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An iterative algorithm for minimization and fixed point problems of two families of pseudononspreading mappings in Hadamard spaces

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Abstract

Using the S-type iteration process, we introduce a modified proximal point algorithm for approximating a common solution of the minimization problem and fixed point problem in Hadamard spaces. In particular, we establish strong convergence of the proposed algorithm to a common solution of a finite family of the minimization problem and the fixed point problem of two finite families of generalized k-strictly pseudononspreading mappings. Numerical example in support of our main result is given to illustrate its applicability. Our work improves and extends some recent results existing in the current literature.

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1 Introduction

Let C be a nonempty subset of a metric space (X, d). A point $x \in C$ is called a fixed point of a mapping $T : C \to C$ if $Tx = x$. The set of all fixed points of T is denoted by $F(T)$.
The mapping $T : C \to C$ is called:

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(i) *contraction* if there exists $k \in (0,1)$ such that

$$d(Tx, Ty) \leq kd(x, y) \ \forall x, y \in C;$$

if $k = 1$, then T is called *nonexpansive*;

(ii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y) \ \forall x, y \in C, \ n \geq 1;$$

(iii) *quasinonexpansive* if $F(T) \neq \emptyset$ and

$$d(p, Tx) \leq d(p, x) \ \forall p \in F(T), \ x \in C;$$

(iv) *nonspreading* (see [24]) if

$$2d^2(Tx, Ty) \leq d^2(Tx, y) + d^2(Ty, x) \ \forall x, y \in C;$$

(v) *k-strictly pseudononspreading* (see [38]) if

$$\begin{aligned}(2-k)d^2(Tx, Ty) \leq\ & kd^2(x, y) + (1-k)d^2(y, Tx) + (1-k)d^2(x, Ty) \\ & + kd^2(x, Tx) + kd^2(y, Ty) \ \forall x, y \in C,\end{aligned} \tag{1.1}$$

which is equivalent to

$$\begin{aligned}||Tx - Ty||^2 \leq\ & ||x - y||^2 + k||x - Tx - (y - Ty)||^2 \\ & + 2\langle x - Tx, y - Ty\rangle \ \forall x, y \in C \subseteq H,\end{aligned} \tag{1.2}$$

where H is a real Hilbert space (see [38, 42]);

(vi) *generalized asymptotically nonspreading* (see [39]) if there exist two mappings $f, g : C \to [0, \gamma], \ \gamma < 1$ such that

$$d^2(T^n x, T^n y) \leq f(x)d^2(T^n x, y) + g(x)d^2(T^n y, x) \ \forall x, y \in C, \quad n \in \mathbb{N},$$

and

$$0 < f(x) + g(x) \leq 1 \ \forall x \in C;$$

if $n = 1$, then T is called *(f, g)-generalized (or simply generalized) nonspreading*;

(vii) *(f, g)-generalized (or simply generalized) k-strictly pseudononspreading* if there exist two mappings $f, g : C \to [0, \gamma], \ \gamma < 1$ and $k \in [0, 1)$ such that

$$\begin{aligned}(1-k)d^2(Tx, Ty) \leq\ & kd^2(x, y) + [f(x) - k]\, d^2(Tx, y) \\ & + [g(x) - k]\, d^2(x, Ty) + kd^2(x, Tx) \\ & + kd^2(y, Ty) \ \forall x, y \in C,\end{aligned} \tag{1.3}$$

and

$$0 < f(x) + g(x) \leq 1 \ \forall x \in C.$$

It is known that nonexpansive mappings and nonspreading mappings with nonempty fixed point set are quasinonexpansive. Every nonspreading mapping is k-strictly pseudononspreading with $k = 0$. We note that a generalized nonspreading mapping T with $f(x) = g(x) = \frac{1}{2}\ \forall x \in C$ reduces to a nonspreading mapping. Moreover, we make the following remarks about generalized k-strictly pseudononspreading mappings.

Remark 1.1. (i) Clearly, every generalized nonspreading mapping is generalized 0-strictly pseudononspreading.

(ii) Every k-strictly pseudononspreading mapping is a generalized k-strictly pseudononspreading mapping. Indeed, if T is a k-strictly pseudononspreading mapping, then by Definition (v), there exists $k \in [0, 1)$ such that

$$\begin{aligned}(2-k)d^2(Tx,Ty) &\leq kd^2(x,y) + (1-k)d^2(Tx,y)\\ &\quad + (1-k)d^2(x,Ty) + kd^2(x,Tx) + kd^2(y,Ty),\end{aligned} \tag{1.4}$$

which implies

$$\begin{aligned}\left(1-\frac{k}{2}\right)d^2(Tx,Ty) &\leq \frac{k}{2}d^2(x,y) + \left(\frac{1}{2}-\frac{k}{2}\right)d^2(y,Tx)\\ &\quad + \left(\frac{1}{2}-\frac{k}{2}\right)d^2(x,Ty) + \frac{k}{2}d^2(x,Tx) + \frac{k}{2}d^2(y,Ty).\end{aligned} \tag{1.5}$$

That is,

$$\begin{aligned}(1-k')\,d^2(Tx,Ty) &\leq k'd^2(x,y) + (f(x)-k')\,d^2(Tx,y)\\ &\quad + (g(x)-k')\,d^2(x,Ty) + k'd^2(x,Tx) + k'd(y,Ty),\end{aligned} \tag{1.6}$$

where $f(x) = g(x) = \frac{1}{2},\ \forall x \in C$ and $k' = \frac{k}{2} \in [0, 1)$. Hence, T is a generalized k-strictly pseudononspreading mapping.

However, converse of the statements given in Remark 1.1 are not always true as indicated by the following examples.

Example 1.2. Let $T : [0, \infty) \to [0, \infty)$ be defined by

$$Tx = \begin{cases} \frac{1}{x+\frac{1}{10}}, & \text{if } x \geq 1,\\ 0, & \text{if } x \in [0,1).\end{cases}$$

Then, T is a generalized k-strictly pseudononspreading mapping, but not k-strictly pseudononspreading mapping.

Example 1.3. Let $T : [0, \infty) \to \mathbb{R}$ defined by

$$Tx = \begin{cases} -3x, & \text{if } x \in [0,1],\\ \frac{1}{x}, & \text{if } x \in (1,\infty).\end{cases}$$

Then, T is a generalized k-strictly pseudononspreading mapping but it is neither a k-strictly pseudononspreading mapping nor a generalized nonspreading mapping.

The approximation of fixed points of nonlinear mappings is one of the most flourishing area of research in mathematics that has enjoyed a prosperous development in the last fifty years or so. Thus, it has attracted and continued to attract the interest of researchers due to its extensive applications in diverse mathematical problems such as inverse problems, signal processing, game theory, fuzzy theory and many others, see [2, 3, 30, 34, 35, 36] and the references therein. Moreover, many mathematical problems emanating from biology, economics, computer science, are among others which can be modelled as a fixed point problem. It is well known that the pivot of the metric fixed point theory is the Banach contraction mapping principle, which states that a contraction mapping T defined on a complete metric space X always has a unique fixed point, and for any starting point $x_1 \in X$, the sequence defined by the Picard iteration process $x_{n+1} = Tx_n,\ n \geq 1$, converges strongly to that fixed point. However, there are several examples in literature (see [10]) which show that for a nonexpansive mapping, its Picard iteration process may not converge to its fixed point, even when the fixed point exists. As a result of this, considerable efforts have been made to approximate fixed points of not only nonexpansive mappings, but more general mappings, by developing different iteration processes. For example, the Mann iteration process is defined in a Hilbert space H as follows: $x_1 \in C \subseteq H$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n,\ \forall n \geq 1, \tag{1.7}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. The Ishikawa iteration process is defined as follows: $x_1 \in C \subseteq H$ and

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n,\ \forall n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. For so many years, many researchers have studied the above iteration processes and their modifications to approximate fixed points of nonexpansive mappings and wider classes of mappings and related optimization problems in both Hilbert spaces and Banach spaces (see, for example [4, 15, 18, 21, 29, 33, 43, 48] and the references therein).

Recently, Agarwal *et al.* [1] introduced and studied the following S-iteration process: $x_1 \in C \subseteq H$ and

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n\ \forall n \geq 1, \end{cases} \tag{1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. It was observed in [1] that iteration process (1.9) is independent of (1.8) and (1.7), and has a better convergence rate than (1.8) and (1.7) for contractions. The study of fixed point problems for nonlinear mappings using the above iteration processes have recently been extended from the framework of Hilbert spaces and Banach spaces to Hadamard spaces (see, for example [17, 32, 31, 46] and the references therein).

On the other hand, approximating solutions of the minimization problems has been of great interest in optimization theory, nonlinear analysis and geometry. Let X be a CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower semi-continuous mapping. The minimization problem is to find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{1.10}$$

Recall that the mapping f is convex if

$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \; \forall x, y \in X, \; \lambda \in (0,1).$$

f is proper if $D(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$, where $D(f)$ denotes the domain of f. The mapping $f : D(f) \to (-\infty, \infty]$ is lower semi-continuous at a point $x \in D(f)$ if

$$f(x) \leq \liminf_{n \to \infty} f(x_n), \tag{1.11}$$

for each sequence $\{x_n\}$ in $D(f)$ such that $\lim\limits_{n \to \infty} x_n = x$; f is said to be lower semi-continuous on $D(f)$ if it is lower semi-continuous at any point in $D(f)$. For any $\lambda > 0$, the resolvent of f in X is defined in [5] as

$$J_\lambda^f(x) = \arg\min_{y \in X} \left[f(y) + \frac{1}{2\lambda} d^2(y, x) \right],$$

where $\arg\min_{y \in X} f$ stands for arguments of minima of f. It was established in [19] that J_λ^f is well defined and that it is a nonexpansive mapping for all $\lambda > 0$. For simplicity, we shall write J_λ for the resolvent of a proper, convex and lower semi-continuous mapping f. Furthermore, we denote the solution set of problem (1.10) by $\operatorname{argmin}_{y \in X} f(y)$. In [5], it was shown that $F(J_\lambda)$ coincides with $\operatorname{argmin}_{y \in X} f(y)$.

The Proximal Point Algorithm (PPA) is known to be one of the most popular and successful methods for solving (1.10). The PPA was introduced by Martinet [28] in 1970 and was further developed by Rockafellar [41] for the approximation of solution of (1.10) in the framework of Hilbert spaces. Later in 2013, Bačák [6] introduced and studied the PPA in CAT(0) spaces. To approximate solution of (1.10), Bačák [6] proposed the following algorithm: For arbitrary $x_1 \in X$, define the sequence $\{x_n\}$ by

$$x_{n+1} = \arg\min_{y \in X} \left(f(y) + \frac{1}{2\lambda_n} d^2(y, x_n) \right), \tag{1.12}$$

where $\lambda_n > 0$ for all $n \geq 1$. Under the conditions that f has a minimizer in X and $\sum_{n=1}^{\infty} \lambda_n = \infty$, he proved that $\{x_n\}$ Δ-converges to a minimizer of f. In 2014, Bačák [7] studied a split version of the PPA for minimizing sum of convex mappings in Hadamard spaces.

Researchers are now beginning to approximate common solution of the minimization problem and the fixed point problem for nonexpansive mappings in

Hadamard spaces. In 2015, Cholamjiak, Abdou and Cho [11] proposed the following modified PPA using the S-type iteration process for two nonexpansive mappings in a Hadamard space: For arbitrary $x_1 \in X$, define the sequence $\{x_n\}$ by

$$\begin{cases} z_n = \arg\min_{y\in X}\left[f(y) + \frac{1}{2\lambda_n}d^2(y, x_n)\right], \\ y_n = (1-\beta_n)x_n \oplus \beta_n T_1 z_n, \\ x_{n+1} = (1-\alpha_n)T_1 x_n \oplus \alpha_n T_2 y_n, \ \forall n \geq 1, \end{cases} \tag{1.13}$$

where $f : X \to (-\infty, \infty]$ is a proper, convex and lower semi-continuous mapping, T_1, T_2 are nonexpansive mappings on X, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying some conditions, and $\{\lambda_n\}$ is a sequence such that $\lambda_n \geq \lambda > 0$ for all $n \geq 1$. They obtained strong convergence results of the iteration process (1.13) to a common solution of the minimization problem and the fixed point problem for two nonexpansive mappings under some compactness conditions. Later in 2016, Chang *et al.* [9] proposed the modified PPA, using the S-type iteration process for four asymptotically nonexpansive mappings in Hadamard spaces, and obtained strong convergence results of their proposed algorithm to a common solution of the minimization problem and the fixed point problem for four asymptotically nonexpansive mappings under some compactness conditions. Very recently, Ugwunnadi *et. al.* [47] studied a hybrid PPA for approximating a common solution of the minimization problem and the fixed point problem for a demicontractive mapping in a Hadamard space, and obtained a strong convergence result.

Motivated by the recent interest on PPA and these ongoing research, it is natural to consider the following question.

Question: Can we propose a modified S-type PPA for two finite families of generalized k-strictly pseudononspreading mappings in Hadamard spaces, and establish its strong convergence without the compactness assumption on the mappings involved?

In this paper, we consider the above question by proposing and studying a modified S-type PPA, and establish its strong convergence for our proposed iteration, to a common solution of a finite family of the minimization problems and the fixed point problems of two finite families of generalized k-strictly pseudononspreading mappings in Hadamard spaces. Numerical example for our main result is also given to illustrate its applicability. Our work improves and extends the results of Bačák [6], Bačák [7], Cholamjiak, Abdou and Cho [11], Chang *et al.* [9], and many other results existing in the literature.

2 Preliminaries

In this section, we recall some definitions and useful results that will be needed in proving our main results.

Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is an isometry $c : [0, d(x,y)] \to X$ such that $c(0) = x$, $c(d(x,y)) = y$. The image of a geodesic path joining x to y is called a geodesic segment between x and y. When it is unique, this geodesic segment is denoted by $[x, y]$. The metric space (X, d) is said to be a geodesic space if every two points of X can be joined by a geodesic segment and it is said to be a uniquely geodesic space if every two points of X are joined by only one geodesic segment. A subset C of a geodesic space X is said to be convex, if for all $x, y \in C$, the segment $[x, y]$ is in C. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 in X (known as the vertices of Δ) and a geodesic segment between each pair of vertices (known as the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in X is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$ for all $i, j \in \{1, 2, 3\}$. A metric space (X, d) is called a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane $\mathbb{R}^2$. Let $t \in [0, 1]$, we write $(1-t)x \oplus ty$ for the unique point z in the geodesic segment joining x to y for each x, y in a CAT(0) space X such that $d(z, x) = td(x, y)$ and $d(z, y) = (1-t)d(x, y)$.

Let X be a CAT(0) space. Denote the pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then, a mapping $\langle ., . \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)\right) \ \forall a, b, c, d \in X$$

is called a quasilinearization mapping (see [8]). It is easy to verify that $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a, b, c, d, e \in X$. A geodesic space X is said to satisfy the Cauchy-Swartz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d) \ \forall a, b, c, d \in X$. It has been established in [8] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality. It is known that CAT(0) spaces are uniquely geodesic spaces (see [31, 45, 46]) and complete CAT(0) spaces are called Hadamard spaces. Examples of CAT(0) spaces includes: Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of non-positive sectional curvature, $\mathbb{R}$-trees, Hilbert ball ([16], [20]), hyperbolic spaces [40]. For more properties of CAT(0) spaces, see [12, 32, 31, 45] and the references therein.

Let $\{x_n\}$ be a bounded sequence in X and $r(., \{x_n\}) : X \to [0, \infty)$ be a continuous mapping defined by $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is given by $r(\{x_n\}) := \inf\{r(x, \{x_n\}) : x \in X\}$ while the asymptotic center of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. It is known that in a Hadamard space X, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ (see [13, 23]).

Definition 2.1. Let C be a nonempty closed and convex subset of a Hadamard space X. A mapping $T : C \to C$ is said to be Δ-demiclosed, if for any bounded sequence $\{x_n\}$ in X such that $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then

$x = Tx$.

Definition 2.2. Let C be a nonempty closed and convex subset of a CAT(0) space X. The metric projection is a mapping $P_C : X \to C$ which assigns to each $x \in X$ the unique point $P_C x$ in C such that

$$d(x, P_C x) = \inf\{d(x, y) : y \in C\}.$$

Recall that a mapping T is *firmly nonexpansive* (see [22]) if we have that

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \ \forall x, y \in X.$$

It follows from the Cauchy-Schwartz inequality that firmly nonexpansive mappings are nonexpansive. An example of a firmly nonexpansive mapping is the metric projection (see [22, Corollary 3.8]).
We will need the following known lemmas.

Lemma 2.3. *([31, 45, 46]). Let X be a CAT(0) space. Then, for $x, y, z \in X$ and $t \in [0, 1]$, the following hold:*

(i) $d(z, tx \oplus (1-t)y) \leq td(z, x) + (1-t)d(z, y)$,

(ii) $d^2(z, tx \oplus (1-t)y) \leq td^2(z, x) + (1-t)d^2(z, y) - t(1-t)d^2(x, y)$,

(iii) $d^2(z, tx \oplus (1-t)y) \leq t^2 d^2(z, x) + (1-t)^2 d^2(z, y) + 2t(1-t)\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle$.

Lemma 2.4. *([44]). Let C be a nonempty, closed and convex subset of a CAT(0) space X. Let $\{x_i, i = 1, 2, \ldots, N\} \subset C$, and $\alpha_i \in (0, 1)$, $i = 1, 2, \ldots, N$ such that $\sum_{i=1}^N \alpha_i = 1$. Then the following inequalities hold:*

$$d\left(z, \bigoplus_{i=1}^N \alpha_i x_i\right) \leq \sum_{i=1}^N \alpha_i d(z, x_i), \ \forall z \in C,$$

$$d^2\left(z, \bigoplus_{i=1}^N \alpha_i x_i\right) \leq \sum_{i=1}^N \alpha_i d^2(z, x_i) - \sum_{i,j=1, i \neq j}^N \alpha_i \alpha_j d^2(x_i, x_j), \ \forall z \in C.$$

Lemma 2.5. *([26]). Every bounded sequence in a Hadamard space has a Δ-convergent subsequence.*

Lemma 2.6. *[31, 45, 46]. Let X be a Hadamard space, $\{x_n\}$ be a bounded sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if* $\limsup_{n \to \infty} \langle \overrightarrow{x_n x}, \overrightarrow{yx} \rangle \leq 0 \ \forall y \in X$.

Lemma 2.7. *[14]. Let X be a Hadamard space and $T : X \to X$ be a nonexpansive mapping. Then T is Δ-demiclosed.*

Lemma 2.8. *[25]. Let X be a Hadamard space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous mapping. Then, for all $x, y \in X$ and $\lambda > 0$, we have*

$$\frac{1}{2\lambda} d^2(J_\lambda x, y) - \frac{1}{2\lambda} d^2(x, y) + \frac{1}{2\lambda} d^2(x, J_\lambda x) + f(J_\lambda x) \leq f(y). \tag{2.1}$$

Lemma 2.9. *[49]. Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:
(i) $\{\alpha_n\} \subset [0,1]$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$,
(iii) $\gamma_n \geq 0 (n \geq 0)$, $\Sigma_{n=0}^{\infty}\gamma_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. *[27]. Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ with $a_{n_j} < a_{n_j+1}$ $\forall j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.

3 Main Results

Lemma 3.1. *Let X be a Hadamard space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous mapping. Then,*
$d(J_\lambda^f x, x) \leq d(J_\mu^f x, x)$ for $0 < \lambda < \mu$ and $x \in X$.

Proof. For $x, y \in X$, we obtain from the definition of the resolvent of f that

$$f(J_\mu x) + \frac{1}{2\mu}d^2(J_\mu x, x) \leq f(y) + \frac{1}{2\mu}d^2(y, x).$$

In particular, we have that

$$f(J_\mu x) + \frac{1}{2\mu}d^2(J_\mu x, x) \leq f(J_\lambda x) + \frac{1}{2\mu}d^2(J_\lambda x, x). \tag{3.1}$$

Similarly, we obtain

$$f(J_\lambda x) + \frac{1}{2\lambda}d^2(J_\lambda x, x) \leq f(J_\mu x) + \frac{1}{2\lambda}d^2(J_\mu x, x) \tag{3.2}$$

Adding (3.1) and (3.2), we obtain that

$$d^2(J_\lambda x, x) - \frac{\lambda}{\mu}d^2(J_\lambda x, x) \leq d^2(J_\mu x, x) - \frac{\lambda}{\mu}d^2(J_\mu x, x).$$

That is,

$$\left(1 - \frac{\lambda}{\mu}\right) d^2(J_\lambda x, x) \leq \left(1 - \frac{\lambda}{\mu}\right) d^2(J_\mu x, x).$$

As $0 < \lambda < \mu$, so we obtain that

$$d(J_\lambda x, x) \leq d(J_\mu x, x).$$

□

Lemma 3.2. *Let C be a closed and convex subset of a Hadamard space X and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \dots, N$ be a finite family of proper convex and lower semi-continuous mappings such that $\cap_{i=1}^N \arg\min_{y\in X} f_i(y) \neq \emptyset$. Let $\{u_n\}$ and $\{z_n\}$ be bounded sequences such that*

$$u_n = P_C(J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \dots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)),$$

where $\{\lambda_n^{(i)}\}$, $i = 1, 2, \dots, N$ is a sequence such that $\lambda_n^{(i)} > \lambda^{(i)} > 0$ for each $i = 1, 2, \dots, N$ and $n \geq 1$. If $\lim\limits_{n\to\infty} d(u_n, z_n) = 0$, then $\lim\limits_{n\to\infty} d(J_{\lambda^{(i)}} z_n, z_n) = 0$, for each $i = 1, 2, \dots, N$.

Proof. Let $p \in \cap_{i=1}^N \arg\min_{y\in X} f_i(y)$.

$$\text{Set } w_n^{(i+1)} = J_{\lambda_n^{(i)}} w_n^{(i)}, \text{ for each } i = 1, 2, \dots, N,$$

where $w_n^{(1)} = z_n$, for all $n \geq 1$. Then,
$w_n^{(2)} = J_{\lambda_n^{(1)}}(z_n)$, $w_n^{(3)} = J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)$, $\dots$, $w_n^{(N+1)} = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \dots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)$.
By Lemma 2.8, we obtain
$\frac{1}{2\lambda_n^{(i)}} d^2(p, w_n^{(i+1)}) - \frac{1}{2\lambda_n^{(i)}} d^2(p, w_n^{(i)}) + \frac{1}{2\lambda_n^{(i)}} d^2(w_n^{(i)}, w_n^{(i+1)}) + f(w_n^{(i+1)}) \leq f(p)$.
As $f(p) \leq f(w_n^{(i+1)})$, we have that

$$d^2(w_n^{(i)}, w_n^{(i+1)}) \leq d^2(p, w_n^{(i)}) - d^2(p, w_n^{(i+1)}).$$

Taking sum in the above inequality from $i = 1$ to $i = N$, we obtain

$$\begin{aligned}
\sum_{i=1}^N d^2(w_n^{(i)}, w_n^{(i+1)}) &\leq d^2(p, z_n) - d^2(p, w_n^{(N+1)}) \\
&\leq d^2(p, z_n) - d^2(p, u_n) \\
&\leq [d(p, u_n) + d(u_n, z_n)]^2 - d^2(p, u_n) \\
&\leq d^2(z_n, u_n) + 2d(z_n, u_n)d(p, u_n) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

which implies

$$\lim_{n\to\infty} d(w_n^{(i)}, w_n^{(i+1)}) = 0, \ i = 1, 2, \dots, N. \tag{3.3}$$

By (3.3) and the triangle inequality, we obtain for each $i = 1, 2, \dots, N$, that

$$\lim_{n\to\infty} d(z_n, w_n^{(i+1)}) = \lim_{n\to\infty} d(w_n^{(1)}, w_n^{(i+1)}) = 0. \tag{3.4}$$

Also, since $\lambda_n^{(i)} > \lambda^{(i)} > 0$ for all $n \geq 1$, we obtain by Lemma 3.1 and (3.3) that

$$\begin{aligned}
d\left(w_n^{(i)}, J_{\lambda^{(i)}} w_n^{(i)}\right) &\leq d\left(w_n^{(i)}, J_{\lambda_n^{(i)}} w_n^{(i)}\right) \\
&\to 0, \text{ as } n \to \infty, \ i = 1, 2, \dots, N. \qquad (3.5)
\end{aligned}$$

Since $J_{\lambda^{(i)}}$ is nonexpansive, we have from (3.3) and (3.4) that

$$\begin{aligned} d(J_{\lambda^{(i)}}z_n, J_{\lambda^{(i)}}w_n^{(i)}) &\leq d(J_{\lambda^{(i)}}z_n, J_{\lambda^{(i)}}w_n^{(i+1)}) + d(J_{\lambda^{(i)}}w_n^{(i+1)}, J_{\lambda^{(i)}}w_n^{(i)}) \\ &\leq d(z_n, w_n^{(i+1)}) + d(w_n^{(i+1)}, w_n^{(i)}) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.6}$$

By (3.3)-(3.5), we obtain

$$\begin{aligned} d\left(J_{\lambda^{(i)}}z_n, z_n\right) \leq d &\left(J_{\lambda^{(i)}}z_n, J_{\lambda^{(i)}}w_n^{(i)}\right) + d\left(J_{\lambda^{(i)}}w_n^{(i)}, w_n^{(i)}\right) \\ &+ d\left(w_n^{(i)}, w_n^{(i+1)}\right) + d\left(w_n^{(i+1)}, z_n\right) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.7}$$

That is,

$$\lim_{n\to\infty} d\left(J_{\lambda^{(i)}}z_n, z_n\right) = 0, \quad i = 1, 2, \ldots, N.$$

□

Lemma 3.3. *Let C be a closed and convex subset of a Hadamard space X and $T : C \to C$ be (f, g)-generalized k-strictly pseudononspreading mapping with $k \in [0, 1)$ such that $F(T) \neq \emptyset$, where $f, g : C \to [0, \gamma]$, $\gamma < 1$ and $0 < f(x) + g(x) \leq 1$ for all $x \in C$. Let $T_\beta : C \to C$ be defined by $T_\beta x = \beta x \oplus (1 - \beta)Tx$ $\forall x \in C$, where $\frac{k}{f(p)} \leq \beta < 1$ with $f(p) \neq 0$ for each $p \in F(T)$. Then,*

(a) $F(T_\beta) = F(T)$,

(b) T_β *is quasinonexpansive.*

Proof. (a) If $\beta = 0$, then $T_\beta = T$. Thus, $F(T) = F(T_\beta)$. Now, let $\beta \neq 0$.
For each $p \in F(T_\beta)$, we have that $p = T_\beta p$ and by Lemma 2.3 (i), we have $d(p, Tp) \leq \beta d(p, Tp)$, which implies $(1 - \beta)d(p, Tp) \leq 0$.
Since $\beta < 1$, it follows that $p \in F(T)$. Thus, $F(T_\beta) \subseteq F(T)$.
We now show that $F(T) \subseteq F(T_\beta)$. Let $p \in F(T)$, then $Tp = p$ and by Lemma 2.3 (i) we have
$d(p, T_\beta p) = d(p, \beta p \oplus (1 - \beta)p) \leq 0$, which implies that $p \in F(T_\beta)$. Thus, $F(T) \subseteq F(T_\beta)$. Therefore, $F(T_\beta) = F(T)$.
(b) First, observe that if T is (f, g)-generalized k-strictly pseudononspreading mapping, then for each $p \in F(T)$ and $x \in C$, we obtain

$$d^2(p, Tx) \leq f(p)d^2(p, x) + g(p)d^2(p, Tx) + kd^2(x, Tx),$$

which implies

$$(1 - g(p))d^2(p, Tx) \leq f(p)d^2(p, x) + kd^2(x, Tx).$$

Since $f(p) + g(p) \leq 1$, we obtain

$$d^2(p, Tx) \leq d^2(p, x) + \frac{k}{f(p)}d^2(x, Tx). \tag{3.8}$$

By Lemma 2.3 (ii) and (3.8), we have for each $x \in C$ and $p \in F(T) = F(T_\beta)$ that

$$\begin{aligned} d^2(T_\beta p, T_\beta x) &= d^2(p, \beta x \oplus (1-\beta)Tx) \\ &\leq \beta d^2(p,x) + (1-\beta)d^2(p,Tx) - \beta(1-\beta)d^2(x,Tx) \\ &\leq \beta d^2(p,x) + (1-\beta)\left[d^2(p,x) + \frac{k}{f(p)}d^2(x,Tx)\right] \\ &\quad -\beta(1-\beta)d^2(x,Tx) \\ &= d^2(p,x) + (1-\beta)\left(\frac{k}{f(p)} - \beta\right)d^2(x,Tx) \\ &\leq d^2(p,x). \end{aligned}$$

Therefore, T_β is quasinonexpansive. □

Theorem 3.4. *Let C be a closed and convex subset of a Hadamard space X and $h_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semi-continuous mappings. For each $j = 1, 2, \ldots, m$, let $T_j : C \to C$ be a finite family of (f_j, g_j)-generalized k_j-strictly pseudononspreading mappings with $k_j \in [0, 1)$, where $f_j, g_j : C \to [0, \gamma]$, $\gamma < 1$ and $0 < f_j(x) + g_j(x) \leq 1$ for all $x \in C$, and $S_j : C \to X$ be a finite family of (f'_j, g'_j)-generalized k'_j-strictly pseudononspreading mappings with $k'_j \in [0, 1)$, where $f'_j, g'_j : C \to [0, \gamma']$, $\gamma' < 1$ and $0 < f'_j(x) + g'_j(x) \leq 1$ for all $x \in C$. Suppose that $\Gamma := \left(\cap_{j=1}^m F(T_j)\right) \cap \left(\cap_{j=1}^m F(S_j)\right) \cap \left(\cap_{i=1}^N \arg\min_{y \in X} h_i(y)\right) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$(3.9) \qquad \begin{cases} z_n = (1-t_n)x_n \oplus t_n u, \\ u_n = P_C(J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)), \\ y_n = \beta_n^{(0)} u_n \oplus \beta_n^{(1)} T_{(\beta,1)} u_n \oplus \beta_n^{(2)} T_{(\beta,2)} u_n \oplus \ldots \\ \oplus \beta_n^{(m)} T_{(\beta,m)} u_n, \\ x_{n+1} = \alpha_n^{(0)} T_{(\beta,m)} u_n \oplus \alpha_n^{(1)} S_{(\alpha,1)} u_n \oplus \alpha_n^{(2)} S_{(\alpha,2)} u_n \\ \oplus \cdots \oplus \alpha_n^{(m)} S_{(\alpha,m)} y_n, \ n \geq 1, \end{cases}$$

where $T_{(\beta,j)}x = \beta x \oplus (1-\beta)T_j x$ and $S_{(\alpha,j)}x = \alpha x \oplus (1-\alpha)S_j x$, $j = 1, 2, \ldots, m$, for all $x \in C$ such that $T_{(\beta,j)}$ and $S_{(\beta,j)}$ are Δ-demiclosed with $\frac{k_j}{f_j(p)} \leq \beta < 1$, $f_j(p) \neq 0$ and $\frac{k'_j}{f'_j(p)} \leq \alpha < 1$, $f'_j(p) \neq 0$ respectively, for each $j = 1, 2, \ldots, m$ and for each $p \in \left(\cap_{j=1}^m F(T_j)\right) \cap \left(\cap_{j=1}^m F(S_j)\right)$, $\{t_n\}, \{\lambda_n^{(i)}\}, \{\beta_n^{(j)}\}$ and $\{\alpha_n^{(j)}\}$ are sequences in $(0, 1)$ satisfying the following conditions:

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^\infty t_n = \infty$,

C3: $0 < a \leq \alpha_n^{(j)}, \ \beta_n^{(j)} \leq b < 1, \ j = 0, 1, 2, \ldots, m$ *such that* $\sum_{j=0}^m \alpha_n^{(j)} = 1$ *and* $\sum_{j=0}^m \beta_n^{(j)} = 1$ *for all* $n \geq 1$, *C4: $\{\lambda_n^{(i)}\}$ is a sequence such that $\lambda_n^{(i)} > \lambda^{(i)}$ for all $n \geq 1$, $i = 1, 2, \ldots, N$ and some $\lambda^{(i)} > 0$.*

Then, $\{x_n\}$ converges strongly to an element of Γ.

Proof. Let $p \in \Gamma$. Then for each $j = 1, 2, \ldots, m$, we have by Lemma 3.3 that $p = T_{(\beta,j)}p = S_{(\alpha,j)}p$, and $T_{(\beta,j)}$ and $S_{(\alpha,j)}$ are quasi-nonexpansive mappings. Set $\Phi_{\lambda_n}^N = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}$, where $\Phi_{\lambda_n}^0 = I$. Now by (3.9) and Lemma 2.4, we have

$$\begin{aligned}
d(p, x_{n+1}) &\leq \alpha_n^{(0)} d(p, T_{(\beta,m)}u_n) + \alpha_n^{(1)} d(p, S_{(\alpha,1)}u_n) + \alpha_n^{(2)} d(p, S_{(\alpha,2)}u_n) \\
&\quad + \cdots + \alpha_n^{(m)} d(p, S_{(\alpha,m)}y_n) \\
&\leq \alpha_n^{(0)} d(p, u_n) + \alpha_n^{(1)} d(p, u_n) + \alpha_n^{(2)} d(p, u_n) + \cdots + \alpha_n^{(m)} d(p, y_n) \\
&\leq \sum_{j=0}^{m-1} \alpha_n^{(j)} d(p, u_n) + \alpha_n^{(m)} [\beta_n^{(0)} d(p, u_n) + \beta_n^{(1)} d(p, T_{(\beta,1)}u_n) \\
&\quad + \beta_n^{(2)} d(p, T_{(\beta,2)}u_n) + \cdots + \beta_n^{(m)} d(p, T_{(\beta,m)}u_n)] \\
&\leq \sum_{j=0}^{m-1} \alpha_n^{(j)} d(p, u_n) + \alpha_n^{(m)} d(p, u_n) \\
&= d(p, u_n) \qquad (3.10) \\
&\leq d(p, \Phi_{\lambda_n}^N z_n) \\
&\leq d(p, \Phi_{\lambda_n}^{N-1} z_n) \\
&\;\;\vdots \\
&\leq d(p, z_n) \qquad (3.11) \\
&\leq (1 - t_n) d(p, x_n) + t_n d(u, p) \\
&\leq \max\{d(p, x_n), d(p, u)\} \\
&\;\;\vdots \\
&\leq \max\{d(p, x_1), d(p, u)\}.
\end{aligned}$$

Therefore, $\{d(p, x_n)\}$ is bounded. Hence, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{y_n\}$ are all bounded.
From (3.9), Lemma 2.3 (i) and condition C1, we obtain that

$$d(z_n, x_n) \leq t_n d(u, x_n) \to 0, \text{ as } n \to \infty. \tag{3.12}$$

We need to consider two cases for our proof.
Case 1: Suppose that $\{d(p, x_n)\}$ is monotonically non-increasing. Then $\lim\limits_{n\to\infty} d(p, x_n)$ exists. Without loss of generality, we may assume that

$$\lim_{n\to\infty} d(p, x_n) = c \geq 0. \tag{3.13}$$

Since P_C is firmly nonexpansive, therefore, we have

$$d^2(p, u_n) \leq \langle \overrightarrow{u_n p}, \overrightarrow{\Phi_{\lambda_n}^N z_n\ p} \rangle = \frac{1}{2}\left(d^2(p, u_n) + d^2(p, \Phi_{\lambda_n}^N z_n) - d^2(u_n, \Phi_{\lambda_n}^N z_n)\right),$$

which together with (3.10), (3.10), (3.12) and (3.13), implies that

$$\begin{aligned} d^2(u_n, \Phi^N_{\lambda_n} z_n) &\leq d^2(p, \Phi^N_{\lambda_n} z_n) - d^2(p, u_n) \\ &\leq d^2(p, \Phi^N_{\lambda_n} z_n) - d^2(p, x_{n+1}) \\ &\vdots \\ &\leq d^2(p, z_n) - d^2(p, x_{n+1}) \\ &\leq d^2(p, x_n) + 2d(p, x_n)d(x_n, z_n) + d^2(x_n, z_n) \\ &\quad -d^2(p, x_{n+1}) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.14}$$

We now show that $\lim_{n\to\infty} d(u_n, J_{\lambda^{(i)}} u_n) = 0,\ i = 1, 2, \ldots, N.$
Indeed, it follows from Lemma 2.8 that

$$\frac{1}{2\lambda_n} d^2(p, \Phi^N_{\lambda_n} z_n) - \frac{1}{2\lambda_n} d^2(p, \Phi^{N-1}_{\lambda_n} z_n) + \frac{1}{2\lambda_n} d^2(\Phi^N_{\lambda_n} z_n, \Phi^{N-1}_{\lambda_n} z_n) + f\left(\Phi^N_{\lambda_n}\right) \leq f(p).$$

Since $f(p) \leq f\left(\Phi^N_{\lambda_n}\right)$, we have by (3.10), (3.13) and (3.12) that

$$\begin{aligned} d^2(\Phi^N_{\lambda_n} z_n, \Phi^{N-1}_{\lambda_n} z_n) &\leq d^2(p, \Phi^{N-1}_{\lambda_n} z_n) - d^2(p, \Phi^N_{\lambda_n} z_n) \\ &\leq d^2(p, \Phi^{N-1}_{\lambda_n} z_n) - d^2(p, x_{n+1}) \\ &\vdots \\ &\leq d^2(p, z_n) - d^2(p, x_{n+1}) \\ &\leq d^2(z_n, x_n) + 2d(z_n, x_n)d(p, x_n) \\ &\quad + \left[d^2(p, x_n) - d^2(p, x_{n+1})\right] \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.15}$$

Similarly, we obtain by Lemma 2.8, (3.10), (3.13) and (3.12) that

$$\begin{aligned} d^2(\Phi^{N-1}_{\lambda_n} z_n, \Phi^{N-2}_{\lambda_n} z_n) &\leq d^2(p, \Phi^{N-2}_{\lambda_n} z_n) - d^2(p, \Phi^{N-1}_{\lambda_n} z_n) \\ &\leq d^2(p, \Phi^{N-2}_{\lambda_n} z_n) - d^2(p, \Phi^N_{\lambda_n} z_n) \\ &\leq d^2(p, \Phi^{N-2}_{\lambda_n} z_n) - d^2(p, x_{n+1}) \\ &\vdots \\ &\leq d^2(p, z_n) - d^2(p, x_{n+1}) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.16}$$

Continuing in this manner, we can show that

$$\begin{aligned} \lim_{n\to\infty} d^2(\Phi^{N-2}_{\lambda_n} z_n, \Phi^{N-3}_{\lambda_n} z_n) &= \cdots = \lim_{n\to\infty} d^2(\Phi^2_{\lambda_n} z_n, \Phi^1_{\lambda_n} z_n) \\ &= \lim_{n\to\infty} d^2(\Phi^1_{\lambda_n} z_n, z_n) = 0. \end{aligned} \tag{3.17}$$

Thus,

$$\begin{aligned} d(u_n, z_n) \leq\ & d(u_n, \Phi^N_{\lambda_n} z_n) + d(\Phi^N_{\lambda_n} z_n, \Phi^{N-1}_{\lambda_n} z_n) \\ & + d(\Phi^{N-1}_{\lambda_n} z_n, \Phi^{N-2}_{\lambda_n} z_n) + \cdots + d(\Phi^1_{\lambda_n} z_n, z_n), \end{aligned} \tag{3.18}$$

which implies by (3.14), (3.15), (3.16) and (3.17) that

$$\lim_{n\to\infty} d(u_n, z_n) = 0. \tag{3.19}$$

It follows from (3.19) and Lemma 3.2 that

$$\begin{aligned} & d(J_{\lambda^{(i)}} u_n, u_n) \\ \leq\ & d(J_{\lambda^{(i)}} u_n, J_{\lambda^{(i)}} z_n) + d(J_{\lambda^{(i)}} z_n, z_n) + d(z_n, u_n) \\ \leq\ & 2d(u_n, z_n) + d(J_{\lambda^{(i)}} z_n, z_n) \to 0, \text{ as } n \to \infty,\ i = 1, 2, \ldots, N. \end{aligned}$$

That is,

$$\lim_{n\to\infty} d(u_n, J_{\lambda^{(i)}} u_n) = 0, \text{ for each } i = 1, 2, \ldots, N. \tag{3.20}$$

Next, we show that $\lim_{n\to\infty} d(u_n, x_n) = 0$ and $\lim_{n\to\infty} d(p, y_n) = c$. By (3.12) and (3.19), we obtain

$$\lim_{n\to\infty} d(u_n, x_n) = 0. \tag{3.21}$$

Again, by (3.9), we have

$$\begin{aligned} d(p, x_{n+1}) \leq\ & \alpha_n^{(0)} d(p, T_{(\beta,m)} u_n) + \alpha_n^{(1)} d(p, S_{(\alpha,1)} u_n) + \alpha_n^{(2)} d(p, S_{(\alpha,2)} u_n) \\ & + \cdots + \alpha_n^{(m)} d(p, S_{(\alpha,m)} y_n) \\ \leq\ & \alpha_n^{(0)} d(p, u_n) + \alpha_n^{(1)} d(p, u_n) + \alpha_n^{(2)} d(p, u_n) + \cdots + \alpha_n^{(m)} d(p, y_n) \\ =\ & (1 - \alpha_n^{(m)}) d(p, u_n) + \alpha_n^{(m)} d(p, y_n) \\ \vdots\ & \\ \leq\ & (1 - \alpha_n^{(m)}) d(p, z_n) + \alpha_n^{(m)} d(p, y_n) \\ \leq\ & (1 - \alpha_n^{(m)}) \left[(1 - t_n) d(p, x_n) + t_n d(p, u)\right] + \alpha_n^{(m)} d(p, y_n) \\ \leq\ & (1 - \alpha_n^{(m)}) d(p, x_n) + t_n (1 - \alpha_n^{(m)}) d(p, u) + \alpha_n^{(m)} d(p, y_n), \end{aligned}$$

which implies

$$d(p, x_n) \leq \frac{1}{\alpha_n^{(m)}} \Big[d(p, x_n) - d(p, x_{n+1}) + (1 - \alpha_n^{(m)}) t_n d(u, p)\Big] + d(p, y_n).$$

It then follows from (3.13) and conditions C1 and C3 that

$$c = \liminf_{n\to\infty} d(p, x_n) \leq \liminf_{n\to\infty} d(p, y_n). \tag{3.22}$$

Also, by (3.9), we have

$$\begin{aligned} d(p, y_n) \leq\ & \beta_n^{(0)} d(p, u_n) + \beta_n^{(1)} d(p, T_{(\beta,1)} u_n) + \beta_n^{(2)} d(p, T_{(\beta,2)} u_n) \\ & + \cdots + \beta_n^{(m)} d(p, T_{(\beta,m)} u_n) \\ \leq\ & d(p, u_n) \qquad (3.23) \\ \leq\ & d(p, z_n) \\ \leq\ & d(p, x_n) + t_n \left[d(p, u) - d(p, x_n)\right], \end{aligned}$$

which implies that

$$\limsup_{n\to\infty} d(p,y_n) \le \limsup_{n\to\infty} \left(d(p,x_n) + t_n \left[d(p,u) - d(p,x_n)\right]\right) = c. \tag{3.24}$$

Thus, by (3.22) and (3.24), we have

$$\lim_{n\to\infty} d(p,y_n) = c. \tag{3.25}$$

We now show that $\lim_{n\to\infty} d(u_n, T_{(\beta,j)}u_n) = 0$, for each $j = 1,2,\ldots,m$ and $\lim_{n\to\infty} d(u_n,y_n) = 0$.
Indeed, by (3.9), Lemma 2.4 and Lemma 3.3, we have

$$\begin{aligned} d^2(p,y_n) &\le \beta_n^{(0)} d^2(p,u_n) + \sum_{j=1}^m \beta_n^{(j)} d^2(p, T_{(\beta,j)}u_n) - \sum_{j=1}^m \beta_n^{(0)}\beta_n^{(j)} d^2(u_n, T_{(\beta,j)}u_n) \\ &\quad - \sum_{j,r=1, j\neq r}^m \beta_n^{(j)}\beta_n^{(r)} d^2(T_{(\beta,j)}u_n, T_{(\beta,r)}u_n) \\ &\le d^2(p,u_n) - \sum_{j=1}^m \beta_n^{(0)}\beta_n^{(j)} d^2(u_n, T_{(\beta,j)}u_n) \\ &\quad - \sum_{j,r=1, j\neq r}^m \beta_n^{(j)}\beta_n^{(r)} d^2(T_{(\beta,j)}u_n, T_{(\beta,r)}u_n), \end{aligned} \tag{3.26}$$

which implies

$$\begin{aligned} \sum_{j=1}^m \beta_n^{(0)}\beta_n^{(j)} d^2(u_n, T_{(\beta,j)}u_n) &\le d^2(p,u_n) - d^2(p,y_n) \\ &\le d^2(u_n,x_n) + 2d(u_n,x_n)d(p,x_n) \\ &\quad + d^2(p,x_n) - d^2(p,y_n). \end{aligned} \tag{3.27}$$

By (3.13), (3.21), (3.25) and condition C3, we obtain that

$$\lim_{n\to\infty} d(u_n, T_{(\beta,j)}u_n) = 0, \ j = 1,2,\ldots,m. \tag{3.28}$$

Thus, by (3.9), (3.28) and Lemma 2.4, we have

$$\begin{aligned} d(u_n,y_n) &\le \beta_n^{(0)} d(u_n,u_n) + \beta_n^{(1)} d(u_n, T_{(\beta,1)}u_n) + \beta_n^{(2)} d(u_n, T_{(\beta,2)}u_n) \\ &\quad + \cdots + \beta_n^{(m)} d(u_n, T_{(\beta,m)}u_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.29}$$

Next, we show that $\lim_{n\to\infty} d(u_n, S_{(\alpha,j)}u_n) = 0$, for each $j = 1,2,\ldots,m-1$, and $\lim_{n\to\infty} d(y_n, S_{(\alpha,m)}y_n) = 0$.

By (3.9), (3.23), Lemma 2.4 and Lemma 3.3, we obtain

$$
\begin{aligned}
d^2(p, x_{n+1}) \leq\ & \alpha_n^{(0)} d^2(p, T_{(\beta,m)} u_n) + \sum_{j=1}^{m-1} \alpha_n^{(j)} d^2(p, S_{(\alpha,j)} u_n) \\
& + \alpha_n^{(m)} d^2(p, S_{(\alpha,m)} y_n) \\
& - \sum_{j=1}^{m-1} \alpha_n^{(0)} \alpha_n^{(j)} d^2(T_{(\beta,m)} u_n, S_{(\alpha,j)} u_n) \\
& - \alpha_n^{(0)} \alpha_n^{(m)} d^2(T_{(\beta,m)} u_n, S_{(\alpha,m)} y_n) \\
& - \sum_{j=1}^{m-1} \alpha_n^{(m)} \alpha_n^{(j)} d^2(S_{(\alpha,m)} y_n, S_{(\alpha,j)} u_n) \\
& - \sum_{j,r=1, j\neq r}^{m-1} \alpha_n^{(j)} \alpha_n^{(r)} d^2(S_{(\alpha,j)} u_n, S_{(\alpha,r)} u_n) \\
\leq\ & d^2(p, u_n) - \sum_{j=1}^{m-1} \alpha_n^{(0)} \alpha_n^{(j)} d^2(T_{(\beta,m)} u_n, S_{(\alpha,j)} u_n) \\
& - \alpha_n^{(0)} \alpha_n^{(m)} d^2(T_{(\beta,m)} u_n, S_{(\alpha,m)} y_n) \\
& - \sum_{j=1}^{m-1} \alpha_n^{(m)} \alpha_n^{(j)} d^2(S_{(\alpha,m)} y_n, S_{(\alpha,j)} u_n) \\
& - \sum_{j,r=1, j\neq r}^{m-1} \alpha_n^{(j)} \alpha_n^{(r)} d^2(S_{(\alpha,j)} u_n, S_{(\alpha,r)} u_n),
\end{aligned} \tag{3.30}
$$

which implies by (3.13) and (3.21) that

$$
\begin{aligned}
\sum_{j=1}^{m-1} \alpha_n^{(0)} \alpha_n^{(j)} d^2(T_{(\beta,m)} u_n, S_{(\alpha,j)} u_n) + \alpha_n^{(0)} \alpha_n^{(m)} d^2(T_{(\beta,m)} u_n, S_{(\alpha,m)} y_n) & \\
\leq d^2(p, u_n) - d^2(p, x_{n+1}) & \\
\to 0 \text{ as } n \to \infty. &
\end{aligned} \tag{3.31}
$$

This together with condition C3, implies that

$$
\lim_{n\to\infty} d(T_{(\beta,m)} u_n, S_{(\alpha,j)} u_n) = 0,\ j = 1, 2, \ldots, m-1 \tag{3.32}
$$

and

$$
\lim_{n\to\infty} d(T_{(\beta,m)} u_n, S_{(\alpha,m)} y_n) = 0. \tag{3.33}
$$

By (3.28), (3.32) and triangle inequality, we obtain

$$
\lim_{n\to\infty} d(u_n, S_{(\alpha,j)} u_n) = 0,\ j = 1, 2, \ldots, m-1. \tag{3.34}
$$

Furthermore,

$$d(y_n, S_{(\alpha,m)}y_n) \leq d(y_n, u_n) + d(u_n, T_{(\beta,m)}u_n) + d(T_{(\beta,m)}u_n, S_{(\alpha,m)}y_n),$$

which implies by (3.28), (3.29) and (3.33) that

$$\lim_{n\to\infty} d(y_n, S_{(\alpha,m)}y_n) = 0. \tag{3.35}$$

Moreover, as $\{x_n\}$ is bounded and X is a Hadamard space so by Lemma 2.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = z \in C$. It follows from (3.21) and (3.29) that there exist subsequences $\{u_{n_k}\}$ of $\{u_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} u_{n_k} = z = \Delta\text{-}\lim_{k\to\infty} y_{n_k}$. Since $T_{(\beta,j)}$ and $S_{(\alpha,j)}$ are Δ-demiclosed, it follows from (3.28), (3.34), (3.35) and Lemma 3.3 that $z \in \left(\cap_{j=1}^m F(T_{(\beta,j)})\right) \cap \left(\cap_{j=1}^m F(S_{(\beta,i)})\right) = \left(\cap_{j=1}^m F(T_j)\right) \cap \left(\cap_{j=1}^m F(S_j)\right)$. Also, since $J_{\lambda^{(i)}}$ is nonexpansive for each $i = 1, 2, \ldots, N$, we obtain by (3.20) and Lemma 2.7 that $z \in \cap_{i=1}^N F(J_{\lambda^{(i)}}) = \left(\cap_{i=1}^N \arg\min_{y\in X} f_i(y)\right)$. Hence $z \in \Gamma$. Furthermore, for arbitrary $u \in X$, we have by Lemma 2.6 that

$$\limsup_{n\to\infty} \langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle \leq 0, \tag{3.36}$$

which implies by condition C1 that

$$\limsup_{n\to\infty} \left(t_n d^2(z,u) + 2(1-t_n)\langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle\right) \leq 0. \tag{3.37}$$

We now show that $\{x_n\}$ converges strongly to z. By (3.10) and Lemma 2.3(ii) (iii), we obtain

$$\begin{aligned} d^2(z, x_{n+1}) &\leq d^2(z, z_n) \\ &\leq (1-t_n)^2 d^2(z, x_n) + t_n^2 d^2(z,u) + 2t_n(1-t_n)\langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle \\ &\leq (1-t_n) d^2(z, x_n) + t_n \left(t_n d^2(z,u) + 2(1-t_n)\langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle\right). \end{aligned} \tag{3.38}$$

Hence, by (3.37) and Lemma 2.9, we conclude that $\{x_n\}$ converges strongly to z.

Case 2: Suppose that $\{d^2(p, x_n)\}$ is monotonically non-decreasing. Then, there exists a subsequence $\{d^2(p, x_{n_i})\}$ of $\{d^2(p, x_n)\}$ such that $d^2(p, x_{n_i}) < d^2(p, x_{n_i+1})$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.10, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$d^2(p, x_{m_k}) \leq d^2(p, x_{m_k+1}) \text{ and } d^2(p, x_k) \leq d^2(p, x_{m_k+1}) \ \forall k \in \mathbb{N}. \tag{3.39}$$

Thus, by (3.10), (3.39) and Lemma 2.3, we obtain

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} \left(d^2(p, x_{m_k+1}) - d^2(p, x_{m_k})\right) \\ &\leq \limsup_{n\to\infty} \left(d^2(p, x_{n+1}) - d^2(p, x_n)\right) \\ &\leq \limsup_{n\to\infty} \left(d^2(p, z_n) - d^2(p, x_n)\right) \\ &\leq \limsup_{n\to\infty} \left((1-t_n) d^2(p, x_n) + t_n d^2(p,u) - d^2(p, x_n)\right) \\ &= \limsup_{n\to\infty} \left[t_n \left(d^2(p,u) - d^2(p, x_n)\right)\right] = 0, \end{aligned}$$

which implies that

$$\lim_{k\to\infty}\left(d^2(p,x_{m_k+1})-d^2(p,x_{m_k})\right)=0. \tag{3.40}$$

Following the arguments as in **Case 1**, we can show that

$$\lim_{k\to\infty}\left(t_{m_k}d^2(z,u)+2(1-t_{m_k})\langle\overrightarrow{zu},\overrightarrow{zx_{m_k}}\rangle\right)\leq 0. \tag{3.41}$$

Also, by (3.38) we have

$$d^2(z,x_{m_k+1})\leq(1-t_{m_k})d^2(z,x_{m_k})+t_{m_k}\left(t_{m_k}d^2(z,u)+2(1-t_{m_k})\langle\overrightarrow{zu},\overrightarrow{zx_{m_k}}\rangle\right).$$

Since $d^2(z,x_{m_k})\leq d^2(z,x_{m_k+1})$, we obtain

$$d^2(z,x_{m_k})\leq\left(t_{m_k}d^2(z,u)+2(1-t_{m_k})\langle\overrightarrow{zu},\overrightarrow{zx_{m_k}}\rangle\right).$$

Thus, by (3.41) we get

$$\lim_{k\to\infty}d^2(z,x_{m_k})=0. \tag{3.42}$$

It then follows from (3.39), (3.40) and (3.42) that $\lim\limits_{k\to\infty}d^2(z,x_k)=0$. Therefore, we conclude by **Case 1** that $\{x_n\}$ converges to $z\in\Gamma$. □

By setting $N=2=m$ in Theorem 3.4, we obtain the following result which extends Theorems 3.1 and 3.2 in [11] and Theorem 3.1 in [9].

Corollary 3.5. *Let C be a closed and convex subset of a Hadamard space X and $h_i:X\to(-\infty,\infty]$, $i=1,2$ be a finite family of proper convex and lower semi-continuous mappings. For each $j=1,2$, let $T_j:C\to C$ be a finite family of (f_j,g_j)-generalized k_j-strictly pseudononspreading mappings with $k_j\in[0,1)$, where $f_j,g_j:C\to[0,\gamma]$, $\gamma<1$ and $0<f_j(x)+g_j(x)\leq 1$ for all $x\in C$, and $S_j:C\to X$ be a finite family of (f'_j,g'_j)-generalized k'_j-strictly pseudononspreading mappings with $k'_j\in[0,1)$, where $f'_j,g'_j:C\to[0,\gamma']$, $\gamma'<1$ and $0<f'_j(x)+g'_j(x)\leq 1$ for all $x\in C$. Suppose that $\Gamma:=\left(\cap_{j=1}^2F(T_j)\right)\cap\left(\cap_{j=1}^2F(S_j)\right)\cap\left(\cap_{i=1}^2 \arg\min_{y\in X}h_i(y)\right)\neq\emptyset$. Let $u,x_1\in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} z_n=(1-t_n)x_n\oplus t_nu,\\ u_n=P_C(J_{\lambda_n^{(2)}}\circ J_{\lambda_n^{(1)}}(z_n)),\\ y_n=\beta_n^{(0)}u_n\oplus\beta_n^{(1)}T_{(\beta,1)}u_n\oplus\beta_n^{(2)}T_{(\beta,2)}u_n,\\ x_{n+1}=\alpha_n^{(0)}T_{(\beta,2)}u_n\oplus\alpha_n^{(1)}S_{(\alpha,1)}u_n\oplus\alpha_n^{(2)}S_{(\alpha,2)}y_n,\ n\geq 1,\end{cases} \tag{3.43}$$

where $T_{(\beta,j)}x=\beta x\oplus(1-\beta)T_jx$ and $S_{(\alpha,j)}x=\alpha x\oplus(1-\alpha)S_jx$, $j=1,2$, for all $x\in C$ such that $T_{(\beta,j)}$ and $S_{(\beta,j)}$ are Δ-demiclosed with $\frac{k_j}{f_j(p)}\leq\beta<1$, $f_j(p)\neq 0$ and $\frac{k'_j}{f'_j(p)}\leq\alpha<1$, $f'_j(p)\neq 0$ respectively, for each $j=1,2$ and for each $p\in\left(\cap_{j=1}^2F(T_j)\right)\cap\left(\cap_{j=1}^2F(S_j)\right)$, $\{t_n\},\{\lambda_n^{(i)}\},\{\beta_n^{(j)}\}$ and $\{\alpha_n^{(j)}\}$ are sequences in $(0,1)$ satisfying the following conditions:

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n^{(j)},\ \beta_n^{(j)} \leq b < 1,\ j = 0,1,2$ *such that* $\sum_{j=0}^{2} \alpha_n^{(j)} = 1$ *and* $\sum_{j=0}^{2} \beta_n^{(j)} = 1$ *for all* $n \geq 1$,

C4: $\{\lambda_n^{(i)}\}$ *is a sequence such that* $\lambda_n^{(i)} > \lambda^{(i)}$ *for all* $n \geq 1,\ i = 1,2$ *and some* $\lambda^{(i)} > 0$.

Then, $\{x_n\}$ *converges strongly to an element of* Γ.

In view of Remark 1.1, we obtain the following corollaries which extend and improve the main results of Osilike and Isiogugu [37], Bačák [7] and Bačák [6].

Corollary 3.6. *Let* C *be a closed and convex subset of a Hadamard space* X *and* $h_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ *be a finite family of proper convex and lower semi-continuous mappings. For each* $j = 1, 2, \ldots, m$, *let* $T_j : C \to C$ *be a finite family of* (f_j, g_j)*-generalized nonspreading mappings, where* $f_j, g_j : C \to [0, \gamma]$, $\gamma < 1$, $0 < f_j(x) + g_j(x) \leq 1$ *for all* $x \in C$, *and* $S_j : C \to X$ *be a finite family of* (f'_j, g'_j)*-generalized nonspreading mappings, where* $f'_j, g'_j : C \to [0, \gamma']$, $\gamma' < 1$, $0 < f'_j(x) + g'_j(x) \leq 1$ *for all* $x \in C$. *Suppose that* $\Gamma := \left(\cap_{j=1}^{m} F(T_j)\right) \cap \left(\cap_{j=1}^{m} F(S_j)\right) \cap \left(\cap_{i=1}^{N} \arg\min_{y\in X} h_i(y)\right) \neq \emptyset$. *Let* $u, x_1 \in X$ *be arbitrary and the sequence* $\{x_n\}$ *be generated by*

$$
\begin{cases}
z_n = (1 - t_n)x_n \oplus t_n u, \\
u_n = P_C(J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)), \\
y_n = \beta_n^{(0)} u_n \oplus \beta_n^{(1)} T_1 u_n \oplus \beta_n^{(2)} T_2 u_n \oplus \ldots \\
\oplus \beta_n^{(m)} T_m u_n, \\
x_{n+1} = \alpha_n^{(0)} T_m u_n \oplus \alpha_n^{(1)} S_1 u_n \oplus \alpha_n^{(2)} S_2 u_n \oplus \ldots \\
\oplus \alpha_n^{(m)} S_m y_n,\ n \geq 1,
\end{cases}
\tag{3.44}
$$

where $\{t_n\}, \{\lambda_n^{(i)}\}, \{\beta_n^{(j)}\}$ *and* $\{\alpha_n^{(j)}\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n^{(j)},\ \beta_n^{(j)} \leq b < 1,\ j = 0,1,2,\ldots,m$ *such that* $\sum_{j=0}^{m} \alpha_n^{(j)} = 1$ *and* $\sum_{j=0}^{m} \beta_n^{(j)} = 1$ *for all* $n \geq 1$,

C4: $\{\lambda_n^{(i)}\}$ *is a sequence such that* $\lambda_n^{(i)} > \lambda^{(i)}$ *for all* $n \geq 1,\ i = 1,2,\ldots,N$ *and some* $\lambda^{(i)} > 0$.

Then, $\{x_n\}$ *converges strongly to an element of* Γ.

Corollary 3.7. *Let C be a closed and convex subset of a Hadamard space X and $h_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semi-continuous mappings. For each $j = 1, 2, \ldots, m$, let $T_j : C \to C$ and $S_j : C \to X$ be finite family of k_j-strictly pseudononspreading mappings with $k_j \in [0, 1)$ and finite family of k'_j-strictly pseudononspreading mappings with $k_j \in [0, 1)$ respectively. Suppose that $\Gamma := \left(\cap_{j=1}^m F(T_j)\right) \cap \left(\cap_{j=1}^m F(S_j)\right) \cap \left(\cap_{i=1}^N arg\min_{y\in X} h_i(y)\right) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$(3.45)\quad \begin{cases} z_n = (1 - t_n)x_n \oplus t_n u, \\ u_n = P_C(J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)), \\ y_n = \beta_n^{(0)} u_n \oplus \beta_n^{(1)} T_{(\beta,1)} u_n \oplus \beta_n^{(2)} T_{(\beta,2)} u_n \oplus \ldots \\ \oplus \beta_n^{(m)} T_{(\beta,m)} u_n, \\ x_{n+1} = \alpha_n^{(0)} T_{(\beta,m)} u_n \oplus \alpha_n^{(1)} S_{(\alpha,1)} u_n \oplus \alpha_n^{(2)} S_{(\alpha,2)} u_n \oplus \ldots \\ \oplus \alpha_n^{(m)} S_{(\beta,m)} y_n, \ n \geq 1, \end{cases}$$

where $T_{(\beta,j)}x = \beta x \oplus (1-\beta)T_j x$ and $S_{(\alpha,j)}x = \alpha x \oplus (1-\alpha)S_j x$, $j = 1, 2, \ldots, m$, for all $x \in C$ such that $k_j \leq \beta < 1$ and $k'_j \leq \alpha < 1$. For each $i, j = 0, 1, 2, \ldots, m$, $\{t_n\}, \{\lambda_n^{(i)}\}, \{\beta_n^{(i)}\}$ and $\{\alpha_n^{(i)}\}$ are sequences in $(0, 1)$ satisfying the following conditions:

- *C1:* $\lim_{n\to\infty} t_n = 0$,
- *C2:* $\sum_{n=1}^{\infty} t_n = \infty$,
- *C3:* $0 < a \leq \alpha_n^{(j)}$, $\beta_n^{(j)} \leq b < 1$ *such that* $\sum_{j=0}^m \alpha_n^{(j)} = 1$ *and* $\sum_{j=0}^m \beta_n^{(j)} = 1$ *for all* $n \geq 1$,
- *C4:* $\{\lambda_n^{(i)}\}$ *is a sequence such that* $\lambda_n^{(i)} > \lambda^{(i)}$ *for all* $n \geq 1$, $i = 1, 2, \ldots, N$ *and some* $\lambda^{(i)} > 0$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

4 Numerical Example

We give numerical example to illustrate Theorem 3.4. Let $X = \mathbb{R}$, endowed with the usual metric and $C = [0, 100]$. Then,

$$P_C(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } x \in [0,\ 100], \\ 100, & \text{if } x > 100 \end{cases}$$

is a metric projection onto C. For $m = 1$, we define $S : C \to \mathbb{R}$ by

$$Sx = \begin{cases} -3x, & \text{if } x \in [0, 1], \\ \frac{1}{x}, & \text{if } x \in (1,\ 100]. \end{cases}$$

Then, S is an (f', g')-generalized k'-strictly pseudononspreading mapping with $k' = \frac{9}{10}$ and $f', g' : [0,\ 100] \to [0,\ \frac{10}{11}]$ defined by

$$f'(x) = \begin{cases} \frac{10}{11}, & \text{if } x \in [0,1], \\ \frac{1}{11}, & \text{if } x \in (1,100] \end{cases} \qquad \text{and} \qquad g'(x) = \begin{cases} \frac{1}{11}, & \text{if } x \in [0,1], \\ \frac{10}{11}, & \text{if } x \in (1,100]. \end{cases}$$

Also, we define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{x+\frac{1}{10}}, & \text{if } x \in [1,100], \\ 0, & \text{if } x \in [0,1). \end{cases}$$

Then, T is an (f, g)-generalized k-strictly pseudocontractive mapping with $k = 0$ and $f, g : [0, 100] \to [0, \frac{9}{10}]$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in [1,100], \\ \frac{9}{10}, & \text{if } x \in [0,1) \end{cases} \quad \text{and } g(x) = \begin{cases} \frac{1}{(x+\frac{1}{10})^2}, & \text{if } x \in [1,\ 100], \\ 0, & \text{if } x \in [0,1). \end{cases}$$

Clearly, $F(T) \cap F(S) = \{0\}$. Thus, we can choose $\alpha = \frac{k'}{f'(0)} = \frac{99}{100}$ and $\beta = 0$. Then, $S_\alpha x = \frac{99}{100}x + \left(1 - \frac{99}{100}\right) Sx$ and $T_\beta x = Tx$.
Let $N = 2$. Then for $i = 1, 2$, we define $h_1, h_2 : \mathbb{R} \to (-\infty, \infty]$ by $h_1(x) = \frac{1}{2}|B_1(x) - b_1|^2$ and $h_2(x) = \frac{1}{2}|B_2(x) - b_2|^2$, where $B_1(x) = 2x$, $B_2(x) = 5x$ and $b_1 = b_2 = 0$. Since B_i is a continuous and linear mapping, so for each $i = 1, 2$, h_i is a proper convex and lower semi-continuous mapping (see [28]). Thus, for $\lambda_n = 1$, we have that (see [28])

$$\begin{aligned} J_{1^{(i)}}(x) &= \operatorname{Prox}_{h_i} x = \arg\min_{y \in C} \left(h_i(y) + \frac{1}{2}|y - x|^2 \right) \\ &= (I + B_i^T B_i)^{-1}(x + B_i^T b_i). \end{aligned}$$

Take $t_n = \frac{1}{4n+3}$, $\alpha_n^{(0)} = \frac{n}{3n+5}$, $\alpha_n^{(1)} = \frac{2n+5}{3n+5}$, $\beta_n^{(0)} = \frac{n}{2n+1}$ and $\beta_n^{(1)} = \frac{n+1}{2n+1}$.
Now, conditions C1-C4 are satisfied.
Hence, for $u, x_1 \in \mathbb{R}$, our Algorithm (3.9) becomes:

$$\begin{cases} z_n = (1 - t_n)x_n + t_n u, \\ u_n = P_C\left(J_{1^{(2)}}\left(J_{1^{(1)}}(z_n)\right)\right), \\ y_n = \beta_n^{(0)} u_n + \beta_n^{(1)} T_\beta u_n, \\ x_{n+1} = \alpha_n^{(0)} T_\beta u_n + \alpha_n^{(1)} S_\alpha y_n, \ n \geq 1. \end{cases} \tag{4.1}$$

Case I: Take $x_1 = 1$ and $u = 0.1$.
Case II: Take $x_1 = 0.5$ and $u = 0.1$.
Case III: Take $x_1 = 0.5$ and $u = 2$.
The following table shows results of our numerical experiment based on MATLAB version R2016a software.
Declaration
The authors declare that they have no competing interests.

Iteration Numbers	**Errors for Case I** u=0.1	**Errors for Case II** u=0.1	**Errors for Case III** u=2
1	1.0000	0.5000	0.5000
2	0.7560	0.3760	0.3000
3	0.1715	0.0857	0.0858
4	0.0485	0.0246	0.0367
5	0.0149	0.0077	0.0192
6	0.0049	0.0027	0.0116
7	0.0017	0.0010	0.0077
8	0.0007	0.0005	0.0055
9	0.0003	0.0003	0.0041
10	0.0002	0.0002	0.0032
11	0.0001	0.0001	0.0026
12	0.0001	0.0001	0.0021
13	0.0001	0.0001	0.0018
14	0.0001	0.0001	0.0015
15	0.0001	0.0001	0.0013

TABLE 1. Showing numerical results for **Case I**, **Case II** and **Case III**.

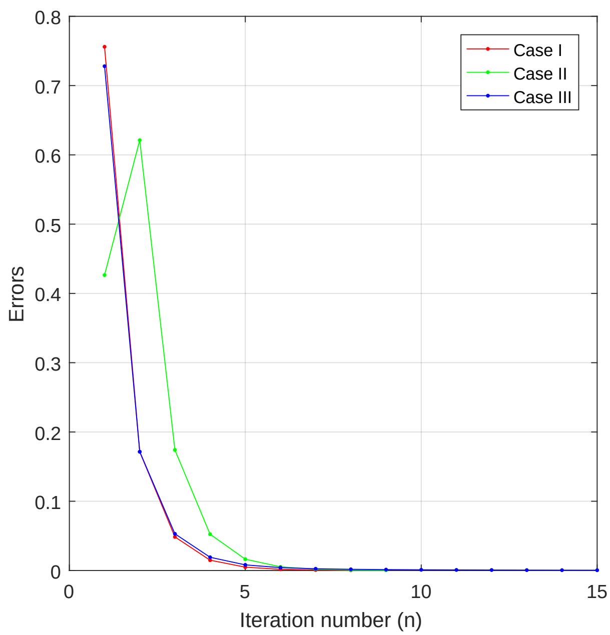

Figure 1: Errors vs number of iterations for **Case I**, **Case II** and **Case III**.

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