Novi Sad J. Math. Vol. 54, No. 1, 2024, 189-207
https://doi.org/10.30755/NSJOM.14508

Joint spectrum and a spectral inclusion theorem for tensor product of semigroups on locally convex spaces

Rachid Ameziane Hassani[1], Aziz Blali[2], Abdelkhalek El Amrani[3], Mouad El Beldi[4][5]

Abstract. In the following paper, we deal with a specific type of joint spectrum, which is the bicommutant joint spectrum. The obtained results are used to establish a relation between the spectrum of the tensor product of semigroups on locally convex spaces and the Cartesian product of its components' spectrums. In addition, a spectral inclusion theorem for the tensor product of semigroups is given.

AMS Mathematics Subject Classification (2010): 47A10; 47D03; 47D99; 47A80

Key words and phrases: Joint spectrum; semigroups; tensor product; locally convex spaces; universally bounded operators

1. Introduction

In this paper, we will look at the bicommutant joint spectrum of operator families on locally convex spaces. Based on the obtained results, we will describe the spectrum of the tensor product of semigroups on locally convex spaces in terms of the Cartesian product of their component spectra. In addition, we will prove a spectral inclusion theorem for the tensor product of semigroups on locally convex spaces.

Many mathematicians have investigated the tensor product of semigroups on Banach spaces [2, 9]. The authors introduced in [3] the tensor product of semigroups on locally convex spaces and developed several properties.

Throughout this paper, X and Y will be two locally convex sequentially complete Hausdorff spaces over the complex field $\mathbb{C}$. Each system of continuous seminorms Γ_X and Γ_Y inducing the topology of X and Y, respectively, is called

[1]Department of Mathematics, Faculty of Sciences Dhar El Mahraz, Sidi Mohammed Ben Abdellah University, Fez, Morocco, e-mail: ra.ameziane@yahoo.fr, ORCID iD: orcid.org/0009-0009-5976-9248

[2]Department of Mathematics, École Normale Supérieure, Sidi Mohammed Ben Abdellah University, Fez, Morocco, e-mail: aziz.blali@usmba.ac.ma, ORCID iD: orcid.org/0000-0001-5788-2399

[3]Department of Mathematics, Faculty of Sciences Dhar El Mahraz, Sidi Mohammed Ben Abdellah University, Fez, Morocco, e-mail: abdelkhalek.elamrani@usmba.ac.ma, ORCID iD: orcid.org/0000-0003-4841-8038

[4]Department of Mathematics, Faculty of Sciences Dhar El Mahraz, Sidi Mohammed Ben Abdellah University, Fez, Morocco, e-mail: mouad.elbeldi@usmba.ac.ma, ORCID iD: orcid.org/0000-0001-6373-8377

[5]Corresponding author

calibration. We will denote by (X, Γ_X), (Y, Γ_Y) the space X, Y endowed with the systems of seminorms Γ_X, Γ_Y, respectively. $\mathcal{L}(X)$ will denote the algebra of the continuous linear operators on a locally convex space X.

According to Moore [13] and Chilana in [4], a linear operator $T : X \to X$ is called universally bounded with respect to a calibration Γ_X if there exists $K > 0$ such that $p(Tx) \leq Kp(x)$, for all $x \in X$ and $p \in \Gamma_X$. We will denote by $B_{\Gamma_X}(X)$ the class of universally bounded operators with respect to a calibration Γ_X.

Universally bounded operators acting on a locally convex space have been studied by Moore, Chilana [13, 4], and Joseph [8]. This class of operators has also been investigated by Giles, Joseph, Koehler, Sims, and others [7, 12, 14].

Let $X\hat{\otimes}_\alpha Y$ be the complemented tensor product of X and Y, where α designates the injective or the projective topology on tensor product and $\Gamma = \Gamma_X\hat{\otimes}_\alpha\Gamma_Y =: \left\{p\hat{\otimes}_\alpha q : p \in \Gamma_X,\ q \in \Gamma_Y\right\}$ the generating family of seminorms for the topology α.

Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$. In the following, we will denote by $\mathfrak{B}$ the bicommutant of the family $\left\{T\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S\right\}$.

The bicommutant joint spectrum of the family $\left\{T\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S\right\}$ on Banach spaces was studied by Dash and Schechter [6, 17].

One of the main problems in the case of locally convex spaces is to describe the bicommutant joint spectrum $\sigma\left(T\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S; \mathfrak{B}\cap B_\Gamma\left(X\hat{\otimes}_\alpha Y\right)\right)$ of the family $\left\{T\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S\right\}$ as the Cartesian product of the spectrum $\sigma(T, B_{\Gamma_1}(X))$, $\sigma(S, B_{\Gamma_2}(Y))$ of T and S, respectively. Wrobel [20, Corollary 2.4.] treated this problem, but it turns out that the obtained families Γ_1 and Γ_2 depend on the initial calibration Γ and a fixed $x_1 \otimes x_2 \in X \otimes Y$.

Inspired by the results of Wrobel [20, Theorem 2.1., Corollary 2.4.], the novelty of our paper is to show that the families Γ_1 and Γ_2 depend only on the initial calibration Γ. Moreover, based on the obtained results, we will prove a spectral inclusion theorem for the tensor product of semigroups on locally convex spaces.

We have organized our paper into three sections.

In Section 2, we give some reminders about the theory of universally bounded operators.

As a result of Theorems 3.6 and 3.10, in Section 3 we will show that if $\Gamma = \Gamma_X\hat{\otimes}_\alpha\Gamma_Y$ is the generating family of seminorms of the topology α, then

$$\sigma\left(T\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S; \mathfrak{B}\cap\mathfrak{B}_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right) = \sigma(T, B_{\Gamma_X}(X)) \times \sigma(S, B_{\Gamma_Y}(Y)).$$

Finally, in Section 4, we will show that the spectrum of the tensor product of semigroups is equal to the Cartesian product of the spectrums of his components (Theorem 4.4). Also, we will be able to prove a spectral inclusion theorem for tensor product of semigroups on locally convex spaces (Theorem 4.6).

2. Preliminaries

A locally convex algebra (A, τ) is an associative linear algebra with a topology τ such that (A, τ) is a Hausdorff locally convex topological vector space and for any element $y \in A$, the maps $x \mapsto xy$ and $x \mapsto yx$ are continuous [1].

Next, we consider the simple convergence topology in $\mathcal{L}(X)$, and we will denote by $\mathcal{L}_s(X)$ the linear space $\mathcal{L}(X)$ with this topology. The multiplication TS $S, T \in \mathcal{L}(X)$ induces a structure of algebra on $\mathcal{L}(X)$, and the algebra $\mathcal{L}_s(X)$ is a locally convex algebra.

Let us recall the following:

Definition 2.1 ([19])**.** If (X, Γ_X) and (Y, Γ_Y) are locally convex spaces, then for all seminorms $p \in \Gamma_X$ and $q \in \Gamma_Y$ the application

$$m_{pq} : L(X,\ Y) \to \mathbb{R}^+,$$

defined by

$$m_{pq}(T) = \sup_{p(x)\neq 0} \frac{q(Tx)}{p(x)},$$

is called the *mixed operator seminorm* of T associated with p and q. When $X = Y$ and $p = q$ we use the notation $\hat{p} = m_{pp}$.

Lemma 2.2 ([19])**.** *If (X, Γ_X) and (Y, Γ_Y) are locally convex spaces and $T \in L(X, Y)$, then*

1. $m_{pq}(T) = \sup_{p(x)=1} q(Tx) = \sup_{p(x)\leq 1} q(Tx), \forall p \in \Gamma_X, \forall q \in \Gamma_Y$.
2. $q(Tx) \leq m_{pq}(T)\, p(x)$, $\forall x \in X$, *whenever* $m_{pq}(T) < \infty$.
3. $m_{pq}(T) = \inf\{M > 0 : q(Tx) \leq Mp(x),\ \forall x \in X\}$, *whenever* $m_{pq}(T) < \infty$.

Definition 2.3. Let X be a locally convex space. An operator $T \in L(X)$ is universally bounded with respect to the calibration Γ_X if there exists $c_0 > 0$ such that

$$p(Tx) \leq c_0 p(x),\ (\forall x \in X,\ \forall p \in \Gamma_X).$$

We denote by $B_{\Gamma_X}(X)$ the class of all universally bounded operators with respect to some calibration Γ_X.

Lemma 2.4 ([4])**.** *$B_{\Gamma_X}(X)$ is a unital normed algebra with respect to the norm $\|\cdot\|_{\Gamma_X}$ defined by*

$$\|T\|_{\Gamma_X} = \inf\{M > 0 : p(Tx) \leq Mp(x),\ \forall x \in X,\ \forall p \in \Gamma_X\},$$

for any $T \in B_{\Gamma_X}(X)$.

Corollary 2.5. *For each $T \in B_{\Gamma_X}(X)$ we have*

$$\|T\|_{\Gamma_X} = \sup\{\hat{p}(T) : p \in \Gamma_X\}.$$

Definition 2.6 ([8]). *Two families $\mathcal{P}_1$ and $\mathcal{P}_2$ of seminorms on a linear space are called B-equivalent (denoted $\mathcal{P}_1 \sim \mathcal{P}_2$)* provided each seminorm in each family is a positive multiple of a seminorm in the other.

Proposition 2.7 ([8]). *Let Γ_X be a calibration on X, then:*

1. *$B_{\Gamma_X}(X)$ is a subalgebra of $L(X)$.*
2. *$\left(B_{\Gamma_X}(X), \|\cdot\|_{\Gamma_X}\right)$ is a unitary normed algebra.*
3. *Let Γ a calibration on X, with the property $\Gamma_X \sim \Gamma$, we have $B_{\Gamma_X}(X) = B_\Gamma(X)$ and $\|\cdot\|_{\Gamma_X} = \|\cdot\|_\Gamma$.*

Proposition 2.8 ([4]). *Let Γ_X be a calibration on X. Then:*

1. *If $(T_n)_n$ is a Cauchy sequence in $\left(B_{\Gamma_X}(X), \|\cdot\|_{\Gamma_X}\right)$ which converges to an operator T, then we have $T \in B_{\Gamma_X}(X)$.*
2. *The algebra $\left(B_{\Gamma_X}(X), \|\cdot\|_{\Gamma_X}\right)$ is complete if X is sequentially complete.*

Definition 2.9 ([20]). Let (X, Γ_X) be a locally convex space. For $T \in \mathcal{L}(X)$ we set

1. $\Lambda(T, \Gamma_X) = \{\lambda \in \mathbb{C} : \exists c > 0 \text{ such that } p(\lambda x - Tx) \geq cp(x), \forall x \in X\, \forall p \in \Gamma_X\}$.
2. We define the approximate point spectrum by setting $\sigma_{ap}(T, \Gamma_X) = (\Lambda(T, \Gamma_X))^c = \mathbb{C} \setminus \Lambda(T, \Gamma_X)$.
3. We define the residual spectrum by
$$\sigma_r(T, \Gamma_X) = \{\lambda \in \Lambda(T, \Gamma_X) : \operatorname{Im}(\lambda I - T) \text{ is not dense in } X\}.$$
4. The point spectrum is defined as follows
$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } T\}.$$
5. We put $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible in } \mathcal{L}(X)\}$.
6. The spectrum of T with respect to $B_{\Gamma_X}(X)$ is defined by
$$\sigma(T, B_{\Gamma_X}(X)) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible in } B_{\Gamma_X}(X)\}.$$
and the resolvent set of T with respect to $B_{\Gamma_X}(X)$ will be the complementary set of $\sigma(T, B_{\Gamma_X}(X))$, i.e
$$\begin{aligned}\rho(T, B_{\Gamma_X}(X)) &= \mathbb{C} \setminus \sigma(T, B_{\Gamma_X}(X))\\ &= \left\{\lambda \in \mathbb{C} : \mathfrak{R}(\lambda, T) = (\lambda I - T)^{-1} \text{ exists and } \mathfrak{R}(\lambda, T) \in B_{\Gamma_X}(X)\right\}.\end{aligned}$$

It is clear that $\sigma_p(T) \subset \sigma(T) \subset \sigma(T, B_{\Gamma_X}(X))$.

The following Proposition is similar to Theorem 4.6 in [18].

Proposition 2.10. *Let $T \in B_{\Gamma_X}(X)$, then we have*

1. *$\lambda \in \sigma_{ap}(T, \Gamma_X)$ if and only if there exist a sequence $(x_n)_n \in X$ and $(p_n)_n \in \Gamma_X$ such that for any $n \in \mathbb{N}$, $p_n(x_n) = 1$ and $p_n((\lambda I - T)x_n) \to 0$ as $n \to +\infty$.*

2. *The set $\sigma_{ap}(T, \Gamma_X)$ is closed.*

3. *$\sigma_{ap}(T, \Gamma_X)$ contains the boundary $\partial\sigma(T, B_{\Gamma_X}(X))$ of $\sigma(T, B_{\Gamma_X}(X))$.*

4. *$\sigma_{ap}(T, \Gamma_X)$ is a nonempty set.*

The following proposition is proven using the same approach as Theorem 4.5 in [18].

Proposition 2.11 ([18])**.** *Let $T \in \mathcal{L}(X)$, then*

$$\sigma(T, B_{\Gamma_X}(X)) = \sigma_{ap}(T, \Gamma_X) \mathring{\cup} \sigma_r(T, \Gamma_X).$$

Let us recall the following

Let $T \in \mathcal{L}(X)$, Let X' denote the topological dual of X. We define the transpose of T

$$\begin{aligned} T' : X' &\to X' \\ y' &\mapsto T'(y') \end{aligned}$$

by setting for any $y' \in X'$ and $x \in X$

$$T'(y')x = \langle x, T'(y') \rangle = \langle T(x), y' \rangle.$$

It is clear that for all $\lambda \in \mathbb{C}$, $(\lambda I - T)' = \lambda I' - T'$ where I' is the identity map on X'.

Lemma 2.12. *Let $T \in \mathcal{L}(X)$ then we have $\sigma(T') \subseteq \sigma(T)$.*

Proof. Let $\lambda \in \mathbb{C} \setminus \sigma_0(T)$, then $\lambda I - T$ is invertible, and for all $x' \in X'$ and $x \in X$ we have

$$\begin{aligned} \langle x, x' \rangle &= \left\langle (\lambda I - T)^{-1} (\lambda I - T) x, x' \right\rangle = \left\langle (\lambda I - T)(\lambda I - T)^{-1} x, x' \right\rangle \\ &= \left\langle x, (\lambda I' - T') \left[(\lambda I - T)^{-1}\right]' x' \right\rangle = \left\langle x, \left[(\lambda I - T)^{-1}\right]' (\lambda I' - T') x' \right\rangle, \end{aligned}$$

then $(\lambda I' - T')$ is invertible and $(\lambda I' - T')^{-1} = \left[(\lambda I - T)^{-1}\right]'$. Hence, $\lambda \in \mathbb{C} \setminus \sigma_0(T')$. ∎

3. The Joint spectrum of tensor product of operators on locally convex spaces

In the following, we will denote by $X \otimes Y$ the algebraic tensor product of X and Y, there exist two main topologies on $X \otimes Y$, the projective topology which is denoted by π, and the injective topology which is denoted by ε. Next, α will denote the injective or the projective topology. We will denote by $X \otimes_\alpha Y$ the tensor product of X and Y equipped with the topology α.

A seminorm r on $X \otimes Y$ is called a cross-semi-norm provided there exist continuous seminorms $p \in \Gamma_X$ and $q \in \Gamma_Y$ such that $r(x \otimes y) = p(x)\, q(y)$ for every $x \otimes y \in X \otimes Y$. The projective and the injective topologies are defined by the family of cross-semi-norms $\{p \otimes_\alpha q : p \in \Gamma_X,\, q \in \Gamma_Y\}$ where $p \otimes_\pi q$ and $p \otimes_\varepsilon q$ are the canonical cross-semi-norms on $X \otimes_\pi Y$ and $X \otimes_\varepsilon Y$, respectively, it is well known that for any $z \in X \otimes_\alpha Y$ we have $p \otimes_\pi q(z) \geq p \otimes_\varepsilon q(z)$. In the following $X \hat{\otimes}_\alpha Y$ will denote the completion of $X \otimes_\alpha Y$. We refer to [3] for more details about tensor product of locally convex spaces.

Next, we shall assume that the space $\mathcal{L}_s\left(X \hat{\otimes}_\alpha Y\right)$ is sequentially complete [15]. One should remark that a sufficient condition for the space of operators $\mathcal{L}_s\left(X \hat{\otimes}_\alpha Y\right)$ to be sequentially complete is that $X \hat{\otimes}_\alpha Y$ is barreled [16, III,4.6], [15].

Let us recall the following:

Lemma 3.1. *Let X be a locally convex Hausdorff space and M a closed subspace of X. Let $x_0 \in X \setminus M$, then there exist a seminorm p on X and $\varphi \in X'$ such that*

1. $\varphi(M) = 0$.
2. $|\varphi(x_0)| \neq 0$.
3. $\varphi \in U_p^0$, *where U_p^0 is the polar of $U_p = \{x \in X : p(x) \leq 1\}$.*

Proof. We consider the quotient space X/M. We have M is closed then X/M is a locally convex Hausdorff space. If Γ_X is a family of seminorms defining the topology of X, we consider $\hat{p} : X/M \to \mathbb{R}^+$ defined by $\hat{p}([x]) = \inf\limits_{y \in M} p(x+y)$, then the family $\bar{\Gamma} = \{\hat{p} : p \in \Gamma_X\}$ defines the topology of X/M. Let $\phi : X \to X/M$ be the canonical surjection defined by $\phi(x) = [x] = x + M$; ϕ is linear and continuous [16].

Let $x_0 \in X \setminus M$, then $\phi(x_0) \neq 0$, so there exists $\hat{p} \in \bar{\Gamma}$ such that $\hat{p}([x_0]) \neq 0$. Let $f_0 : \mathbb{C}[x_0] \to \mathbb{C}$ be defined by $f_0(\lambda [x_0]) = \lambda \hat{p}([x_0])$, f_0 is a linear form and we have $|f_0(\lambda [x_0])| = |\lambda \hat{p}([x_0])| = |\lambda|\, \hat{p}([x_0]) = \hat{p}(\lambda [x_0])$, therefore f_0 is continuous. Hence, the Hahn Banach extension theorem gives that f_0 can be extended to a continuous linear form $\bar{f} : X/M \to \mathbb{C}$ such that $\bar{f}_{|\mathbb{C}[x_0]} = f_0$ and $\left|\bar{f}([x])\right| \leq \hat{p}([x])$, $\forall [x] \in X/M$.

Let $\varphi = \bar{f} \circ \phi : X \to \mathbb{C}$; φ is linear and continuous and we have

1. For any $x \in M$, $\varphi(x) = \bar{f}(\phi(x)) = \bar{f}(0) = 0$, so $\varphi(M) = 0$.

2. $|\varphi(x_0)| = |\bar{f}(\phi(x_0))| = |\bar{f}([x_0])| = |f_0([x_0])| = \hat{p}([x_0]) \neq 0.$
3. If $x \in U_p$, then $p(x) \leq 1$ and we have
$$|\varphi(x)| = |\bar{f}([x])| \leq \hat{p}([x]) \leq p(x) \leq 1,$$
therefore $\varphi \in U_p^0$.

□

Definition 3.2 ([21])**.** For a given n-tuple $a = (a_1, ..., a_n)$ of pairwise commuting operators from $\mathcal{L}(X)$, we define
$$\{a\}^c := \{b \in \mathcal{L}(X) : ba_j = a_j b \text{ for } 1 \leq j \leq n\}$$
and
$$\{a\}^{cc} := \{b \in \mathcal{L}(X) : bc = cb \text{ for all } c \in \{a\}^c\}.$$
$\{a\}^c$ and $\{a\}^{cc}$ are called the commutant of a and the bicommutant of a, respectively.

It is clear that the bicommutant $\{a\}^{cc}$ is a commutative subalgebra of $\mathcal{L}(X)$.

Definition 3.3 ([20])**.** Let A denote an algebra with a unit element e over the complex numbers field $\mathbb{C}$, and let M be a subset of A. For $a_i \in A$ $(i = 1, .., n)$ denote by $\rho(a_1, ..., a_n; M)$ the set of all those $(\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ such that there exist $b_i \in M$ $(i = 1, .., n)$ with
$$\sum_{i=1}^{n} b_i(\lambda_i e - a_i) = e.$$

The set $\rho(a_1, ..., a_n; M)$ is called the joint resolvent of $(a_1, ..., a_n)$ with respect to M.

The set $\sigma(a_1, ..., a_n; M) = \mathbb{C}^n \setminus \rho(a_1, ..., a_n; M)$ is called the joint spectrum of $(a_1, ..., a_n)$ with respect to M.

Next, the spectrum of an element x of A, denoted $\sigma(x, A)$, is the set of complex numbers for which $\lambda e - x$ is not invertible in A.

Let us recall the spectral mapping theorem for joint spectra in Banach algebras.

Theorem 3.4 ([6])**.** *Let A be a commutative unital Banach algebra and let $a_i \in A$ $(i = 1, .., n)$ and let $P(z_1, ..., z_n)$ be a polynomial in n variables then we have*
$$P[\sigma(a_1, ..., a_n; A)] = \sigma(P(a_1, ..., a_n), A).$$

The following proposition is proved with the same method as Proposition 4 in [3].

Proposition 3.5. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, if there exist two calibrations Γ_X and Γ_Y on X and Y, respectively, such that $T \in B_{\Gamma_X}(X)$ and $S \in B_{\Gamma_Y}(Y)$, then $T\hat{\otimes}_\alpha S \in B_{\Gamma_X \hat{\otimes}_\alpha \Gamma_Y}(X\hat{\otimes}_\alpha Y)$.*

The following theorem characterizes the spectrum of operators of the form $T\hat{\otimes}_{\alpha}I$ with $T \in \mathcal{L}(X)$ in terms of the spectrum of T; however, it is not a straightforward corollary of the previous proposition. It is also crucial to prove Theorem 3.10.

Theorem 3.6. *Let $T \in \mathcal{L}(X)$, then we have*

$$\sigma\left(T\hat{\otimes}_{\alpha}I, B_{\Gamma_X\hat{\otimes}_{\alpha}\Gamma_Y}\left(X\hat{\otimes}_{\alpha}Y\right)\right) = \sigma\left(T, B_{\Gamma_X}(X)\right).$$

Proof. First of all, we show that $\sigma\left(T\hat{\otimes}_{\alpha}I, B_{\Gamma_X\hat{\otimes}_{\alpha}\Gamma_Y}\left(X\hat{\otimes}_{\alpha}Y\right)\right) \subseteq \sigma\left(T, B_{\Gamma_X}(X)\right)$.

Let $\lambda \in \rho\left(T, B_{\Gamma_X}(X)\right)$, then $\lambda I - T$ is invertible and $(\lambda I - T)^{-1} \in B_{\Gamma_X}(X)$, and we have $I \in B_{\Gamma_Y}(Y)$, therefore by Proposition 3.5, $(\lambda I - T)^{-1}\hat{\otimes}_{\alpha}I \in B_{\Gamma_X\hat{\otimes}_{\alpha}\Gamma_Y}\left(X\hat{\otimes}_{\alpha}Y\right)$, and we have:

$$\begin{aligned}\left((\lambda I - T)^{-1}\hat{\otimes}_{\alpha}I\right)\left(\lambda I\hat{\otimes}_{\alpha}I - T\hat{\otimes}_{\alpha}I\right) &= \left(\lambda I\hat{\otimes}_{\alpha}I - T\hat{\otimes}_{\alpha}I\right)\left((\lambda I - T)^{-1}\hat{\otimes}_{\alpha}I\right)\\ &= I\hat{\otimes}_{\alpha}I.\end{aligned}$$

Hence, $\lambda \in \rho\left(T\hat{\otimes}_{\alpha}I, B_{\Gamma_X\hat{\otimes}_{\alpha}\Gamma_Y}\left(X\hat{\otimes}_{\alpha}Y\right)\right)$.

For the reverse inclusion, let $\lambda \in \sigma\left(T, B_{\Gamma_X}(X)\right) = \sigma_{ap}(T, \Gamma_X)\mathring{\cup}\sigma_r(T, \Gamma_X)$, then there are two cases.

If $\lambda \in \sigma_{ap}(T, \Gamma_X)$, then for any $\varepsilon > 0$ there exist $x \in X$ and $p \in \Gamma_X$ such that

$$p\left((\lambda I - T)x\right) < \varepsilon p(x).$$

Let $y \in Y \setminus \{0\}$ then there exists $q \in \Gamma_Y$ such that $q(y) \neq 0$, so one obtains

$$\begin{aligned}p\hat{\otimes}_{\alpha}q\left(\left(\lambda I\hat{\otimes}_{\alpha}I - T\hat{\otimes}_{\alpha}I\right)(x\otimes y)\right) &= p\hat{\otimes}_{\alpha}q\left(\left(\lambda I\otimes_{\alpha}I - T\otimes_{\alpha}I\right)(x\otimes y)\right)\\ &= p\otimes_{\alpha}q\left((\lambda I - T)x\otimes_{\alpha}y\right)\\ &= p\left((\lambda I - T)x\right)q(y)\\ &< \varepsilon p(x)q(y)\\ &= \varepsilon p\hat{\otimes}_{\alpha}q(x\otimes y),\end{aligned}$$

therefore, $\lambda \in \sigma_{ap}\left(T\hat{\otimes}_{\alpha}I, \Gamma_X\hat{\otimes}_{\alpha}\Gamma_Y\right)$.

Now, if $\lambda \in \sigma_r(T, \Gamma_X)$ then $\mathrm{Im}(\lambda I - T)$ is not dense in (X, Γ_X). By Lemma 3.1, we can find $\varphi \in X'$ such that $\varphi\left(\mathrm{Im}(\lambda I - T)\right) = 0$. Then we have, for any $x \in X$, $0 = \langle(\lambda I - T)x, \varphi\rangle = \langle x, (\lambda I' - T')\varphi\rangle$. Therefore $(\lambda I' - T')\varphi = 0 \in X'$, which is equivalent to $T'\varphi = \lambda\varphi$, and hence $\lambda \in \sigma_p(T')$.

Let $\psi \in Y' \setminus \{0\}$ arbitrary, then for all $x \otimes y \in X \otimes Y$

$$\begin{aligned}\left\langle x\otimes y, \left(T\hat{\otimes}_{\alpha}I\right)'\varphi\otimes\psi\right\rangle &= \langle Tx\otimes y, \varphi\otimes\psi\rangle = \langle Tx, \varphi\rangle\langle y, \psi\rangle\\ &= \langle x, T'\varphi\rangle\langle y, \psi\rangle = \langle x, \lambda\varphi\rangle\langle y, \psi\rangle\\ &= \lambda\langle x, \varphi\rangle\langle y, \psi\rangle = \lambda\langle x\otimes y, \varphi\otimes\psi\rangle\end{aligned}$$

then by linearity, we get for all $z \in X \otimes Y$

$$\left\langle z, \left(T\hat{\otimes}_{\alpha}I\right)'\varphi\otimes\psi\right\rangle = \lambda\langle z, \varphi\otimes\psi\rangle.$$

Let $\varphi\hat{\otimes}_\alpha\psi$ be the continuous extension of $\varphi\otimes_\alpha\psi$ to $X\hat{\otimes}_\alpha Y$, we have $\varphi\hat{\otimes}_\alpha\psi \in \left(X\hat{\otimes}_\alpha Y\right)'$ and for all $z \in X\otimes_\alpha Y$

$$\left\langle z, \left(T\hat{\otimes}_\alpha I\right)' \varphi\hat{\otimes}_\alpha\psi\right\rangle = \lambda \left\langle z, \varphi\hat{\otimes}_\alpha\psi\right\rangle.$$

By density of $X\otimes_\alpha Y$ in $X\hat{\otimes}_\alpha Y$, it follows that for any $z \in X\hat{\otimes}_\alpha Y$

$$\begin{aligned}\left\langle z, \left(T\hat{\otimes}_\alpha I\right)' \varphi\hat{\otimes}_\alpha\psi\right\rangle &= \lambda \left\langle z, \varphi\hat{\otimes}_\alpha\psi\right\rangle \\ &= \left\langle z, \lambda\varphi\hat{\otimes}_\alpha\psi\right\rangle.\end{aligned}$$

Finally, we have found $\varphi\hat{\otimes}_\alpha\psi \in \left(X\hat{\otimes}_\alpha Y\right)'$ such that

$$\left(T\hat{\otimes}_\alpha I\right)' \varphi\hat{\otimes}_\alpha\psi = \lambda\varphi\hat{\otimes}_\alpha\psi,$$

therefore $\lambda \in \sigma_p\left(\left(T\hat{\otimes}_\alpha I\right)'\right)$, but we have $\sigma_p\left(\left(T\hat{\otimes}_\alpha I\right)'\right) \subset \sigma\left(\left(T\hat{\otimes}_\alpha I\right)'\right) \subset \sigma\left(T\hat{\otimes}_\alpha I\right)$ and $\sigma\left(T\hat{\otimes}_\alpha I\right) \subset \sigma\left(T\hat{\otimes}_\alpha I, B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)$.

Hence, $\lambda \in \sigma\left(T\hat{\otimes}_\alpha I, B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)$. □

Lemma 3.7. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, and let $\mathfrak{B}$ denote the bicommutant of $T\hat{\otimes}_\alpha I$ and $I\hat{\otimes}_\alpha S$ in the algebra $\mathcal{L}\left(X\hat{\otimes}_\alpha Y\right)$, then we have*

$$\sigma_{ap}\left(T, \Gamma_X\right) \times \sigma_{ap}\left(S, \Gamma_Y\right) \subseteq \sigma\left(T\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S; \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right).$$

Proof. If $(\lambda, \mu) \in \sigma_{ap}\left(T, \Gamma_X\right) \times \sigma_{ap}\left(S, \Gamma_Y\right)$.

Suppose that $(\lambda, \mu) \in \rho\left(T\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S; \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)$, then there exist $b_1, b_2 \in \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)$ such that

$$I\hat{\otimes}_\alpha I = b_1\left(\lambda I\hat{\otimes}_\alpha I - T\hat{\otimes}_\alpha I\right) + b_2\left(\mu I\hat{\otimes}_\alpha I - I\hat{\otimes}_\alpha S\right). \tag{3.1}$$

The fact that $b_1, b_2 \in \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)$ gives that for all $p\hat{\otimes}_\alpha q \in \Gamma_X\hat{\otimes}_\alpha\Gamma_Y$, there exist $C_1, C_2 > 0$ such that

$$\begin{cases} p\hat{\otimes}_\alpha q\left(b_1 z\right) \leq C_1 p\hat{\otimes}_\alpha q\left(z\right) \\ p\hat{\otimes}_\alpha q\left(b_2 z\right) \leq C_2 p\hat{\otimes}_\alpha q\left(z\right)\end{cases}, \forall z \in X\hat{\otimes}_\alpha Y \tag{3.2}$$

Let $0 < \varepsilon < \left[4\left(C_1 + C_2\right)\right]^{-1}$, we have $(\lambda, \mu) \in \sigma_{ap}\left(T, \Gamma_X\right) \times \sigma_{ap}\left(S, \Gamma_Y\right)$ then there exist $\left(x_0, y_0\right) \in X \times Y$ and $\left(p_0, q_0\right) \in \Gamma_X \times \Gamma_Y$ such that

$$\begin{cases} p_0\left(\left(\lambda I - T\right) x_0\right) < \varepsilon p_0\left(x_0\right) \\ q_0\left(\left(\mu I - S\right) y_0\right) < \varepsilon q_0\left(y_0\right).\end{cases} \tag{3.3}$$

Then by applying successively (3.1), (3.2) and (3.3), we get

$$\begin{aligned}
p_0 \otimes_\alpha q_0 (x_0 \otimes y_0) &= p_0 \otimes_\alpha q_0 \begin{pmatrix} b_1 \left(\lambda I \hat{\otimes}_\alpha I - T \hat{\otimes}_\alpha I\right)(x_0 \otimes y_0) \\ +b_2 \left(\mu I \hat{\otimes}_\alpha I - I \hat{\otimes}_\alpha S\right)(x_0 \otimes y_0) \end{pmatrix} \\
&\leq p_0 \otimes_\alpha q_0 \left(b_1 \left(\lambda I \hat{\otimes}_\alpha I - T \hat{\otimes}_\alpha I\right)(x_0 \otimes y_0)\right) \\
&\quad + p_0 \otimes_\alpha q_0 \left(b_2 \left(\mu I \hat{\otimes}_\alpha I - I \hat{\otimes}_\alpha S\right)(x_0 \otimes y_0)\right) \\
&\leq C_1 p_0 \otimes_\alpha q_0 \left[(\lambda I - T) x_0 \otimes y_0\right] \\
&\quad + C_2 p_0 \otimes_\alpha q_0 \left[x_0 \otimes_\alpha (\mu - S) y_0\right] \\
&= C_1 p_0 \left[(\lambda I - T) x_0\right] q_0 (y_0) + C_2 p_0 (x_0) q_0 \left[(\mu - S) y_0\right] \\
&< C_1 \varepsilon p_0 (x_0) q_0 (y_0) + C_2 \varepsilon p_0 (x_0) q_0 (y_0) \\
&= \varepsilon (C_1 + C_2) p_0 \otimes_\alpha q_0 (x_0 \otimes y_0) \\
&\leq 4^{-1} p_0 \otimes_\alpha q_0 (x_0 \otimes y_0).
\end{aligned}$$

Finally, we have found that there exist $x_0 \otimes y_0 \in X \otimes Y$ and $p_0 \otimes_\alpha q_0 \in \Gamma_X \times \Gamma_Y$ such that

$$p_0 \otimes_\alpha q_0 (x_0 \otimes y_0) < 4^{-1} p_0 \otimes_\alpha q_0 (x_0 \otimes y_0),$$

and this is a contradiction. □

Lemma 3.8. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, and let $\mathfrak{B}$ denote the bicommutant of $T \hat{\otimes}_\alpha I$ and $I \hat{\otimes}_\alpha S$ in the algebra $\mathcal{L}\left(X \hat{\otimes}_\alpha Y\right)$, then we have*

$$\sigma_r (T, \Gamma_X) \times \sigma_r (S, \Gamma_Y) \subseteq \sigma\left(T \hat{\otimes}_\alpha I, I \hat{\otimes}_\alpha S; \mathfrak{B} \cap B_{\Gamma_X \hat{\otimes}_\alpha \Gamma_Y}\left(X \hat{\otimes}_\alpha Y\right)\right).$$

Proof. If $(\lambda, \mu) \in \sigma_r (T, \Gamma_X) \times \sigma_r (S, \Gamma_Y)$.

Then $\operatorname{Im}(\lambda I - T)$ and $\operatorname{Im}(\mu I - S)$ are not dense in X and Y, respectively. Then by Lemma 3.1 and with a slight modification, we can find $\varphi \in X'$ and $x_0 \in X$ such that

$$\varphi\left(\operatorname{Im}(\lambda I - T)\right) = 0 \text{ and } \varphi(x_0) = 1, \tag{3.4}$$

and similarly, we find $\psi \in Y'$ and $y_0 \in Y$ such that

$$\psi\left(\operatorname{Im}(\mu I - S)\right) = 0 \text{ and } \psi(y_0) = 1. \tag{3.5}$$

Therefore, because of density we have:

$$\varphi \hat{\otimes}_\alpha \psi \left[\left((\lambda I - T) \hat{\otimes}_\alpha I\right)\left(X \hat{\otimes}_\alpha Y\right)\right] = 0 \tag{3.6}$$

and

$$\varphi \hat{\otimes}_\alpha \psi \left[\left(I \hat{\otimes}_\alpha (\mu I - S)\right)\left(X \hat{\otimes}_\alpha Y\right)\right] = 0. \tag{3.7}$$

Indeed, for $z = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$ we have

$$\begin{aligned}\varphi\hat{\otimes}_\alpha\psi\left[\left((\lambda I - T)\,\hat{\otimes}_\alpha I\right)(z)\right] &= \varphi \otimes_\alpha \psi\left[\left((\lambda I - T) \otimes_\alpha I\right)(z)\right] \\ &= \varphi \otimes_\alpha \psi\left[\sum_{i=1}^{n} (\lambda I - T)\, x_i \otimes_\alpha y_i\right] \\ &= \sum_{i=1}^{n} \varphi\left((\lambda I - T)\, x_i\right)\psi\left(y_i\right),\end{aligned}$$

but we have $\varphi\left(\operatorname{Im}\left(\lambda I - T\right)\right) = 0$, so for all $z \in X \otimes Y$

$$\varphi\hat{\otimes}_\alpha\psi\left[\left((\lambda I - T)\,\hat{\otimes}_\alpha I\right)(z)\right] = 0,$$

and by the density of $X \otimes Y$ in $X\hat{\otimes}_\alpha Y$ we get for all $z \in X\hat{\otimes}_\alpha Y$

$$\varphi\hat{\otimes}_\alpha\psi\left[\left((\lambda I - T)\,\hat{\otimes}_\alpha I\right)(z)\right] = 0.$$

Hence,

$$\varphi\hat{\otimes}_\alpha\psi\left[\left((\lambda I - T)\,\hat{\otimes}_\alpha I\right)\left(X\hat{\otimes}_\alpha Y\right)\right] = 0$$

and similarly, we show that

$$\varphi\hat{\otimes}_\alpha\psi\left[\left(I\hat{\otimes}_\alpha\left(\mu I - S\right)\right)\left(X\hat{\otimes}_\alpha Y\right)\right] = 0.$$

Now, if we suppose that $(\lambda, \mu) \in \rho\left(T\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S; \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)$, then there exist b_1, $b_2 \in \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)$ such that

$$I\hat{\otimes}_\alpha I = b_1\left(\lambda I\hat{\otimes}_\alpha I - T\hat{\otimes}_\alpha I\right) + b_2\left(\mu I\hat{\otimes}_\alpha I - I\hat{\otimes}_\alpha S\right),$$

then

$$\begin{aligned}x_0 \otimes y_0 &= b_1\left(\lambda I\hat{\otimes}_\alpha I - T\hat{\otimes}_\alpha I\right)\left(x_0 \otimes y_0\right) + b_2\left(\mu I\hat{\otimes}_\alpha I - I\hat{\otimes}_\alpha S\right)\left(x_0 \otimes y_0\right) \\ &= b_1\left[(\lambda I - T)\,\hat{\otimes}_\alpha I\right]\left(x_0 \otimes y_0\right) + b_2\left[I\hat{\otimes}_\alpha\left(\mu I - S\right)\right]\left(x_0 \otimes y_0\right) \\ &= b_1\left[(\lambda I - T)\, x_0 \otimes y_0\right] + b_2\left[x_0 \otimes \left(\mu I - S\right) y_0\right].\end{aligned}$$

We apply $\varphi\hat{\otimes}_\alpha\psi$ to both sides of the equality.
For the left side, by using (3.4) and (3.5) we get:

$$\varphi\hat{\otimes}_\alpha\psi\left(x_0 \otimes y_0\right) = \varphi \otimes_\alpha \psi\left(x_0 \otimes y_0\right) = \varphi\left(x_0\right)\psi\left(y_0\right) = 1.$$

For the right side:
we have $(\lambda I - T)\,\hat{\otimes}_\alpha I$ and $I\hat{\otimes}_\alpha\left(\mu I - S\right)$ are in the commutant of $\left\{T\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha T\right\}$, and since b_1, $b_2 \in \mathfrak{B}$, we have

$$\begin{aligned}&b_1\left[(\lambda I - T)\, x_0 \otimes y_0\right] + b_2\left[x_0 \otimes \left(\mu I - S\right) y_0\right] \\ &= b_1\left[(\lambda I - T)\,\hat{\otimes}_\alpha I\right]\left(x_0 \otimes y_0\right) + b_2\left[I\hat{\otimes}_\alpha\left(\mu I - S\right)\right]\left(x_0 \otimes y_0\right) \\ &= \left[(\lambda I - T)\,\hat{\otimes}_\alpha I\right] b_1\left(x_0 \otimes y_0\right) + \left[I\hat{\otimes}_\alpha\left(\mu I - S\right)\right] b_2\left(x_0 \otimes y_0\right).\end{aligned}$$

Therefore, by using (3.6) and (3.7) we have

$$\begin{aligned}
&\varphi\hat{\otimes}_\alpha\psi\left(b_1\left[(\lambda I-T)x_0\otimes y_0\right]+b_2\left[x_0\otimes(\mu I-S)y_0\right]\right)\\
&=\varphi\hat{\otimes}_\alpha\psi\left(\left[(\lambda I-T)\hat{\otimes}_\alpha I\right]b_1\left(x_0\otimes y_0\right)+\left[I\hat{\otimes}_\alpha(\mu I-S)\right]b_2\left(x_0\otimes y_0\right)\right)\\
&=\varphi\hat{\otimes}_\alpha\psi\left(\left[(\lambda I-T)\hat{\otimes}_\alpha I\right]b_1\left(x_0\otimes y_0\right)\right)+\varphi\hat{\otimes}_\alpha\psi\left(\left[I\hat{\otimes}_\alpha(\mu I-S)\right]b_2\left(x_0\otimes y_0\right)\right)\\
&=0.
\end{aligned}$$

Finally, we get $1=0$, which is a contradiction. □

Lemma 3.9. *Let $T\in\mathcal{L}(X)$ and $S\in\mathcal{L}(Y)$, and let $\mathfrak{B}$ denote the bicommutant of $T\hat{\otimes}_\alpha I$ and $I\hat{\otimes}_\alpha S$ in the algebra $\mathcal{L}\left(X\hat{\otimes}_\alpha Y\right)$, then we have*

$$\sigma_{ap}(T,\Gamma_X)\times\sigma_r(S,\Gamma_Y)\subseteq\sigma\left(T\hat{\otimes}_\alpha I,I\hat{\otimes}_\alpha S;\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right),$$

and

$$\sigma_r(T,\Gamma_X)\times\sigma_{ap}(S,\Gamma_Y)\subseteq\sigma\left(T\hat{\otimes}_\alpha I,I\hat{\otimes}_\alpha S;\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right).$$

Proof. We will only treat the first inclusion since the second inclusion is similar because of symmetry.

Let $(\lambda,\mu)\in\sigma_{ap}(T,\Gamma_X)\times\sigma_r(S,\Gamma_Y)$.

Suppose that $(\lambda,\mu)\in\rho\left(T\hat{\otimes}_\alpha I,I\hat{\otimes}_\alpha S;\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)$, then there exist b_1, $b_2\in\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)$ such that

$$I\hat{\otimes}_\alpha I=b_1\left(\lambda I\hat{\otimes}_\alpha I-T\hat{\otimes}_\alpha I\right)+b_2\left(\mu I\hat{\otimes}_\alpha I-I\hat{\otimes}_\alpha S\right).$$

Let $\varepsilon>0$, we have $\lambda\in\sigma_{ap}(T,\Gamma_X)$, then there exist $x_0\in X$ and $p_0\in\Gamma_X$ such that

$$p_0(\lambda x_0-Tx_0)<\varepsilon p_0(x_0),$$

by Hahn Banach theorem, there exists $\varphi\in(U_{p_0})^0$ such that $|\varphi(x_0)|=p_0(x_0)$ and $|\varphi(x)|\le p_0(x)$ for all $x\in X$.

We have $\mu\in\sigma_r(S,\Gamma_Y)$, then $\operatorname{Im}(\mu I-S)$ is not dense in Y. Let $y_0\in Y\backslash\overline{\operatorname{Im}(\mu I-S)}$, by Lemma 3.1 there exists $\psi\in Y'$ such that $\psi(\operatorname{Im}(\mu I-S))=0$ and $|\psi(y_0)|=c>0$ and a seminorm $q_0\in\Gamma_Y$ such that $\psi\in(U_{q_0})^0$. Similarly to (3.7), we have

$$\varphi\hat{\otimes}_\alpha\psi\left[\left(I\hat{\otimes}_\alpha(\mu I-S)\right)\left(X\hat{\otimes}_\alpha Y\right)\right]=0.$$

Thus, we get

$$\begin{aligned}
&p_0(x_0)c\\
&=|\varphi(x_0)||\psi(y_0)|=|\varphi(x_0)\psi(y_0)|=|\varphi\otimes_\alpha\psi(x_0\otimes y_0)|=\left|\varphi\hat{\otimes}_\alpha\psi(x_0\otimes y_0)\right|\\
&=\left|\varphi\hat{\otimes}_\alpha\psi\left(b_1\left[(\lambda I-T)\hat{\otimes}_\alpha I\right](x_0\otimes y_0)\right)+\varphi\hat{\otimes}_\alpha\psi\left(b_2\left[I\hat{\otimes}_\alpha(\mu I-S)\right](x_0\otimes y_0)\right)\right|\\
&=\left|\varphi\hat{\otimes}_\alpha\psi\left(b_1\left[(\lambda I-T)\hat{\otimes}_\alpha I\right](x_0\otimes y_0)\right)+\varphi\hat{\otimes}_\alpha\psi\left(\left[I\hat{\otimes}_\alpha(\mu I-S)\right]b_2(x_0\otimes y_0)\right)\right|\\
&=\left|\varphi\hat{\otimes}_\alpha\psi\left(b_1\left[(\lambda I-T)\hat{\otimes}_\alpha I\right](x_0\otimes y_0)\right)\right|,
\end{aligned}$$

then if $\alpha = \varepsilon$ we get

$$\begin{aligned} p_0(x_0)\, c &= \left|\varphi\hat{\otimes}_\varepsilon\psi\left(b_1\left[(\lambda I - T)\,\hat{\otimes}_\alpha I\right](x_0 \otimes y_0)\right)\right| \\ &= \left|\varphi\hat{\otimes}_\varepsilon\psi\left(b_1\left[(\lambda I - T)\, x_0 \otimes y_0\right]\right)\right| \\ &\leq \sup_{\substack{\varphi\in U^0_{p_0}\\ \psi\in U^0_{q_0}}} \left\{\left|\varphi\hat{\otimes}_\varepsilon\psi\left(b_1\left[(\lambda I - T)\, x_0 \otimes y_0\right]\right)\right|\right\} \\ &\leq p_0\hat{\otimes}_\varepsilon q_0\left(b_1\left[(\lambda I - T)\, x_0 \otimes y_0\right]\right) \\ &\leq p_0\hat{\otimes}_\pi q_0\left(b_1\left[(\lambda I - T)\, x_0 \otimes y_0\right]\right). \end{aligned}$$

Therefore, for $\alpha = \pi$ and $\alpha = \varepsilon$ we have

$$\begin{aligned} p_0(x_0)\, c &\leq p_0\hat{\otimes}_\alpha q_0\left(b_1\left[(\lambda I - T)\, x_0 \otimes y_0\right]\right) \\ &\leq \|b_1\|_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\, p_0\hat{\otimes}_\alpha q_0\left(\left[(\lambda I - T)\, x_0 \otimes y_0\right]\right) \\ &\leq \|b_1\|_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\, p_0\left((\lambda I - T)\, x_0\right) q_0(y_0) \\ &< \|b_1\|_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\, \varepsilon p_0(x_0)\, q_0(y_0) = \|b_1\|_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\, \varepsilon p_0(x_0)\, c. \end{aligned}$$

Thus, taking into consideration that $c \neq 0$, we have that for all $\varepsilon > 0$ there exist $x_0 \in X$, $p_0 \in \Gamma_X$ such that

$$p_0(x_0) < \|b_1\|_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\, \varepsilon p_0(x_0).$$

We have $\|b_1\|_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y} \neq 0$, if not we obtain $p_0(x_0) < 0$, which is a contradiction so we can take $0 < \varepsilon < \left(2\,\|b_1\|_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\right)^{-1}$. Therefore,

$$p_0(x_0) < \frac{p_0(x_0)}{2}.$$

Contradiction. □

Theorem 3.10. *Let $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, and let $\mathfrak{B}$ denote the bicommutant of $T\hat{\otimes}_\alpha I$ and $I\hat{\otimes}_\alpha S$ in the algebra $\mathcal{L}\left(X\hat{\otimes}_\alpha Y\right)$, then we have*

$$\begin{aligned} &\sigma\left(T\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S; \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right) \\ &= \sigma\left(T\hat{\otimes}_\alpha I; \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right) \times \sigma\left(I\hat{\otimes}_\alpha S; \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right) \\ &= \sigma\left(T, B_{\Gamma_X}(X)\right) \times \sigma\left(T, B_{\Gamma_X}(X)\right). \end{aligned}$$

Proof. Let $(\lambda, \mu) \in \rho\left(T\hat{\otimes}_\alpha I; \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right) \times \rho\left(I\hat{\otimes}_\alpha S; \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)$, by Theorem 3.6, we have $(\lambda, \mu) \in \rho\left(T; B_{\Gamma_X}(X)\right) \times \rho\left(S; B_{\Gamma_Y}(Y)\right)$, then $(\lambda I - T)^{-1}$ exists, but we have

$$\begin{aligned} (\lambda I - T)^{-1}\,\hat{\otimes}_\alpha I &= \left((\lambda I - T)^{-1}\,\hat{\otimes}_\alpha I\right)\left((\lambda I - T)\,\hat{\otimes}_\alpha I\right)\left((\lambda I - T)\,\hat{\otimes}_\alpha I\right)^{-1} \\ &= \left((\lambda I - T)\,\hat{\otimes}_\alpha I\right)^{-1} \\ &= \left(\lambda I\hat{\otimes}_\alpha I - T\hat{\otimes}_\alpha I\right)^{-1}, \end{aligned}$$

then

$$(\lambda I - T)^{-1} \hat{\otimes}_\alpha I \in \mathfrak{B} \cap B_{\Gamma_X \hat{\otimes}_\alpha \Gamma_Y} \left(X \hat{\otimes}_\alpha Y\right).$$

With the same procedure, we get

$$I \hat{\otimes}_\alpha (\mu I - S)^{-1} \in \mathfrak{B} \cap B_{\Gamma_X \hat{\otimes}_\alpha \Gamma_Y} \left(X \hat{\otimes}_\alpha Y\right).$$

We can take $b_1 = (\lambda I - T)^{-1} \hat{\otimes}_\alpha I$ and $b_2 = 0$ then

$$b_1 \left(\lambda I \hat{\otimes}_\alpha I - T \hat{\otimes}_\alpha I\right) + b_2 \left(\mu I \hat{\otimes}_\alpha I - I \hat{\otimes}_\alpha S\right) = I \hat{\otimes}_\alpha I.$$

Hence, $(\lambda, \mu) \in \rho \left(T \hat{\otimes}_\alpha I, I \hat{\otimes}_\alpha S; \mathfrak{B} \cap B_{\Gamma_X \hat{\otimes}_\alpha \Gamma_Y} \left(X \hat{\otimes}_\alpha Y\right)\right)$.

For the inverse, from Theorem 3.6, we have to show that:

$$\sigma\left(T, B_{\Gamma_X}(X)\right) \times \sigma\left(S, B_{\Gamma_Y}(Y)\right) \subseteq \sigma\left(T \hat{\otimes}_\alpha I, I \hat{\otimes}_\alpha S; \mathfrak{B} \cap B_{\Gamma_X \hat{\otimes}_\alpha \Gamma_Y} \left(X \hat{\otimes}_\alpha Y\right)\right).$$

Let $(\lambda, \mu) \in \sigma\left(T, B_{\Gamma_X}(X)\right) \times \sigma\left(S, B_{\Gamma_Y}(Y)\right)$. There are four cases:

1) $(\lambda, \mu) \in \sigma_{ap}(T, \Gamma_X) \times \sigma_{ap}(S, \Gamma_Y)$.
2) $(\lambda, \mu) \in \sigma_r(T, \Gamma_X) \times \sigma_r(S, \Gamma_Y)$.
3) $(\lambda, \mu) \in \sigma_{ap}(T, \Gamma_X) \times \sigma_r(S, \Gamma_Y)$.
4) $(\lambda, \mu) \in \sigma_r(T, \Gamma_X) \times \sigma_{ap}(S, \Gamma_Y)$.

Then the result follows from Lemmas 3.7, 3.8 and 3.9. □

Remark 3.11. Let $p \hat{\otimes}_\alpha q \in \Gamma$ with $\Gamma = \Gamma_X \hat{\otimes}_\alpha \Gamma_Y$, and let $r_1 \in \Gamma_1$ and $r_2 \in \Gamma_2$ as defined in [20, Corollary 2.4.]. Let $x \in X$ and $y \in Y$, we have $r_1(x) = p \hat{\otimes}_\alpha q (x \otimes x_2) = p(x) q(x_2)$ and similarly $r_2(y) = p(x_1) q(y)$.

From [20, Corollary 2.4.] we have $x_1 \otimes x_2 \neq 0$ then there exist $p \in \Gamma_X$ and $q \in \Gamma_Y$ such that $p(x_1) \neq 0$ and $q(x_2) \neq 0$.

If we add the hypothesis that: $\forall p \in \Gamma_X, p(x_1) \neq 0$ and $\forall q \in \Gamma_Y, q(x_2) \neq 0$, then from Definition 2.6 we have $\Gamma_1 \sim \Gamma_X$ and $\Gamma_2 \sim \Gamma_Y$. Therefore from Proposition 2.7 we get $B_{\Gamma_1}(X) = B_{\Gamma_X}(X)$ and $\|.\|_{\Gamma_1} = \|.\|_{\Gamma_X}$ and likewise $B_{\Gamma_2}(Y) = B_{\Gamma_Y}(Y)$ and $\|.\|_{\Gamma_2} = \|.\|_{\Gamma_Y}$, so under this assumption, we can deduce our result from the result of Wrobel [20, Corollary 2.4.]. But otherwise, we cannot deduce the previous result directly.

4. A spectral inclusion theorem for tensor product of semigroups

Lemma 4.1 ([13, 5]). *Let $(T(s))_{s \geq 0}$ be a semigroup on X and let Γ_X a calibration on X. The semigroup $(T(s))_{s \geq 0}$ of linear operators is equicontinuous if and only if there is a calibration $\tilde{\Gamma}_X$ for X such that $(T(s))_{s \geq 0} \subset B_{\tilde{\Gamma}_X}(X)$.*

Remark 4.2 ([5]). The calibration $\tilde{\Gamma}_X$ in the previous Lemma is defined by

$$\tilde{\Gamma}_X := \{\tilde{p} : p \in \Gamma_X\}$$

where for each $p \in \Gamma_X$, $\tilde{p}$ is defined on X by

$$\tilde{p}(x) := \sup_{s \geq 0} p(T(s)x), \quad x \in X.$$

$\tilde{p}$ is well defined, is a seminorm and $\tilde{\Gamma}_X := \{\tilde{p} : p \in \Gamma_X\}$ is also a system of continuous seminorms generating the topology of X, with the additional property that

$$\tilde{p}\left(T\left(s\right)x\right) = \sup_{t\geq 0} p\left(T\left(t\right)T\left(s\right)x\right) = \sup_{t\geq 0} p\left(T\left(t+s\right)x\right) \leq \tilde{p}(x)\ ,\ x \in X,\ s \geq 0.$$

Now, let $\left(T\left(s\right)\hat{\otimes}_\alpha S\left(t\right)\right)_{s,t\geq 0}$ be an equicontinuous tensor product of semigroups on $X\hat{\otimes}_\alpha Y$. Then $(T\left(s\right))_{s\geq 0}$ and $(S\left(t\right))_{t\geq 0}$ are one parameter equicontinuous semigroups on X and Y [3].

Then Lemma 4.1 gives that there exist two calibrations Γ_X and Γ_Y such that $(T\left(s\right))_{s\geq 0} \subset B_{\Gamma_X}\left(X\right)$ and $(S\left(t\right))_{t\geq 0} \subset B_{\Gamma_Y}\left(Y\right)$, ie, for all $p \in \Gamma_X$ and $q \in \Gamma_X$, there exist $c_1, c_2 > 0$ such that $p\left(T\left(s\right)x\right) \leq c_1 p\left(x\right)$ and $q\left(S\left(t\right)y\right) \leq c_2 q\left(y\right)$ for all $x \in X$, $y \in Y$, $s \geq 0$ and $t \geq 0$. Then applying the same steps of Theorem 2 in [3], we get the following result.

Theorem 4.3 ([3])**.** *Let $\left(T\left(s\right)\hat{\otimes}_\alpha S\left(t\right)\right)_{s,t\geq 0}$ be an equicontinuous tensor product of semigroups on $X\hat{\otimes}_\alpha Y$, then there exist two calibrations Γ_X and Γ_Y on X and Y, respectively, such that the following are equivalent :*

1. $\left(T\left(s\right)\hat{\otimes}_\alpha S\left(t\right)\right)_{s,t\geq 0} \subset B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right).$
2. $\left(T\left(s\right)\hat{\otimes}_\alpha I\right)_{s\geq 0}$ *and* $\left(I\hat{\otimes}_\alpha S\left(t\right)\right)_{t\geq 0} \subset B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right).$
3. $\left(T\left(s\right)\right)_{s\geq 0} \subset B_{\Gamma_X}\left(X\right)$ *and* $\left(S\left(t\right)\right)_{t\geq 0} \subset B_{\Gamma_Y}\left(Y\right).$

Theorem 4.4. *Let $\left(T\left(s\right)\hat{\otimes}_\alpha S\left(t\right)\right)_{s,t\geq 0}$ be an equicontinuous tensor product of semigroups on $X\hat{\otimes}_\alpha Y$, and let $\mathfrak{B}$ denote the bicommutant of $\left(T\left(s\right)\hat{\otimes}_\alpha I\right)_{s\geq 0}$ and $\left(I\hat{\otimes}_\alpha S\left(t\right)\right)_{t\geq 0}$ in the algebra $\mathcal{L}_s\left(X\hat{\otimes}_\alpha Y\right)$, then there exist two calibrations Γ_X and Γ_Y on X and Y, respectively, such that*

$$\begin{aligned}&\sigma\left(T\left(s\right)\hat{\otimes}_\alpha S\left(t\right), \mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)\\ &\quad= \sigma\left(T\left(s\right), B_{\Gamma_X}\left(X\right)\right)\times\sigma\left(S\left(t\right), B_{\Gamma_Y}\left(Y\right)\right).\end{aligned}$$

Proof. From Theorems 3.10 and 4.3, we have that there exist two calibrations Γ_X and Γ_Y on X and Y, respectively, such that

$$\begin{aligned}&\sigma\left(T\left(s\right)\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S\left(t\right); \mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)\\ &\quad= \sigma\left(T\left(s\right), B_{\Gamma_X}\left(X\right)\right)\times\sigma\left(S\left(t\right), B_{\Gamma_Y}\left(Y\right)\right)\end{aligned}$$

and

$$\begin{aligned}&\sigma\left(T\left(s\right)\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S\left(t\right); \mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)\\ =&\sigma\left(T\left(s\right)\hat{\otimes}_\alpha I, \mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)\\ &\times\sigma\left(I\hat{\otimes}_\alpha S\left(t\right), \mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right).\end{aligned}$$

We have that $\left(\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right),\|\cdot\|_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\right)$ is a commutative unital Banach algebra. Let P be a polynomial in two variables, then the spectral mapping theorem for joint spectra in Banach algebras (Theorem 3.4) gives

$$\begin{aligned}&P\left(\sigma\left(T(s)\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S(t);\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)\right)\\&=\sigma\left(P\left(T(s)\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S(t)\right),\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right).\end{aligned}$$

Let $P(z_1,z_2)=z_1z_2$ be a polynomial in two variables. Then we get

$$\begin{aligned}&P\left(\sigma\left(T(s)\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S(t);\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)\right)\\&=P\left(\sigma\left(T(s),B_{\Gamma_X}(X)\right),\sigma\left(S(t),B_{\Gamma_Y}(Y)\right)\right)\\&=\sigma\left(T(s),B_{\Gamma_X}(X)\right)\times\sigma\left(S(t),B_{\Gamma_Y}(Y)\right),\end{aligned}$$

on the other hand we have

$$\begin{aligned}&\sigma\left(P\left(T(s)\hat{\otimes}_\alpha I, I\hat{\otimes}_\alpha S(t)\right),\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)\\&=\sigma\left(\left(T(s)\hat{\otimes}_\alpha I\right)\left(I\hat{\otimes}_\alpha S(t)\right),\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)\\&=\sigma\left(T(s)\hat{\otimes}_\alpha S(t),\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right).\end{aligned}$$

Finally

$$\begin{aligned}&\sigma\left(T(s)\hat{\otimes}_\alpha S(t),\mathfrak{B}\cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right)\\&\quad=\quad\sigma\left(T(s),B_{\Gamma_X}(X)\right)\times\sigma\left(S(t),B_{\Gamma_Y}(Y)\right).\end{aligned}$$

□

Let us recall the following (see [11], [22, page 241]).

If A is the infinitesimal generator of an equicontinuous C_0-semigroup $(T(s))_{s\geq 0}$ on X, then the resolvent set defined by

$$\rho(A)=\left\{\lambda\in\mathbb{C}:\mathfrak{R}(\lambda,A)=(\lambda I-A)^{-1}\text{ exists and }\mathfrak{R}(\lambda,A)\in\mathcal{L}(X)\right\}$$

is not empty and $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}\subset\rho(A)$. We consider $\sigma(A)=\mathbb{C}\setminus\rho(A)$.

Lemma 4.5. *Let $(T(s))_{s\geq 0}$ be an equicontinuous C_0-semigroup of one parameter on (X,Γ_X) with the generator A. Then we have the spectral inclusion relation :*

$$\sigma\left(T(s),B_{\Gamma_X}(X)\right)\supset e^{s\sigma\left(A,B_{\Gamma_X}(X)\right)}\supset e^{s\sigma(A)},\ \forall s\geq 0.$$

Proof. We consider $F(s)=e^{-\lambda s}T(s),\ s\geq 0$. It is clear that $(F(s))_{s\geq 0}$ is an equicontinuous C_0-semigroup on X with the infinitesimal generator $A-\lambda$. Using [10, Proposition 1.2], we have for all $s\geq 0$ and $\lambda\in\mathbb{C}$

$$(\lambda-A)\int_0^s e^{\lambda(s-t)}T(t)\,xdt=\left(e^{\lambda s}-T(s)\right)x,\forall x\in X,$$

and

$$\int_0^s e^{\lambda(s-t)}T(t)(\lambda - A)\,xdt = \left(e^{\lambda s} - T(s)\right)x, \forall x \in \mathcal{D}(A),$$

where the integral above is an integral of Riemann on X.

Suppose that $e^{\lambda s} \in \rho(T(s); B_{\Gamma_X}(X))$ for some $\lambda \in \mathbb{C}$ and $s \geq 0$, and let $Q_{\lambda,s} := \left(e^{\lambda s} - T(s)\right)^{-1}$. Since $Q_{\lambda,s} \in B_{\Gamma_X}(X)$ and $Q_{\lambda,s}$ commutes with $T(s)$, then $Q_{\lambda,s}$ commutes also with A, and we have

$$(\lambda - A)\int_0^s e^{\lambda(s-t)}T(t)\,Q_{\lambda,s}xdt = x, \forall x \in X,$$

and

$$\int_0^s e^{\lambda(s-t)}T(t)\,Q_{\lambda,s}(\lambda - A)\,xdt = x, \forall x \in \mathcal{D}(A).$$

This last point shows that the operator N_λ defined by

$$N_\lambda x = \int_0^s e^{\lambda(s-t)}T(t)\,Q_{\lambda,s}xdt$$

is the inverse of $\lambda - A$, and we have $N_\lambda \in B_{\Gamma_X}(X)$. It follows that $\lambda \in \rho(A, B_{\Gamma_X}(X))$. Hence we have shown that

$$\sigma(T(s), B_{\Gamma_X}(X)) \supset e^{s\sigma\left(A, B_{\Gamma_X}(X)\right)}, \forall s \geq 0.$$

We have $B_{\Gamma_X}(X) \subset \mathcal{L}(X)$ then $\sigma(A) \subset \sigma(A, B_{\Gamma_X}(X))$. Finally

$$\sigma(T(s), B_{\Gamma_X}(X)) \supset e^{s\sigma\left(A, B_{\Gamma_X}(X)\right)} \supset e^{s\sigma(A)}, \ \forall s \geq 0.$$

□

The main conclusion of this section is the following Theorem, in which a spectral inclusion theorem for tensor product of semigroups over locally convex spaces is announced.

Theorem 4.6. *Let $\left(T(s)\hat{\otimes}_\alpha S(t)\right)_{s,t\geq 0}$ be an equicontinuous C_0 tensor product of semigroups on $X\hat{\otimes}_\alpha Y$, and let $\mathfrak{B}$ denote the bicommutant of $\left(T(s)\hat{\otimes}_\alpha I\right)_{s\geq 0}$ and $\left(I\hat{\otimes}_\alpha S(t)\right)_{t\geq 0}$ in the algebra $\mathcal{L}_s\left(X\hat{\otimes}_\alpha Y\right)$, then there exist two calibrations Γ_X and Γ_Y on X and Y, respectively, such that*

$$\sigma\left(T(s)\hat{\otimes}_\alpha S(t), \mathfrak{B} \cap B_{\Gamma_X\hat{\otimes}_\alpha\Gamma_Y}\left(X\hat{\otimes}_\alpha Y\right)\right) \supset e^{s\sigma\left(A_1, B_{\Gamma_X}(X)\right)+t\sigma\left(A_2, B_{\Gamma_Y}(Y)\right)}$$
$$\supset e^{s\sigma(A_1)+t\sigma(A_2)}$$

where A_1 and A_2 are the infinitesimal generators of $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$, respectively.

Proof. From Theorem 2 and Theorem 3 in [3], we have $(T(s))_{s\geq 0}$ and $(S(t))_{t\geq 0}$ are equicontinuous C_0-semigroups, therefore by Lemma 4.5, we have

$$\sigma(T(s), B_{\Gamma_X}(X)) \supset e^{s\sigma\left(A_1, B_{\Gamma_X}(X)\right)} \supset e^{s\sigma(A_1)}, \ \forall s \geq 0.$$

and

$$\sigma(S(t), B_{\Gamma_Y}(Y)) \supset e^{t\sigma\left(A_2, B_{\Gamma_Y}(Y)\right)} \supset e^{t\sigma(A_2)}, \ \forall t \geq 0.$$

Using Theorem 4.4, we get the result. □

Acknowledgement

The authors are grateful to the editor and the referee for many thoughtful comments and suggestions that helped improve the initial draft of this paper.

References

[1] Allan, G. R. A spectral theory for locally convex algebras. *Proc. London Math. Soc. 3*, 15 (1965), 399–421.

[2] Arendt, W., Grabosch, A., Greiner, G., Groh, U., Lotz, H. P., Moustakas, U., Nagel, R., Neubrander, F., and Schlotterbeck, U. *One-parameter semigroups of positive operators*, vol. 1184 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.

[3] Blali, A., Hassani, R. A., El Amrani, A., and El Beldi, M. Tensor product semigroups on locally convex spaces. *Note Mat. 41*, 2 (2021), 31–60.

[4] Chilana, A. K. Invariant subspaces for linear operators in locally convex spaces. *J. London Math. Soc. (2) 2* (1970), 493–503.

[5] Choe, Y. H. C_0-semigroups on a locally convex space. *J. Math. Anal. Appl. 106*, 2 (1985), 293–320.

[6] Dash, A. T., and Schechter, M. Tensor products and joint spectra. *Israel J. Math. 8* (1970), 191–193.

[7] Giles, J. R., Joseph, G., Koehler, D. O., and Sims, B. On numerical ranges of operators on locally convex spaces. *J. Austral. Math. Soc. Ser. A 20*, 4 (1975), 468–482.

[8] Joseph, G. A. Boundedness and completeness in locally convex spaces and algebras. *J. Austral. Math. Soc. Ser. A 24*, 1 (1977), 50–63.

[9] Khalil, R., Al-Mirbati, R., and Drissi, D. Tensor product semi-groups. *Eur. J. Pure Appl. Math. 3*, 5 (2010), 881–898.

[10] Kōmura, T. Semigroups of operators in locally convex spaces. *J. Functional Analysis 2* (1968), 258–296.

[11] Kostić, M. *Abstract Volterra integro-differential equations*. CRC Press, Boca Raton, FL, 2015.

[12] Kramar, E. On the numerical range of operators on locally and on H-locally convex spaces. *Comment. Math. Univ. Carolin. 34*, 2 (1993), 229–237.

[13] Moore, R. T. Banach algebras of operators on locally convex spaces. *Bull. Amer. Math. Soc. 75* (1969), 68–73.

[14] Pater, F., and Bînzar, T. On some ergodic theorems for a universally bounded operator. *Carpathian J. Math. 26*, 1 (2010), 97–102.

[15] Ricker, W. Remarks on completeness in spaces of linear operators. *Bull. Austral. Math. Soc. 34*, 1 (1986), 25–35.

[16] Schaefer, H. H., and Wolff, M. P. *Topological vector spaces*, second ed., vol. 3 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[17] Schechter, M. On the spectra of operators on tensor products. *J. Functional Analysis 4* (1969), 95–99.

[18] Stoian, S. M. A functional calculus for quotient bounded operators. *Surv. Math. Appl. 1* (2006), 61–69.

[19] Troitsky, V. G. Spectral radii of bounded operators on topological vector spaces. *PanAmer. Math. J. 11*, 3 (2001), 1–35.

[20] Wrobel, V. Tensor products of linear operators in locally convex spaces. *Rend. Circ. Mat. Palermo (2)*, Suppl. 2 (1982), 299–315.

[21] Wrobel, V. Tensor products of linear operators in Banach spaces and Taylor's joint spectrum. *J. Operator Theory 16*, 2 (1986), 273–283.

[22] Yosida, K. *Functional analysis.* Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the sixth (1980) edition.

Received by the editors March 23, 2022
First published online September 5, 2022