1^2}{2}\right)B_1+\tfrac{1}{4}q_1^2 B_2\right]z^2+....$$

On the other hand,

$$\begin{aligned}&(1-\eta)\frac{\Re_{\lambda,p,q}^{\alpha,\beta,\gamma}f(z)}{z}+\eta D_{pq}\Re_{\lambda,p,q}^{\alpha,\beta,\gamma}f(z)\\ =\ & 1+\left[1+\eta\left([2,p,q]-1\right)\right]\psi_2 a_2 z+\left[1+\eta\left([3,p,q]-1\right)\right]\psi_3 a_3 z^2+...,\end{aligned}$$

and

$$\begin{aligned}(1-\eta)\frac{g(w)}{w}+\eta D_{pq}g(w) &= 1-\left[1+\eta\left([2,p,q]-1\right)\right]\psi_2 a_2 w\\ &\quad+\left[1+\eta\left([3,p,q]-1\right)\right]\left(2\psi_2^2 a_2^2-\psi_3 a_3\right)w^2+....\end{aligned}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$\left[1+\eta\left([2,p,q]-1\right)\right]\psi_2 a_2=\frac{B_1 p_1}{2}, \tag{2.12}$$

$$\left[1+\eta\left([3,p,q]-1\right)\right]\psi_3 a_3=\frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)B_1+\frac{1}{4}p_1^2 B_2, \tag{2.13}$$

$$-\left[1+\eta\left([2,p,q]-1\right)\right]\psi_2 a_2=\frac{B_1 q_1}{2} \tag{2.14}$$

and

$$\left[1+\eta\left([3,p,q]-1\right)\right]\left(2\psi_2^2 a_2^2-\psi_3 a_3\right)=\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)B_1+\frac{1}{4}q_1^2 B_2. \tag{2.15}$$

From (2.12) and (2.14), we get

$$p_1=-q_1 \tag{2.16}$$

and

$$2\left[1+\eta\left([2,p,q]-1\right)\right]^2\psi_2^2 a_2^2=\frac{B_1^2}{4}\left(p_1^2+q_1^2\right). \tag{2.17}$$

Now from (2.13), (2.15) and (2.17), we obtain

$$\begin{aligned}&2\left[1+\eta\left([3,p,q]-1\right)\right]\psi_2^2 a_2^2\\ &\quad=\frac{B_1}{2}\left(p_2+q_2\right)+\frac{B_2-B_1}{4}\left(p_1^2+q_1^2\right)\\ &\quad=\frac{B_1}{2}\left(p_2+q_2\right)+\frac{2\left(B_2-B_1\right)\left[1+\eta\left([2,p,q]-1\right)\right]^2\psi_2^2 a_2^2}{B_1^2}.\end{aligned}$$

Therefore, we have

$$a_2^2 = \frac{B_1^3\left(p_2+q_2\right)}{4\psi_2^2\left[\left[1+\eta\left([3,p,q]-1\right)\right]B_1^2+\left(B_1-B_2\right)\left[1+\eta\left([2,p,q]-1\right)\right]^2\right]} \tag{2.18}$$

Applying Lemma1.3 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\psi_2\sqrt{\left|\left[1+\eta\left([3,p,q]-1\right)\right]B_1^2+\left[1+\eta\left([2,p,q]-1\right)\right]^2\left(B_1-B_2\right)\right|}}.$$

This gives the bound on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.15) from (2.13) and using (2.16), we get

$$\begin{aligned} &2\left[1+\eta\left([3,p,q]-1\right)\right]\psi_3 a_3 - 2\left[1+\eta\left([3,p,q]-1\right)\right]\psi_2^2 a_2^2 \\ =\ & \frac{1}{2}\left(p_2-\frac{p_1^2}{2}\right)B_1+\frac{1}{4}p_1^2B_2-\frac{1}{2}\left(q_2-\frac{q_1^2}{2}\right)B_1-\frac{1}{4}q_1^2B_2 \\ =\ & \frac{1}{2}B_1\left(p_2-q_2\right). \end{aligned} \tag{2.19}$$

It follows from (2.17) and (2.19) that

$$2\left[1+\eta\left([3,p,q]-1\right)\right]\psi_3 a_3 = \frac{B_1^2\left[1+\eta\left([3,p,q]-1\right)\right]\left(p_1^2+q_1^2\right)}{4\left[1+\eta\left([2,p,q]-1\right)\right]^2}+\frac{1}{2}B_1\left(p_2-q_2\right)$$

and then,

$$a_3 = \frac{B_1^2\left(p_1^2+q_1^2\right)}{8\left[1+\eta\left([2,p,q]-1\right)\right]^2\psi_3}+\frac{B_1\left(p_2-q_2\right)}{4\left[1+\eta\left([3,p,q]-1\right)\right]\psi_3}$$

Applying Lemma1.3 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{B_1^2}{\left[1+\eta\left([2,p,q]-1\right)\right]^2\psi_3}+\frac{B_1}{\left[1+\eta\left([3,p,q]-1\right)\right]\psi_3}.$$

This completes the proof of Theorem 2.2. □

Taking $\eta = 1$ in Theorem 2.2, we get the following corollary:

Corollary 2.3. *If f, given by* (1.1), *is in the class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi]$, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\psi_2\sqrt{\left|[3,p,q]\,B_1^2+[2,p,q]^2\left(B_1-B_2\right)\right|}},$$

and

$$|a_3| \leq \frac{B_1}{[3,p,q]\,\psi_3}+\frac{B_1^2}{[2,p,q]^2\,\psi_3},$$

where ψ_k, $k \in \{2,3\}$, is given by (1.5).

Taking $\lambda = p = 1$ and $q \longrightarrow 1^{-}$ in Theorem 2.2, we get the following corollary:

Corollary 2.4. *If f, given by* (1.1)*, is in the class* $\Sigma^{\alpha,\beta,\gamma}[\varphi,\eta]$*, then*

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\psi_2\sqrt{\left|(1+2\eta)B_1^2+(1+\eta)^2(B_1-B_2)\right|}},$$

and

$$|a_3| \leq \frac{B_1}{(1+2\eta)\psi_3}+\frac{B_1^2}{(1+\eta)^2\psi_3},$$

where

$$\psi_k = \frac{\Gamma(\alpha+\beta-\gamma+2)\,\Gamma(\beta+k+1)}{\Gamma(\beta+1)\,\Gamma(\alpha+\beta+k-\gamma+2)},\ k=2,3.$$

Remark 2.5.

(i) Taking $\gamma = 1$ and $\alpha = 0$ in Corollary2.4, we obtain the result obtained by Omar et al. [[25], Theorem 2.2];

(ii) Taking $\gamma = \eta = 1$ and $\alpha = 0$ in Corollary 2.4, we obtain the result obtained by Ali et al. [[3], Theorem 2.1];

(iii) Taking $\gamma = 1$, $\alpha = 0$ and $\varphi = \left(\frac{1+z}{1-z}\right)^{\zeta}$ $(0 < \zeta \leq 1)$ in Corollary 2.4, we obtain the result obtained by Frasin and Aouf [[14], Theorem 2.2];

(iv) Taking $\gamma = 1$, $\alpha = 0$ and $\varphi = \frac{1+(1-2\sigma)z}{1-z}$ $(0 \leq \sigma < 1)$ in Corollary 2.4, we obtain the result obtained by Frasin and Aouf [[14], Theorem 3.2];

(v) Taking $\gamma = \eta = 1$, $\alpha = 0$ and $\varphi = \left(\frac{1+z}{1-z}\right)^{\zeta}$ $(0 < \zeta \leq 1)$ in Corollary 2.4, we obtain the result obtained by Srivastava et al. [[29], Theorem 1];

(vi) Taking $\gamma = \eta = 1$, $\alpha = 0$ and $\varphi = \frac{1+(1-2\sigma)z}{1-z}$ $(0 \leq \sigma < 1)$ in Corollary 2.4, we obtain the result obtained by Srivastava et al. [[29], Theorem 2].

3. Fekete-Szegő Problem

Theorem 3.1. *If f, given by* (1.1)*, is in the class* $\Sigma^{\alpha,\beta,\gamma}_{\lambda,p,q}[\varphi,\eta]$*, then*

$$\left|a_3-\mu a_2^2\right| \leq$$

$$\begin{cases} \frac{B_1}{[1+\eta([3,p,q]-1)]\psi_3} & \text{for} \quad \left|1-\mu\frac{\psi_3}{\psi_2^2}\right| \leq \left|1+\frac{[1+\eta([2,p,q]-1)]^2(B_1-B_2)}{[1+\eta([3,p,q]-1)]B_1^2}\right|; \\ \frac{B_1^3\left|1-\mu\frac{\psi_3}{\psi_2^2}\right|}{\psi_3\left|\begin{array}{c}[1+\eta([3,p,q]-1)]B_1^2+\\ [1+\eta([2,p,q]-1)]^2(B_1-B_2)\end{array}\right|} & \text{for} \quad \left|1-\mu\frac{\psi_3}{\psi_2^2}\right| \geq \left|1+\frac{[1+\eta([2,p,q]-1)]^2(B_1-B_2)}{[1+\eta([3,p,q]-1)]B_1^2}\right|, \end{cases}$$

where $\mu \in \mathbb{C}$, and ψ_k, $k \in \{2,3\}$, is given by(1.5).

Proof. Let $f \in \Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$ and from (2.19), we have

$$a_3 - \frac{\psi_2^2}{\psi_3}a_2^2 = \frac{B_1(p_2 - q_2)}{4[1+\eta([3,p,q]-1)]\psi_3}. \tag{3.1}$$

Multiplying (2.18) by $\left(\frac{\psi_2^2}{\psi_3} - \mu\right)$ we get

$$\left(\frac{\psi_2^2}{\psi_3} - \mu\right)a_2^2 = \frac{B_1^3\left(1-\mu\frac{\psi_3}{\psi_2^2}\right)(p_2+q_2)}{4\psi_3\left[[1+\eta([3,p,q]-1)]B_1^2+(B_1-B_2)[1+\eta([2,p,q]-1)]^2\right]}. \tag{3.2}$$

Adding (3.1) and (3.2), it follows that

$$a_3 - \mu a_2^2 = \frac{B_1}{4\psi_3}\left[\left(L(\mu) + \frac{1}{[1+\eta([3,p,q]-1)]}\right)p_2 + \left(L(\mu) - \frac{1}{[1+\eta([3,p,q]-1)]}\right)q_2\right], \tag{3.3}$$

and it follows

$$\left|a_3 - \mu a_2^2\right| = \frac{B_1}{4\psi_3}\left|\left(L(\mu) + \frac{1}{[1+\eta([3,p,q]-1)]}\right)p_2 + \left(L(\mu) - \frac{1}{[1+\eta([3,p,q]-1)]}\right)q_2\right|, \tag{3.4}$$

where

$$L(\mu) = \frac{B_1^2\left(1-\mu\frac{\psi_3}{\psi_2^2}\right)}{[1+\eta([3,p,q]-1)]B_1^2 + [1+\eta([2,p,q]-1)]^2(B_1-B_2)}.$$

Since $B_1, B_2, B_3, ...$ are real, $B_1 > 0$, $|p_2| \le 2$, $|q_2| \le 2$ and from Lemma1.4, we conclude that

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{B_1}{[1+\eta([3,p,q]-1)]\psi_3} & for \quad o \le |L(\mu)| \le \frac{1}{1+\eta([3,p,q]-1)}; \\ \frac{B_1}{\psi_3}|L(\mu)| & for \quad |L(\mu)| \ge \frac{1}{1+\eta([3,p,q]-1)}. \end{cases}$$

This completes the proof of Theorem 3.1. □

Taking $\eta = 1$ in Theorem 3.1, we get the following corollary:

Corollary 3.2. *If f, given by (1.1), is in the class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi]$, then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{B_1}{[3,p,q]\psi_3} & for \quad \left|1-\mu\frac{\psi_3}{\psi_2^2}\right| \le \left|1+\frac{[2,p,q]^2(B_1-B_2)}{[3,p,q]B_1^2}\right|; \\ \frac{B_1^3\left|1-\mu\frac{\psi_3}{\psi_2^2}\right|}{\psi_3\left|[3,p,q]B_1^2+[2,p,q]^2(B_1-B_2)\right|} & for \quad \left|1-\mu\frac{\psi_3}{\psi_2^2}\right| \ge \left|1+\frac{[2,p,q]^2(B_1-B_2)}{[3,p,q]B_1^2}\right|, \end{cases}$$

where $\mu \in \mathbb{C}$, and ψ_k, $k \in \{2,3\}$, is given by (1.5).

Taking $\mu = 0$ in Theorem 3.1, we obtain the following corollary which improves the corresponding result in Theorem 2.2:

Corollary 3.3. *If f, given by* (1.1)*, is in the class* $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$, *then*

$$|a_3| \le \begin{cases} \frac{B_1}{[1+\eta([3,p,q]-1)]\psi_3} & for \quad \frac{B_1-B_2}{B_1^2} \in \left(-\infty, \frac{-2[1+\eta([3,p,q]-1)]}{[1+\eta([2,p,q]-1)]^2}\right] \cup [0,\infty); \\ \frac{B_1^3}{\psi_3 \left| \begin{array}{c} [1+\eta([3,p,q]-1)]B_1^2 + \\ [1+\eta([2,p,q]-1)]^2 \\ (B_1-B_2) \end{array} \right|} & for \quad \frac{B_1-B_2}{B_1^2} \in \left(\frac{-[1+\eta([3,p,q]-1)]}{[1+\eta([2,p,q]-1)]^2}, 0\right] \cup \left[\frac{-2[1+\eta([3,p,q]-1)]}{[1+\eta([2,p,q]-1)]^2}, \frac{-[1+\eta([3,p,q]-1)]}{[1+\eta([2,p,q]-1)]^2}\right), \end{cases}$$

where ψ_k , $k \in \{2,3\}$, *are given by*(1.5).

Taking $\lambda = p = 1$ and $q \longrightarrow 1^-$ in Theorem 3.1, the class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$ reduces to the class $\Sigma^{\alpha,\beta,\gamma}[\varphi,\eta]$ and we get the following corollary:

Corollary 3.4. *If f, given by* (1.1)*, is in the class* $\Sigma^{\alpha,\beta,\gamma}[\varphi,\eta]$, *then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{B_1}{(1+2\eta)\psi_3} & for \quad \left|1-\mu\frac{\psi_3}{\psi_2^2}\right| \le \left|1+\frac{(1+\eta)^2(B_1-B_2)}{(1+2\eta)B_1^2}\right|; \\ \frac{B_1^3\left|1-\mu\frac{\psi_3}{\psi_2^2}\right|}{\psi_3\left|(1+2\eta)B_1^2+(1+\eta)^2(B_1-B_2)\right|} & for \quad \left|1-\mu\frac{\psi_3}{\psi_2^2}\right| \ge \left|1+\frac{(1+\eta)^2(B_1-B_2)}{(1+2\eta)B_1^2}\right|, \end{cases}$$

where $\mu \in \mathbb{C}$,*and* $\psi_k = \frac{\Gamma(\alpha+\beta-\gamma+2)\Gamma(\beta+k+1)}{\Gamma(\beta+1)\Gamma(\alpha+\beta+k-\gamma+2)}$, $k = 2,3$.

Remark 3.5. (i) Taking $\gamma = 1$ and $\alpha = 0$ in Corollary3.4, we obtain the result obtained by Omar et al. [[25], Theorem 2.7];

(ii) Taking $\gamma = \eta = 1$ and $\alpha = 0$ in Corollary3.4, we obtain the result obtained by Zaprawa [[31], Theorem 1].

4. Conclusion

In our results, by using the (p, q)-derivative operator, the generalized class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$ was introduced. We find estimates for the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in this class. Moreover, the Fekete-Szegő inequalities of the analytic function belonging to this introduced class was investigated. The results obtained in this study generalize some of the previously obtained results. We mention that all the above estimations for the first two Taylor-Maclaurin coefficients and Fekete-Szegő problem for the function class $\Sigma_{\lambda,p,q}^{\alpha,\beta,\gamma}[\varphi,\eta]$ are not sharp. To find the sharp upper bounds for the above function class, it still is an interesting open problem, as well as for $|a_n|$, $n \geq 4$.

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