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A remark on Pixley-Roy hyperspaces

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Abstract. In this paper, we study the relation between a space X satisfying certain generalized metric properties and the Pixley-Roy hyperspace $\mathtt{PR}[X]$ over X satisfying the same properties. We prove that a regular space X is a countable stric $\mathfrak{B}_0$-space if and only if $\mathtt{PR}_2[X]$ is a stric $\mathfrak{B}_0$-space. However, there exists a countable stric $\mathfrak{B}_0$-space X such that $\mathtt{PR}_n[X]$ with $n \geq 3$ and $\mathtt{PR}[X]$ are not stric $\mathfrak{B}_0$-spaces. Moreover, we show that $\mathtt{PR}[X]$ is a compact space if and only if X is finite, and there exists a compact subset K of a space X such that $[\{x\}, K]$ with $x \in K$ is not a compact subset of $\mathtt{PR}[X]$. On the other hand, X is a P-space if and only if so is $\mathtt{PR}[X]$. Finally, we prove that if $\mathtt{PR}[X]$ of a regular space X is an r-space, then X is also an r-space.

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1. Introduction

The generalized metric properties on Pixley-Roy hyperspaces have been studied by many authors ([2], [13], [5], [7], [8], [9], [10], [11], [12], for example). They considered several generalized metric properties and studied the relation between a space X satisfying such property and its Pixley-Roy hyperspaces satisfying the same property.

In this paper, we study concepts such as stric $\mathfrak{B}_0$-space, P-space, r-space and compactness on Pixley-Roy hyperspaces. We obtain some new results about Pixley-Roy hyperspaces.

Throughout this paper, all spaces are assumed to be at least T_1, $\mathbb{N}$ denotes the set of all positive integers, the first infinite ordinal denoted by ω.

2. Definitions

The *Pixley-Roy hyperspace* $\mathtt{PR}[X]$ over a space X, defined by C. Pixley and P. Roy in [9], is the set of all non-empty finite subsets of X with the topology generated by the sets of the form

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$$[F, V] = \{G \in \mathtt{PR}[X] : F \subset G \subset V\},$$

where $F \in \mathtt{PR}[X]$ and V is an open subset in X containing F. It is known that $\mathtt{PR}[X]$ is always zero-dimensional, completely regular (see [13]).

For each $n \in \mathbb{N}$, let $\mathtt{PR}_n[X] = \{F \in \mathtt{PR}[X] : |F| \leq n\}$.

Remark 2.1 ([12], p. 305). For each $n \in \mathbb{N}$, $\mathtt{PR}_n[X]$ is a closed subspace of $\mathtt{PR}[X]$ and, in particular, $\mathtt{PR}_1[X]$ is a closed discrete subspace of $\mathtt{PR}[X]$.

Remark 2.2 ([5], Remark 1.2). Every $\mathtt{PR}_m[X]$ is a closed subspace of $\mathtt{PR}_n[X]$ for each $m, n \in \mathbb{N}$, $m < n$.

For each $F \in \mathtt{PR}[X]$ and $A \subset X$, denote

$$[F, A] = \{H \in \mathtt{PR}[X] : F \subset H \subset A\}.$$

Definition 2.3. Let $\mathcal{P}$ be a family of subsets of a space X.

1. $\mathcal{P}$ is called *point-countable* [3], if the family $\{P \in \mathcal{P} : x \in P\}$ is countable for each $x \in X$.
2. $\mathcal{P}$ is an *sp-network* [6] for X, if for each $x \in U \cap \overline{A}$ with U open and A subset in X, there is a set $P \in \mathcal{P}$ such that $x \in P \subset U$ and $x \in \overline{P \cap A}$.

Definition 2.4 ([3])**.** 1. A space X is called a *k-space* if X is a Hausdorff space and X is an image of a locally compact space under a quotient mapping.

2. A space X is called a *sequential space* if a set $A \subset X$ is closed if and only if together with any sequence it contains all its limits.
3. A space X is called a *Fréchet-Urysohn space* if for each $A \subset X$ and each $x \in \overline{A}$, there is a sequence in A converging to the point x in X.

Remark 2.5 ([3]). 1. Every Fréchet-Urysohn space is a sequential space.

2. Every sequential Hausdorff space is a k-space.

Definition 2.6 ([6])**.** Let

$$X = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\},$$

where every x_n, $x_n(m)$ and ∞ are different from each other. The set X endowed with the following topology is called the *Arens space* and denoted briefly as S_2: each $x_n(m)$ is isolated; a basic neighborhood of x_n has the form $\{x_n\} \cup \{x_n(m) : m > k\}$ for some $k \in \mathbb{N}$; a basic neighborhood of ∞ has the form $\{\infty\} \cup (\bigcup\{V_n : n \geq k\})$ for some $k \in \mathbb{N}$, where each V_n is a neighborhood of x_n.

Definition 2.7 ([6])**.** A topological space X is called the *sequential fan*, which is denoted briefly as S_ω if X is the quotient space by identifying all the limit points of ω many non-trivial convergent sequences.

Definition 2.8. Let X be a space.

(1) X is said to be a *stric* $\mathfrak{B}_0$*-space* [6], if X is regular and has a countable *sp*-network.

(2) X is called a P*-space* [1], if every G_δ-set in X is open.

Definition 2.9 ([4])**.** Let X be a regular space. A point x of X is an r*-point* if it has a sequence $\{U_m : m \in \mathbb{N}\}$ of neighborhoods of x such that if $x_m \in U_m$, then $\{x_m : m \in \mathbb{N}\}$ is contained in a compact subset of X. The space X is an r*-space* if all of its points are r-points.

3. Main results

Lemma 3.1 ([6], Theorem 4.11)**.** *A regular space has a countable sp-network if and only if it is separable and has a point-countable sp-network.*

Theorem 3.2. *Let X be a regular space. Then X is a countable stric $\mathfrak{B}_0$-space if and only if $\mathtt{PR}_2[X]$ is a stric $\mathfrak{B}_0$-space.*

Proof. Necessity. Let X be a countable stric $\mathfrak{B}_0$-space. Then X is regular and has a countable *sp*-network. By Lemma 3.1, X is separable and has a point-countable *sp*-network. It follows from [5, Theorem 2.11] that $\mathtt{PR}_2[X]$ has a point-countable *sp*-network. On the other hand, because X is countable, $\mathtt{PR}_2[X]$ is countable. This implies that $\mathtt{PR}_2[X]$ is separable. Hence, $\mathtt{PR}_2[X]$ is a stric $\mathfrak{B}_0$-space by Lemma 3.1.

Sufficiency. Assume that $\mathtt{PR}_2[X]$ is a stric $\mathfrak{B}_0$-space. Then $\mathtt{PR}_2[X]$ has a countable *sp*-network. By Lemma 3.1, $\mathtt{PR}_2[X]$ is separable. Similar to the proof of [5, Theorem 2.16], we claim that X has a countable *sp*-network. Moreover, since X is regular, X is a stric $\mathfrak{B}_0$-space. On the other hand, it is known that if X is uncountable, then $\mathtt{PR}_2[X]$ is not separable (see [11]). This is a contradiction. Therefore, X is countable. □

Corollary 3.3. *Let X be a space.*

1. *If X is uncountable, then $\mathtt{PR}_2[X]$ is not a stric $\mathfrak{B}_0$-space.*

2. *If $\mathtt{PR}[X]$ is a stric $\mathfrak{B}_0$-space, then X is a countable stric $\mathfrak{B}_0$-space.*

Example 3.4. There exists a countable stric $\mathfrak{B}_0$-space X such that $\mathtt{PR}_n[X]$ with $n \geq 3$ and $\mathtt{PR}[X]$ are not stric $\mathfrak{B}_0$-spaces.

Proof. It follows from [5, Example 2.13] that the sequential fan S_ω is a countable regular space with a countable *sp*-network but $\mathtt{PR}_n[X]$ with $n \geq 3$ and $\mathtt{PR}[X]$ do not have point-countable *sp*-networks. Thus, S_ω is a countable stric $\mathfrak{B}_0$-space but $\mathtt{PR}_n[X]$ with $n \geq 3$ and $\mathtt{PR}[X]$ are not stric $\mathfrak{B}_0$-spaces. □

Remark 3.5. Let X be a space. By Remark 2.1, $\mathtt{PR}[X]$ is a compact space if and only if X is finite.

Example 3.6. There exists a compact subset K of a space X such that $[\{x\}, K]$ with $x \in K$ is not a compact subset of $\mathtt{PR}[X]$.

Proof. We consider the Arens space

$$S_2 = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_n(m) : m, n \in \mathbb{N}\}.$$

Put $L_n = \{x_n(m) : m \in \mathbb{N}\}$ for each $n \in \mathbb{N}$ and $K = \{\infty\} \cup \{x_1\} \cup L_1$. Then observe that K is a compact subset of X. However, $[\{\infty\}, K]$ is not a compact subset of $\mathtt{PR}[S_2]$. Indeed, we have

$$[\{\infty\}, K] \cap \mathtt{PR}_2[S_2] = \Big\{\{\infty\}, \{\infty, x_1\}\Big\} \cup \Big\{\{\infty, x_1(m)\} : m \in \mathbb{N}\Big\}.$$

Put $M = \bigcup_{n \geq 2} (L_n \cup \{x_n\})$. Since $\{\infty\} \cup M$ and S_2 are open subsets of S_2, we claim that

$$\Big\{\Big[\{\infty\}, \{\infty\} \cup M\Big] \cap \mathtt{PR}_2[S_2]\Big\} \cup \Big\{\Big[\{\infty, x_1\}, S_2\Big] \cap \mathtt{PR}_2[S_2]\Big\} \cup \Big\{\Big[\{\infty, x_1(m)\}, S_2\Big] \cap \mathtt{PR}_2[S_2] : m \in \mathbb{N}\Big\}$$

is an open cover of $[\{\infty\}, K] \cap \mathtt{PR}_2[S_2]$ in $\mathtt{PR}_2[S_2]$ without any finite subcover. This implies that $[\{\infty\}, K] \cap \mathtt{PR}_2[S_2]$ is not a compact subset of $\mathtt{PR}_2[S_2]$. Since $\mathtt{PR}_2[S_2]$ is closed in $\mathtt{PR}[S_2]$ by Remark 2.1, we conclude that $[\{\infty\}, K]$ is not a compact subset of $\mathtt{PR}[S_2]$. □

Theorem 3.7. *Let X be a space. Then X is a P-space if and only if so is $\mathtt{PR}[X]$.*

Proof. Necessity. Let X be a P-space and $\mathcal{U}$ be a G_δ-set in $\mathtt{PR}[X]$. Then there exists a sequence $\{\mathcal{U}_m : m \in \mathbb{N}\}$ consisting of open subsets of $\mathtt{PR}[X]$ such that $\mathcal{U} = \bigcap_{m \in \mathbb{N}} \mathcal{U}_m$. We prove that $\mathcal{U}$ is open in $\mathtt{PR}[X]$. In fact, let $F \in \mathcal{U}$. Then $F \in \mathcal{U}_m$ for each $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, since $\mathcal{U}_m$ is open in $\mathtt{PR}[X]$, there is an open set U_m in X satisfying $F \in [F, U_m] \subset \mathcal{U}_m$. This implies that

$$F \in [F, \bigcap_{m \in \mathbb{N}} U_m] = \bigcap_{m \in \mathbb{N}} [F, U_m] \subset \bigcap_{m \in \mathbb{N}} \mathcal{U}_m = \mathcal{U}.$$

Since X is a P-space, $\bigcap_{m \in \mathbb{N}} U_m$ is open in X. It shows that $[F, \bigcap_{m \in \mathbb{N}} U_m]$ is open in $\mathtt{PR}[X]$. Therefore, $\mathcal{U}$ is open in $\mathtt{PR}[X]$.

Sufficiency. Assume that $\mathtt{PR}[X]$ is a P-space and U is a G_δ-set in X. Then there exists a sequence $\{U_m : m \in \mathbb{N}\}$ consisting of open subsets of X such that $U = \bigcap_{m \in \mathbb{N}} U_m$. We will prove that U is open in X. Given a point $x \in U$. Then we have that

$$\bigcap_{m \in \mathbb{N}} [\{x\}, U_m] = [\{x\}, \bigcap_{m \in \mathbb{N}} U_m] = [\{x\}, U].$$

Since $\mathtt{PR}[X]$ is a P-space and $[\{x\}, U_m]$ is open in $\mathtt{PR}[X]$ for each $m \in \mathbb{N}$, $\bigcap_{m \in \mathbb{N}}[\{x\}, U_m]$ is open in $\mathtt{PR}[X]$. It follows from [5, Lemma 2.1] that $\bigcup(\bigcap_{m \in \mathbb{N}}[\{x\}, U_m])$ is open in X. This implies that

$$\bigcup(\bigcap_{m\in\mathbb{N}}[\{x\},U_m]) = \bigcup[\{x\},U] = U.$$

Thus, U is open in X. This shows that X is a P-space. □

Lemma 3.8. *Every closed subspace of an r-space is an r-space.*

Proof. Let Y be a closed subspace of an r-space X and $x \in Y$. Then since X is an r-space, there exists a sequence $\{U_m : m \in \mathbb{N}\}$ of open neighborhoods of x in X such that if $x_m \in U_m$ for each $m \in \mathbb{N}$, then $\{x_m : m \in \mathbb{N}\}$ is contained in a compact subset of X. For each $m \in \mathbb{N}$, put $V_m = U_m \cap Y$, then $\{V_m : m \in \mathbb{N}\}$ is a sequence of open neighborhoods of x in Y. Now, for each $m \in \mathbb{N}$, take $y_m \in V_m$, then $y_m \in U_m$. This implies that there is a compact subset K of X satisfying $\{y_m : m \in \mathbb{N}\} \subset K$. Put $K_1 = K \cap Y$. Then since Y is closed, K_1 is a compact subset of Y which contains $\{y_m : m \in \mathbb{N}\}$. Therefore, x is an r-point in Y, and Y is an r-space. □

Theorem 3.9. *Let X be a regular space. If $\mathtt{PR}[X]$ is an r-space, then X is an r-space.*

Proof. Assume that $\mathtt{PR}[X]$ is an r-space and $x \in X$. Then $\{x\} \in \mathtt{PR}[X]$ and there exists a sequence $\{\mathcal{U}_m : m \in \mathbb{N}\}$ of open neighborhoods of $\{x\}$ satisfying the definition of an r-point. For each $m \in \mathbb{N}$, put $U_m = \bigcup \mathcal{U}_m$. By [5, Lemma 2.1], $\{U_m : m \in \mathbb{N}\}$ is a sequence of open neighborhoods of x. For each $m \in \mathbb{N}$, take $x_m \in U_m$. Then there is a set $A_m \in \mathcal{U}_m$ satisfying $x_m \in A_m$. Because $\mathtt{PR}[X]$ is an r-space, $\{A_m : m \in \mathbb{N}\}$ is contained in a compact subset $\mathcal{K}$ of $\mathtt{PR}[X]$. This implies that $x_m \in A_m \subset \bigcup \mathcal{K}$ for each $m \in \mathbb{N}$. It follows from [5, Lemma 2.2] that $\bigcup \mathcal{K}$ is a compact subset of X. Thus, $\{x_m : m \in \mathbb{N}\}$ is contained in a compact subset $\bigcup \mathcal{K}$ of X. It shows that X is an r-space. □

Question 1. If X is an r-space, then is $\mathtt{PR}_n[X]$ an r-space for some $n \in \mathbb{N}$?

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