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Multiplicity results for some Steklov problems involving $p(x)$-Laplacian operator

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Abstract. This paper deals with the existence and multiplicity of solutions for a class of $p(x)$-Laplacian problems. The main results are obtained on variable exponent Sobolev spaces, by using mountain pass theorem and fountain theorem.

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1. Introduction

This paper is devoted to the study of the following class of Steklov boundary value problems involving the $p(x)$-Laplacian operator

$$(1.1) \qquad \begin{cases} -div\left(a(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + |u|^{p(x)-2}\,u = f(x,u) \text{ in } \Omega, \\[2ex] a(x)\,|\nabla u|^{p(x)-2}\,\frac{\partial u}{\partial v} + b(x)|u|^{q(x)-2}u = g(x,u) \text{ on } \partial\Omega, \end{cases}$$

and

(1.2)

$$\begin{cases} -div\left(a(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + |u|^{p(x)-2}\,u = f(x,u) + \lambda\,|u|^{\gamma(x)-2}\,u \text{ in } \Omega, \\[2ex] a(x)\,|\nabla u|^{p(x)-2}\,\frac{\partial u}{\partial v} + b(x)|u|^{q(x)-2}u = g(x,u) \text{ on } \partial\Omega, \end{cases}$$

where, through this work, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $\frac{\partial}{\partial v}$ is the outer unit normal derivative, $p(x), \gamma(x) \in C(\overline{\Omega})$, $q(x) \in C(\partial\Omega)$, $p(x), q(x), \gamma(x) > 1, p(x) \neq q(y)$, for any $x \in \overline{\Omega}, y \in \partial\Omega$, λ is a positive parameter, $f : \Omega \times \mathbb{R} \to \mathbb{R}$, $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions that satisfy some suitable assumptions which will be stated later, a and b are continuous functions such that

$$a_1 \leq a(x) \leq a_2, \text{ and } b_1 \leq b(x) \leq b_2,$$

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where a_1, a_2, b_1 and b_2 are positive constants. $(-\Delta)_{p(x)}u = -div(|\nabla u|^{p(x)-2}\nabla u)$ denotes the $p(x)$-Laplacian. When $p(x) \equiv p$ (a positive constant), $p(x)$-Laplacian is the usual p-Laplacian. The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian (see [16]).

The study of partial differential equations with variable exponent is a new and interesting subject see [9, 12]. This type of problems is interesting in applications such as, the modeling of image processing , electrorhegological fluids (see for example [11, 19, 25]), and raises many difficult mathematical problems. For advances and references in this field, see [13, 26].

In the past decay, many authors have studied the Steklov problems involving the p-Laplacian, (see for example [3, 7, 20–23, 30]). More recently, problems of type (1.1) involving the $p(x)$-Laplacian, have been investigated by many papers (see [1, 2, 4–6, 8, 9, 12, 14, 17, 18, 24, 27, 29]). For example, Z. Yücedag [29] studied problem (1.1) with $b\,(x) = -1$, and $g\,(x, u) = 0$ and proved that problem (1.1) in this case has at least one nontrivial weak solution $u \in W^{1,p(x)}\,(\Omega)$. In a recent paper [10], Chammem et al. considered problems (1.1) and (1.2) in the case:

$$f(x, u) = v_1(x)h_1(u) \text{ and } g(x, u) = v_2(x)h_2(u),$$

and they obtained results on existence and multiplicity of solutions via the mountain pass theorem and Ekeland's variational principle. Motivated by the results mentioned above, our objective is to prove the existence of nontrivial weak solutions for problem ((1.1) by applying the mountain pass theorem. moreover, we show the existence of infinite solutions to the problem (1.2) via the fountain theorem.

This paper is structured as follows. In Section 2, we introduce some necessary preliminary knowledge on variable exponent Lebesgue and Sobolev spaces. In Section 3, we prove the existence of a nontrivial weak solution of problem (1.1) by using the mountain pass theorem. In Section 4, we show that problem (1.2) has infinitely many pairs of weak solutions by means of fountain theorem.

2. Preliminaries

In order to study problems (1.1) and (1.2), we recall some necessary properties and definitions of variable exponent spaces. For more details, we refer the reader to [9, 12, 18, 27, 29] and the references therein. Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$. Denote

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}), p(x) > 1, \forall\, x \in \overline{\Omega}\}.$$

For all $p \in C_+(\overline{\Omega})$, we set

$$p^- = \inf_{\overline{\Omega}} p(x),\ \ p^+ = \sup_{\overline{\Omega}} p(x).$$

We also denote $C_+(\partial\Omega)$ and p^-, p^+ for every $p(x) \in C(\partial\Omega)$. We define,

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R},\ \text{is measurable} : \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

and

$$L^{p(x)}(\partial\Omega) = \{u : \partial\Omega \to \mathbb{R}, \text{ is measurable} : \int_{\partial\Omega} |u(x)|^{p(x)} d\sigma < \infty\},$$

with norms on $L^{p(x)}(\Omega)$ and $L^{p(x)}(\partial\Omega)$ defined respectively by

$$|u|_{L^{p(x)}(\Omega)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1\},$$

and

$$|u|_{L^{p(x)}(\partial\Omega)} = \inf\{\gamma > 0 : \int_{\partial\Omega} |\frac{u(x)}{\gamma}|^{p(x)} d\sigma \leq 1\},$$

where $d\sigma$ is the surface measure on $\partial\Omega$. The spaces $(L^{p(x)}(\Omega), |.|_{L^{p(x)}(\Omega)})$ and $(L^{p(x)}(\partial\Omega), |.|_{L^{p(x)}(\partial\Omega)})$ become Banach spaces, which we call variable exponent Lebesgue spaces.

Now, we define the Sobolev space with variable exponent as follows:

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

endowed with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \inf\{\kappa > 0 : \int_\Omega \left(|\tfrac{u(x)}{\kappa}|^{p(x)} + |\tfrac{\nabla u(x)}{\kappa}|^{p(x)}\right) dx \leq 1\}.$$

For $u \in W^{1,p(x)}(\Omega)$, if we define

$$\|u\| = \inf\{\kappa > 0 : \int_\Omega (a(x)|\tfrac{u(x)}{\kappa}|^{p(x)} + b(x)|\tfrac{\nabla u(x)}{\kappa}|^{p(x)}) dx \leq 1\},$$

then, from the assumptions on a and b, it is easy to see that $\|u\|$ is an equivalent norm on $W^{1,p(x)}(\Omega)$.

Let $W_0^{1,p(x)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1. (*see [9, 10, 12]*).

(1) *The space $(L^{p(x)}(\Omega), |.|_{L^{p(x)}(\Omega)})$ is a separable, uniformly convex Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Moreover, the Hölder inequality holds, that is, for any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, one has*

$$|\int_\Omega uv dx| \leq (\frac{1}{p^-} + \frac{1}{(p')^-})|u|_{p(x)}|v|_{p'(x)}.$$

(2) *If $p_1, p_2 \in C_+(\overline{\Omega})$ and $p_1(x) \leq p_2(x)$, for any $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.*

Proposition 2.2. (*see [10, 27, 29]*).

(1) *$W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable reflexive Banach spaces.*

(2) *If* $q \in C_+(\overline{\Omega})$ *with* $q(x) < p^*(x)$, *for all* $x \in \overline{\Omega}$, *then the embedding from* $W^{1,p(x)}(\Omega)$ *to* $L^{q(x)}(\Omega)$, *is compact and continuous, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

(3) *If* $q \in C_+(\partial\Omega)$ *with* $q(x) < p_*(x)$ *for all* $x \in \partial\Omega$, *then the trace embedding from* $W^{1,p(x)}(\Omega)$ *to* $L^{q(x)}(\partial\Omega)$, *is compact and continuous, where*

$$p_*(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

Denote

$$\Gamma(u) = \int_\Omega (a(x)|\nabla u|^{p(x)} + |u|^{p(x)})dx, \ \forall u \in W^{1,p(x)}(\Omega).$$

Proposition 2.3. (*see [10, 27, 29]*) *There exist positive constants* ξ_1, ξ_2, *such that*

$(i)\Gamma(u) \geq 1 \Longrightarrow \xi_1||u||^{p^-} \leq \Gamma(u) \leq \xi_2||u||^{p^+}$,
(ii) $\Gamma(u) \leq 1 \Longrightarrow \xi_1||u||^{p^+} \leq \Gamma(u) \leq \xi_2||u||^{p^-}$.

Put

$$\rho(u) = \int_\Omega |u(x)|^{p(x)}dx.$$

Then, we have the following result.

Proposition 2.4. (*see [9, 12, 18]*) *Let* $u \in L^{p(x)}(\Omega)$, *then we have*

(1) $|u|_{L^{p(x)}(\Omega)} < 1$ $(resp = 1, > 1) \Leftrightarrow \rho(u) < 1$ $(resp = 1, > 1)$,
(2) $|u|_{L^{p(x)}(\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq |u|^{p^+}_{L^{p(x)}(\Omega)}$,
(3) $|u|_{L^{p(x)}(\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq |u|^{p^-}_{L^{p(x)}(\Omega)}$.

Denote

$$\rho_\partial(u) = \int_{\partial\Omega} |u(x)|^{p(x)}dx.$$

Proposition 2.5. (*see [9, 12, 18]*) *For all* $u \in L^{p(x)}(\partial\Omega)$, *we have*

(1) $|u|_{L^{p(x)}(\partial\Omega)} > 1 \Rightarrow |u|^{p^-}_{L^{p(x)}(\partial\Omega)} \leq \rho_\partial(u) \leq |u|^{p^+}_{L^{p(x)}(\partial\Omega)}$,
(2) $|u|_{L^{p(x)}(\partial\Omega)} < 1 \Rightarrow |u|^{p^+}_{L^{p(x)}(\partial\Omega)} \leq \rho_\partial(u) \leq |u|^{p^-}_{L^{p(x)}(\partial\Omega)}$.

Proposition 2.6. (*see [9, 12, 18]*) *If* p *and* q *are measurable functions, such that* $p \in L^\infty(\mathbb{R}^N)$ *and* $1 \leq p(x).q(x) \leq \infty$, *for all* $x \in \mathbb{R}^N$, *then, for all* $u \in L^{q(x)}(\mathbb{R}^N)$ *with* $u \neq 0$, *we have*

(1) $|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{q^+}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{q^-}_{p(x)q(x)}$,
(2) $|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{q^-}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{q^+}_{p(x)q(x)}$.

Corollary 2.1. *Let X be a Banach space and $\Psi \in C^1(X,\mathbb{R}), c \in \mathbb{R}$. We say that Ψ satisfies the (PS) condition at level c if any sequence $\{u_n\}$ in X, such that*

$$\Psi(u_n) \to c \text{ and } \Psi'(u_n) \to 0 \text{ in } X^*, \text{ as } n \to \infty,$$

has a convergent subsequence.

Theorem 2.1. (*Mountain pass theorem*). *Let X be a Banach space, $\Psi \in C^1(X,\mathbb{R}), e \in X$ and $||e|| > r$ for some $r > 0$, and assume that*

$$\inf_{||u||=r} \Psi(u) > \Psi(0) \geq \Psi(e).$$

If Ψ satisfies the (PS) condition at level c, with

$$\begin{aligned} c &= \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Psi(\gamma(t)), \\ \text{and } \Gamma &= \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e\}, \end{aligned}$$

then c is a critical value of Ψ.

Throughout the rest of the paper, letters $c_i, i = 1, 2, ...,$ denote positive constants which may change from line to line.

3. Existence result

In this section, we will state and prove the first main result of the paper. More precisely, we will apply the mountain pass theorem to prove the existence of a nontrivial weak solution for the problem (1.1). To this aim, we assume the following hypothesis

(A_0) There exist $C_1 > 0, \alpha \in C_+(\overline{\Omega})$, such that for all $(x,u) \in \Omega \times \mathbb{R}$, we have

$$|f(x,u)| \leq C_1 \left(1 + |u|^{\alpha(x)-1}\right),$$

and

$$1 < \alpha(x) < p^*(x). \tag{3.1}$$

(A_1) There exist $C_2 > 0$, $\beta \in C_+(\partial\Omega)$, such that for all $(x,u) \in \partial\Omega \times \mathbb{R}$,

$$|g(x,u)| \leq C_2 \left(1 + |u|^{\beta(x)-1}\right),$$

and

$$1 < \beta(x) < p_*(x), q(x) < p_*(x). \tag{3.2}$$

(A_2) $f(x,u) = o\left(|u|^{p^+-1}\right)$ as $u \to 0$ and for all $x \in \Omega$.

(A_3) $g(x,u) = o\left(|u|^{p^+-1}\right)$ as $u \to 0$ and for all $x \in \partial\Omega$.

(A_4) There exist $K_1 > 0, \theta_1 > p^+$ such that for all $x \in \Omega$, we have

$$0 < \theta_1 F(x,u) \leq f(x,u)u, \ |u| \geq K_1,$$

where $F(x,t) = \int_0^t f(x,s)ds$.
(A_5) There exist $K_2 > 0, \theta_2 > p^+$ such that for all $x \in \partial\Omega$,

$$0 < \theta_2 G(x,u) \leq g(x,u)u, \ |u| \geq K_2,$$

where $G(x,t) = \int_0^t g(x,s)ds$.

Corollary 3.1. *We say that $u \in X := W^{1,p(x)}(\Omega)$ is a weak solution for the problem* (1.1)*, if*

$$\begin{aligned} 0 &= \int_\Omega a(x) |\nabla u|^{p(x)-2} \nabla u \nabla v + \int_\Omega |u|^{p(x)-2} uv dx \\ &\quad - \int_\Omega f(x,u) v dx + \int_{\partial\Omega} b(x)|u|^{q(x)-2} uv d\sigma - \int_{\partial\Omega} g(x,u) v d\sigma. \end{aligned}$$

for any $v \in X$.

We give below our main result that we will prove.

Theorem 3.1. *Assume that $(A_0) - (A_5)$ hold. If $\min(\alpha^-, \beta^-) > p^+$ and $\min(\theta_1, \theta_2) > q^+$, then problem* (1.1) *has a nontrivial solution.*

The proof of Theorem 3.1, is divided into several lemmas. First, we define the energy functional $\Psi : X \to \mathbb{R}$, associated to the problem (1.1), as follows:

$$\Psi(u) = I(u) + J(u) - \phi(u),$$

where

$$\begin{aligned} I(u) &= \int_\Omega \frac{a(x) |\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} dx, \\ J(u) &= \int_{\partial\Omega} \frac{b(x)|u|^{q(x)}}{q(x)} d\sigma, \end{aligned}$$

and

$$\phi(u) = \int_\Omega F(x,u)dx + \int_{\partial\Omega} G(x,u)d\sigma.$$

Proposition 3.1. *(see [29]) Assume that $I \in C^1(X, \mathbb{R})$, and its derivative $I^{'} : X \to X^*$ is given by*

$$\langle I^{'}(u), v \rangle = \int_\Omega \left(a(x) |\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv \right) dx,$$

moreover, the functional I' satisfies the following properties:
(i) $I^{'} : X \to X^*$ is a continuous, bounded and strictly monotone operator.

(ii) $I^{'}$ *is a mapping of* (S_+) *type, that is, if* $u_n \rightharpoonup u$ *in* X *and*

$$\limsup_{n\to\infty}\langle I^{'}(u_n) - I^{'}(u), u_n - u\rangle \leq 0.$$

Then $u_n \to u$ *strongly in* X.

(iii) $I' : X \to X^*$ *is a homeomorphism.*

Proposition 3.2. (*see [27, 29]*) $J \in C^1(X, \mathbb{R})$, *and for all* $u, v \in X$, *we have*

$$\langle J^{'}(u), v\rangle = \int_{\partial\Omega} b(x)|u|^{q(x)-2} uv d\sigma.$$

Moreover, $J : X \to \mathbb{R}$ *and* $J^{'} : X \to X^*$, *are sequentially weakly-strongly continuous, namely,* $u_n \rightharpoonup u$ *in* X *implies that* $J(u_n) \to J(u)$ *and* $J^{'}(u_n) \to J^{'}(u)$.

Remark 3.1. *By Propositions 2.2, 2.6 and assumptions* $(A_1), (A_2)$, *it is not difficult to prove that* $\phi \in C^1(X, \mathbb{R})$ *and for all* $u, v \in X$, *we get*

$$\langle \phi^{'}(u), v\rangle = \int_\Omega f(x, u(x)) v(x) dx + \int_{\partial\Omega} g(x, u(x)) v(x) d\sigma.$$

Therefore, from Propositions 3.1, 3.2 and Remark 3.1, it is easy to see that $\Psi \in C^1(X, \mathbb{R})$ and for any $u, v \in X$, we obtain

$$\begin{aligned}\langle \Psi^{'}(u), v\rangle &= \int_\Omega \left(a(x)|\nabla u|^{p(x)-2}\nabla u\nabla v + |u|^{p(x)-2}uv\right)dx \\ &\quad + \int_{\partial\Omega} b(x)|u|^{q(x)-2}uvd\sigma \\ &\quad - \int_\Omega f(x, u(x)) v(x) dx - \int_{\partial\Omega} g(x, u(x)) v(x) d\sigma.\end{aligned}$$

So, the critical points of Ψ are weak solutions of problem (1.1).

Lemma 3.1. *Suppose that* $\min(\alpha^-, \beta^-) > p^+$, *and* $(A_0) - (A_3)$ *are satisfied. Then there exist* $\rho, r > 0$ *such that, for* $u \in X$,

$$\textit{if } \|u\| = r, \textit{ then } \Psi(u) \geq \rho.$$

Proof. Let $u \in X$, with $\|u\| < 1$. Then, by (A_0) and (A_2), we know there exist an arbitrary constant $0 < \varepsilon < 1$ and a positive constant $C(\varepsilon_1)$ such that

$$|F(x, u)| \leq \varepsilon_1 |u|^{p^+} + C(\varepsilon_1)|u|^{\alpha(x)}, \ (x, u) \in \Omega \times \mathbb{R}. \tag{3.3}$$

Similarly, hypothesis (A_1) and (A_3) assure that

$$|G(x, u)| \leq \varepsilon_2 |u|^{p^+} + C(\varepsilon_2)|u|^{\beta(x)}, \ (x, u) \in \partial\Omega \times \mathbb{R}. \tag{3.4}$$

Therefore, from Propositions 2.4 and 2.5, we get

$$\Psi(u) \geq \frac{1}{p^+}\Gamma(u) - \int_\Omega F(x, u) dx - \int_{\partial\Omega} G(x, u) d\sigma$$

$$\begin{aligned}
&\geq \frac{1}{p^+}\Gamma(u) - \int_\Omega \left(\varepsilon_1 |u|^{p^+} + C(\varepsilon_1)|u|^{\alpha(x)}\right) dx \\
&\quad - \int_{\partial\Omega} \left(\varepsilon_2 |u|^{p^+} + C(\varepsilon_2)|u|^{\beta(x)}\right) d\sigma \\
&\geq \frac{1}{p^+}\Gamma(u) - \int_\Omega \varepsilon_1 |u|^{p^+} dx - \int_{\partial\Omega} \varepsilon_2 |u|^{p^+} d\sigma \\
&\quad - C(\varepsilon_1)\max\left(|u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}\right) \\
&\quad - C(\varepsilon_2)\max\left(|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}\right).
\end{aligned}$$

Since $1 < p^+ < \alpha^- < \alpha(x) < p^*(x), 1 < p^+ < \beta^- < \beta(x) < p_*(x)$, then, from Proposition 2.2, there exist $c_1, c_2, c_3, c_4 > 0$, such that

$$|u|_{L^{p^+}(\Omega)} \leq c_1 \|u\|, |u|_{L^{p^+}(\partial\Omega)} \leq c_2 \|u\|, \tag{3.5}$$

$$|u|_{L^{\alpha(x)}(\Omega)} \leq c_3 \|u\|, |u|_{L^{\beta(x)}(\partial\Omega)} \leq c_4 \|u\|. \tag{3.6}$$

So, using Proposition 2.3, we get

$$\begin{aligned}
&\Psi(u) \\
&\geq \frac{\xi_1}{p^+}\|u\|^{p^+} - (\varepsilon_1 c_1 + \varepsilon_2 c_2)\|u\|^{p^+} - c_3 C(\varepsilon_1)\|u\|^{\alpha^-} - c_4 C(\varepsilon_2)\|u\|^{\beta^-} \\
&\geq \|u\|^{p^+}\left(\frac{\xi_1}{p^+} - \varepsilon_1 c_1 - \varepsilon_2 c_2 - c_3 C(\varepsilon_1)\|u\|^{\alpha^- - p^+} - c_4 C(\varepsilon_2)\|u\|^{\beta^- - p^+}\right).
\end{aligned}$$

Choose $\varepsilon_1, \varepsilon_2$ small enough that $0 < \varepsilon_1 c_1 + \varepsilon_2 c_2 < \frac{\xi_1}{2p^+}$, we obtain

$$\begin{aligned}
\Psi(u) &\geq \|u\|^{p^+}\left(\frac{\xi_1}{2p^+} - c_3 C(\varepsilon_1)\|u\|^{\alpha^- - p^+} - c_4 C(\varepsilon_2)\|u\|^{\beta^- - p^+}\right) \\
&\geq \|u\|^{p^+}\left(\frac{\xi_1}{2p^+} - \eta\|u\|^{\min\left(\alpha^- - p^+, \beta^- - p^+\right)}\right).
\end{aligned}$$

$$\eta = c_3 C(\varepsilon_1) + c_4 C(\varepsilon_2).$$

Using the fact that $\alpha^-, \beta^- > p^+$, we can choose $\|u\| = r$, small enough such that

$$\frac{\xi_1}{2p^+} - \eta r^{\min\left(\alpha^- - p^+, \beta^- - p^+\right)} > 0.$$

Finally, we conclude that

$$\Psi(u) \geq r^{p^+}\left(\frac{\xi_1}{2p^+} - \eta r^{\min\left(\alpha^- - p^+, \beta^- - p^+\right)}\right) := \rho > 0.$$

□

Lemma 3.2. *If* $\min(\theta_1, \theta_2) > q^+$, *and* $(A_0), (A_1), (A_4)$ *and* (A_5) *are fulfilled, then* Ψ *satisfies* (PS) *condition.*

Proof. Suppose that $\{u_n\}$ is a sequence in X, such that

$$\Psi(u_n) \to c, \Psi'(u_n) \to 0, \text{in } X^*, \text{as } n \to \infty,$$

where c is a positive constant.

Since $\Psi(u_n) \to c$, then there exists $M_1 > 0$, such that

$$|\Psi(u_n)| \leq M_1. \tag{3.7}$$

On the other hand, the fact that $\Psi'(u_n) \to 0$ in X^*, implies that $\langle \Psi'(u_n), u_n \rangle \to 0$, in particular, $\langle \Psi'(u_n), u_n \rangle$ is bounded. Thus, there exists $M_2 > 0$, such that

$$\left| \langle \Psi'(u_n), u_n \rangle \right| \leq M_2. \tag{3.8}$$

We claim that $\{u_n\}$ is bounded. If it is not true, by passing to a subsequence, if necessary, we may assume that $\|u_n\| \to \infty$. Without loss of generality, we assume that $\|u_n\| \geq 1$.

From (3.7) and (3.8), we obtain for $\theta := \min(\theta_1, \theta_2)$

$$\begin{aligned} M_1 &\geq \Psi(u_n) = I(u_n) + J(u_n) - \phi(u_n) \\ &\geq \frac{1}{p^+}\Gamma(u_n) + \frac{1}{q^+}\int_{\partial\Omega} b(x)|u_n|^{q(x)}\, d\sigma - \phi(u_n) \\ &\geq \frac{1}{p^+}\Gamma(u_n) + \frac{1}{\theta}\int_{\partial\Omega} b(x)|u_n|^{q(x)}\, d\sigma - \phi(u_n), \end{aligned} \tag{3.9}$$

and

$$M_2 \geq -\langle \Psi'(u_n), u_n \rangle = -\Gamma(u_n) - \int_{\partial\Omega} b(x)|u_n|^{q(x)}\, d\sigma + \langle \phi'(u_n), u_n \rangle. \tag{3.10}$$

By combining (3.9), (3.10) and using Proposition 2.3, we have

$$\begin{aligned} \theta M_1 + M_2 &\geq \left(\frac{\theta}{p^+} - 1\right)\Gamma(u_n) - \theta\phi(u_n) + \langle \phi'(u_n), u_n \rangle \\ &\geq \left(\frac{\theta}{p^+} - 1\right)\xi_1 \|u_n\|^{p^-} + \int_{\Omega} (f(x,u_n)u_n - \theta_1 F(x,u_n))\, dx \\ &\quad + \int_{\partial\Omega} (g(x,u_n)u_n - \theta_2 G(x,u_n))\, d\sigma. \end{aligned}$$

Hence, assumptions $(A_4) - (A_5)$ imply,

$$\theta M_1 + M_2 \geq \left(\frac{\theta}{p^+} - 1\right)\xi_1 \|u_n\|^{p^-}.$$

Note that $\theta = \min(\theta_1, \theta_2) > p^+$. So, by letting n tend to infinity, we obtain a contradiction.

Therefore, the sequence $\{u_n\}$ is bounded in X. Thus, up to a subsequence, there exists $u \in X$ such that, $\{u_n\}$ converges weakly to u in X.

Because $q(x) < p_*(x)$ and $\alpha(x) < p^*(x)$, we deduce by Proposition 2.2 that

$$\begin{cases} u_n \to u, \text{ strongly in } L^{\alpha(x)}(\Omega), \\ u_n \to u, \text{ strongly in } L^{p^+}(\Omega), \\ u_n \to u, \text{ strongly in } L^{q(x)}(\Omega). \end{cases}$$

To complete the proof, it remains to show that $u_n \to u$ strongly in X. For that, we have

$$\begin{aligned} \langle \Psi'(u_n), u_n - u \rangle &= \langle I'(u_n), u_n - u \rangle + \int_{\partial\Omega} b(x) |u_n|^{q(x)-2} u_n (u_n - u) \, d\sigma \\ &\quad - \int_\Omega f(x, u_n)(u_n - u) \, dx - \int_{\partial\Omega} g(x, u_n)(u_n - u) \, d\sigma. \end{aligned}$$

Using Hölder's inequality and Propositions 2.2, 2.6, we obtain

$$\begin{aligned} &\int_{\partial\Omega} b(x) |u_n|^{q(x)-1} |u_n - u| \, d\sigma \\ &\leq b_2 |u_n - u|_{L^{q(x)}} \left| |u_n|^{q(x)-1} \right|_{L^{\frac{q(x)}{q(x)-1}}} \\ &\leq b_2 |u_n - u|_{L^{q(x)}} \max\left(|u_n|^{q^+-1}_{L^{q(x)}}, |u_n|^{q^--1}_{L^{q(x)}} \right) \\ &\leq c_1 |u_n - u|_{L^{q(x)}} \max\left(\|u_n\|^{q^+-1}, \|u_n\|^{q^--1} \right). \end{aligned}$$

So, we get

$$\lim_{n\to\infty} \int_{\partial\Omega} b(x) |u_n|^{q(x)-2} u_n (u_n - u) \, d\sigma = 0. \tag{3.11}$$

On the other hand, using (A_0), Propositions 2.2, 2.6, and the Hölder inequality, one has

$$\begin{aligned} &\left| \int_\Omega f(x, u_n)(u_n - u) \, dx \right| \\ &\leq \int_\Omega C_1 |u_n - u| \, dx + \int_\Omega C_1 |u_n|^{\alpha(x)-1} |u_n - u| \, dx \\ &\leq C_1 |\Omega|^{\frac{p^+-1}{p^+}} |u_n - u|_{L^{p^+}(\Omega)} + C_1 |u_n - u|_{L^{\alpha(x)}} \left| |u_n|^{\alpha(x)-1} \right|_{L^{\frac{\alpha(x)}{\alpha(x)-1}}} \\ &\leq C_1 |\Omega|^{\frac{p^+-1}{p^+}} |u_n - u|_{L^{p^+}(\Omega)} + C_1 |u_n - u|_{L^{\alpha(x)}} \max\left(|u_n|^{\alpha^+-1}_{L^{\alpha(x)}}, |u_n|^{\alpha^--1}_{L^{\alpha(x)}} \right) \\ &\leq C_1 |\Omega|^{\frac{p^+-1}{p^+}} |u_n - u|_{L^{p^+}(\Omega)} + C_1 |u_n - u|_{L^{\alpha(x)}} \max\left(||u_n||^{\alpha^+-1}, ||u_n||^{\alpha^--1} \right) \end{aligned}$$

So, we obtain

$$\lim_{n\to\infty} \int_\Omega f(x, u_n)(u_n - u) \, dx = 0. \tag{3.12}$$

Similarly, we have

$$\lim_{n\to\infty} \int_{\partial\Omega} g(x, u_n)(u_n - u) \, d\sigma = 0. \tag{3.13}$$

By combining (3.11) – (3.13), and using the fact that $\langle \Psi' (u_n), u_n - u\rangle \to 0$, we conclude that

$$\begin{aligned}&\langle I'(u_n), u_n - u\rangle \\ &= \int_\Omega \left(a(x)\left|\nabla u_n\right|^{p(x)-2} \nabla u_n \nabla (u_n - u) + \left|u_n\right|^{p(x)-2} u_n (u_n - u)\right) dx \to 0,\end{aligned}$$

Passing to the limit as n tends to infinity, and using the fact that u_n converges weakly to u, we get

$$\langle I'(u), u_n - u\rangle \to 0.$$

Hence

$$\lim_{n\to\infty} \langle I'(u_n) - I'(u), u_n - u\rangle = 0.$$

Since I' is of type (S_+) (see Proposition 3.1), we deduce that $u_n \to u$ strongly in X. This completes the proof. □

Lemma 3.3. *Assume that* $\min(\theta_1, \theta_2) > q^+$, *and* $(A_4), (A_5)$ *hold. Then there exists* $e_1 \in X$ *such that*

$$\|e_1\| > r, \text{ and } \Psi(e_1) < 0,$$

where r *is given by Lemma 3.1.*

Proof. By $(A_4), (A_5)$, there exist $m_1 > 0, m_2 > 0$, such that

$$F(x,t) \geq m_1 |t|^{\theta_1}, (x,t) \in \Omega \times \mathbb{R} \tag{3.14}$$

and

$$G(x,t) \geq m_2 |t|^{\theta_2}, (x,t) \in \partial\Omega \times \mathbb{R} \tag{3.15}$$

Let $e \in X$, such that $\int_\Omega |e|^{\theta_1} dx > 0$, and $t > 1$ large enough. Then we obtain

$$\begin{aligned}\Psi(te) &= \int_\Omega \frac{a(x)\left|\nabla(te)\right|^{p(x)} + |te|^{p(x)}}{p(x)} dx + \int_{\partial\Omega} b(x)\frac{|te|^{q(x)}}{q(x)} d\sigma \\ &\quad - \int_\Omega F(x, te)\, dx - \int_{\partial\Omega} G(x, te)\, d\sigma.\end{aligned}$$

So, from (3.14) and (3.15), we get

$$\begin{aligned}\Psi(te) &\leq \frac{t^{p^+}}{p^-}\int_\Omega a(x)\left|\nabla(e)\right|^{p(x)} + |e|^{p(x)} dx + \frac{t^{q^+}}{q^-} b_2 \int_{\partial\Omega} |e|^{q(x)} d\sigma \\ &\quad - m_1 t^{\theta_1} \int_\Omega |e|^{\theta_1} dx - m_2 t^{\theta_2} \int_{\partial\Omega} |e|^{\theta_2} d\sigma.\end{aligned}$$

Since $\min(\theta_1, \theta_2) > \max(q^+, p^+)$, we conclude that

$$\Psi(te) \to -\infty, \text{ as } t \to \infty.$$

Then, we can choose $t_1 > 0, e_1 = t_1 e$, such that $\|e_1\| > r$ and $\Psi(e_1) < 0$. □

Proof of Theorem3.1 From Lemma 3.3, there exists $e_1 \in X$ with $\|e_1\| > r$, for some $r > 0$, and

$$\Psi\left(e_1\right) < 0 = \Psi\left(0\right). \tag{3.16}$$

On the other hand, Lemma 3.1 implies that

$$\inf_{\|u\|=r} \Psi\left(u\right) \geq \rho > 0 = \Psi\left(0\right). \tag{3.17}$$

By combining Equations (3.16), (3.17) with Lemma 3.2, we deduce that all conditions of the mountain pass theorm are satisfied. So, from Theorem 2.1, we deduce the existence of a critical point u of Ψ, which is a weak solution for problem (1.1). Moreover, Equation (3.17) implies that u is a nontrivial solution of problem (1.1). The proof of Theorem 3.1, is now completed.

4. Multiplicity of solutions via the fountain theorem

Through this section, we assume that $\gamma \in C_+\left(\overline{\Omega}\right)$ is such that

$$1 < \gamma^- \leq \gamma^+ < p^-.$$

Now, in order to prove the multiplicity of solutions for problem (1.2), we state the following assumptions.

(A_6) $f\left(x, -u\right) = -f\left(x, u\right)$, for all $(x, u) \in (\Omega \times \mathbb{R})$.

(A_7) $g\left(x, -u\right) = -g\left(x, u\right)$, for all $(x, u) \in (\partial\Omega \times \mathbb{R})$.

The energy functional $\Phi_\lambda : X \to \mathbb{R}$ associated with problem (1.2) is given by:

$$\Phi_\lambda\left(u\right) = \Psi\left(u\right) - \lambda \int_\Omega \frac{|u|^{\gamma(x)}}{\gamma\left(x\right)},$$

where Ψ is the functional associated with problem (1.1) which is introduced in Section 3.

Remark 4.1. $\Phi_\lambda \in C^1\left(X, \mathbb{R}\right)$. *Moreover, weak solutions of problem* (1.2) *correspond to critical points of the functional* Φ_λ.

Since X is a reflexive and separable Banach space, there exist $\{e_i\}_{i=1}^\infty \subset X$ and $\{e_j^*\}_{j=1}^\infty \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_i, i = 1, 2, \dots\}},\ X^* = \overline{\operatorname{span}\{e_j^*, j = 1, 2, \dots\}} \tag{4.1}$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{4.2}$$

For $k = 1, 2\dots$, denote

$$X_k = \operatorname{span}\{e_k\},\ Y_k = \bigoplus_{i=1}^k X_i,\ Z_k = \overline{\bigoplus_{i \geq k} X_i}. \tag{4.3}$$

Proposition 4.1. *(Fountain theorem, see [15, 27]). Assume that*

(H_1) *X is a Banach space, $\Psi \in C^1(X, \mathbb{R})$ is an even functional, subspaces X_k, Y_k and Z_k are defined by (4.3). Suppose that, for every $k \in \mathbb{N}$, there exist $\rho_k > \gamma_k > 0$, such that*

(H_2) $\inf_{u \in Z_k, \|u\|=\gamma_k} \Psi(u) \to \infty$ *as* $k \to \infty$,

(H_3) $\max_{u \in Y_k, \|u\|=\rho_k} \Psi(u) \leq 0$,

(H_4) *Ψ satisfies (PS) condition for every $c > 0$.*

Then Ψ has a sequence of critical values tending to $+\infty$.

Proposition 4.2. *(see [28]). If $\alpha(x) \in C_+(\overline{\Omega}), \alpha(x) < p^*(x)$, for all $x \in \overline{\Omega}$, and $q(x) \in C_+(\partial\Omega), q(x) < p_*(x)$, for all $x \in \partial\Omega$, denote*

$$\begin{aligned} \alpha_k &= \sup\{|u|_{L^{\alpha(x)}(\Omega)} : ||u|| = 1, u \in Z_k\}; \\ q_k &= \sup\{|u|_{L^{q(x)}(\partial\Omega)} : ||u|| = 1, u \in Z_k\}, \end{aligned} \tag{4.4}$$

then $\lim_{k\to\infty} \alpha_k = 0, \lim_{k\to\infty} q_k = 0.$

Let us introduce the following lemma that will be useful in the proof of our main result.

Lemma 4.1. *Let* $\min(\theta_1, \theta_2) > q^+$ *and assume that* $(A_0), (A_1), (A_4)$ *and* (A_5) *hold. Then, for all $\lambda > 0, \Phi_\lambda$ satisfies (PS) condition.*

Proof. Let $\{u_n\} \subset X$ be a sequence such that

$$\Phi_\lambda(u_n) \to c, \Phi_\lambda'(u_n) \to 0, \text{ in } X^*, \text{ as } n \to \infty,$$

where c is a positive constant.

As in the proof of Lemma 3.2, we can find two positive constants M_1 and M_2, such that

$$|\Phi_\lambda(u_n)| \leq M_1, \tag{4.5}$$

and

$$\left|\langle \Phi_\lambda'(u_n), u_n \rangle\right| \leq M_2. \tag{4.6}$$

We claim that the sequence $\{u_n\}$ is bounded. If it is not true, by passing to a subsequence if necessary, we may assume that $\|u_n\| \to \infty$.
Without loss of generality, we assume that $\|u_n\| \geq 1$.

From (4.5) and (4.6), we get

$$\begin{aligned} M_1 &\geq \Phi_\lambda(u_n) = I(u_n) + J(u_n) - \phi(u_n) - \lambda \int_\Omega \frac{|u_n|^{\gamma(x)}}{\gamma(x)} dx \\ &\geq \frac{1}{p^+}\Gamma(u_n) + \frac{1}{q^+}\int_{\partial\Omega} b(x)|u_n|^{q(x)} d\sigma - \phi(u_n) - \lambda \int_\Omega \frac{|u_n|^{\gamma(x)}}{\gamma^-} dx \\ &\geq \frac{1}{p^+}\Gamma(u_n) + \frac{1}{\theta}\int_{\partial\Omega} b(x)|u_n|^{q(x)} d\sigma - \phi(u_n) - \lambda \int_\Omega \frac{|u_n|^{\gamma(x)}}{\gamma^-} dx. \end{aligned} \tag{4.7}$$

where $\theta = \min(\theta_1, \theta_2)$.

On the other hand, we have

$$\begin{aligned} M_2 &\geq -\langle \Phi_\lambda^{'}(u_n), u_n\rangle \\ &= -\Gamma(u_n) - \int_{\partial\Omega} b(x)|u_n|^{q(x)}\, d\sigma + \langle \phi^{'}(u_n), u_n\rangle + \lambda \int_\Omega |u_n|^{\gamma(x)}\, dx. \end{aligned} \tag{4.8}$$

So, by combining Equations (4.7), (4.8) with assumptions $(A_4)-(A_5)$ and Proposition 2.3, we obtain

$$\begin{aligned} &\theta M_1 + M_2 \\ &\geq \left(\frac{\theta}{p^+} - 1\right)\Gamma(u_n) - \theta\left(\phi(u_n) + \lambda\int_\Omega \frac{|u|^{\gamma(x)}}{\gamma^-}dx\right) \\ &\quad + \langle \phi^{'}(u_n), u_n\rangle + \lambda\int_\Omega |u_n|^{\gamma(x)}\, dx \\ &\geq \left(\frac{\theta}{p^+} - 1\right)\xi_1\|u_n\|^{p^-} + \int_\Omega (f(x,u_n)u_n - \theta_1 F(x,u_n))\, dx \\ &\quad + \int_{\partial\Omega}(g(x,u_n)u_n - \theta_2 G(x,u_n))\, d\sigma + \lambda\int_\Omega\left(1 - \frac{\theta}{\gamma^-}\right)|u_n|^{\gamma(x)}\, dx \\ &\geq \left(\frac{\theta}{p^+} - 1\right)\xi_1\|u_n\|^{p^-} + \lambda\int_\Omega\left(1 - \frac{\theta}{\gamma^-}\right)|u_n|^{\gamma(x)}\, dx. \end{aligned}$$

Hence,

$$\theta M_1 + M_2 \geq \left(\frac{\theta}{p^+} - 1\right)\xi_1\|u_n\|^{p^-} + \lambda\left(1 - \frac{\theta}{\gamma^-}\right)\int_\Omega |u_n|^{\gamma(x)}\, dx. \tag{4.9}$$

According to Proposition 2.2, there exists $c > 0$ such that

$$\int_\Omega |u_n|^{\gamma(x)}\, dx \leq |u_n|^l_{L^{\gamma(x)}(\Omega)} \leq c\|u_n\|^l, \tag{4.10}$$

where $l = \gamma^-$ or γ^+.

Therefore, from (4.9) and (4.10), we deduce that

$$\theta M_1 + M_2 \geq \left(\frac{\theta}{p^+} - 1\right)\xi_1\|u_n\|^{p^-} - c\lambda\left(\frac{\theta}{\gamma^-} - 1\right)\|u_n\|^l.$$

Since, $\theta > p^+ \geq p^- > l$, then, by letting n tend to infinity, we obtain a contradiction. So, we conclude that $\{u_n\}$ is bounded in X. Thus, there exists $u \in X$ such that, up to a subsequence, $\{u_n\}$ converges weakly to $u \in X$. The rest of the proof is very similar to the one in Lemma 3.2, so we omit it here. □

The second main result of this paper is as follows.

Theorem 4.1. *If* $\min(\alpha^-, \beta^-) > p^+, \min(\theta_1, \theta_2) > q^+, (A_0)-(A_1)$ *and* $(A_4)-(A_7)$ *are satisfied, then* Φ_λ *has a sequence of critical points* $\{\pm u_n\}$ *such that* $\Phi_\lambda(\pm u_n) \to \infty$ *as* $n \to \infty$.

Proof. Obviously, assumptions (A_6) and (A_7) imply that Φ_λ is an even functional and satisfies the (PS) condition (see Lemma 3.2). We will prove that if k is large enough, then there exist $\rho_k > \gamma_k > 0$ such that (H_2) and (H_3) hold.

Let $u \in Z_k$ with $\|u\| > 1$, by conditions (A_0) and (A_1), we have

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{1}{p^+}\Gamma(u) - \int_\Omega C_1(1+|u|^{\alpha(x)})dx - \int_{\partial\Omega} C_2(1+|u|^{\beta(x)})d\sigma \\ &\quad -\frac{\lambda}{\gamma^-}\int_\Omega |u|^{\gamma(x)}dx \\ &\geq \frac{\xi_1}{p^+}||u||^{p^-} - C_1 \max\left(|u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}\right) \\ &\quad -C_2 \max\left(|u|^{\beta^-}_{L^{\beta(x)}(\partial\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\partial\Omega)}\right) - c_1 \\ &\quad -\frac{\lambda}{\gamma^-}\max\left(|u|^{\gamma^-}_{L^{\gamma(x)}(\Omega)}, |u|^{\gamma^+}_{L^{\gamma(x)}(\Omega)}\right).\end{aligned}$$

If $\max\{|u|^{\alpha^-}_{L^{\alpha(x)}(\Omega)}, |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}, |u|^{\beta^-}_{L^{\beta(x)}(\Omega)}, |u|^{\beta^+}_{L^{\beta(x)}(\Omega)}, |u|^{\gamma^-}_{L^{\gamma(x)}(\Omega)}, |u|^{\gamma^+}_{L^{\gamma(x)}(\Omega)}\} = |u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)}$. Then, by Proposition 4.2, we have

$$\begin{aligned}\Phi_\lambda(u) &\geq \frac{\xi_1}{p^+}||u||^{p^-} - c_2(\lambda,\gamma^-)\,|u|^{\alpha^+}_{L^{\alpha(x)}(\Omega)} - c_1 \\ &\geq \frac{\xi_1}{p^+}||u||^{p^-} - c_2(\lambda,\gamma^-)\alpha_k^{\alpha^+}||u||^{\alpha^+} - c_1.\end{aligned}$$

Choose $\gamma_k = (\frac{c_2(\lambda,\gamma^-)\alpha^+\alpha_k^{\alpha^+}}{\xi_1})^{\frac{1}{p^- - \alpha^+}}$. For $u \in Z_k$ with $||u|| = \gamma_k$, we have

$$\Phi_\lambda(u) \geq \xi_1(\frac{1}{p^+} - \frac{1}{\alpha^+})\gamma_k^{p^-} - c_1. \tag{4.11}$$

Since $\alpha_k \to 0$ as $k \to \infty$ and $p^+ < \alpha^- \leq \alpha^+$, we have $1/p^+ - 1/\alpha^+ > 0$ and $\gamma_k \to \infty$. Thus, for sufficiently large k, we have $\Phi_\lambda(u) \to \infty$ with $u \in Z_k$ and $||u|| = \gamma_k$ as $k \to \infty$. In other cases, similarly, we can deduce

$$\Phi_\lambda(u) \to \infty, \text{ since } \quad q_k \to 0, \alpha_k \to 0, k \to \infty. \tag{4.12}$$

So, (H_2) holds.

Let $u \in Y_k$ such that $||u|| = \rho_k > \gamma_k > 1$, then from (3.14) and (3.15), we have

$$\begin{aligned}\Phi_\lambda(u) &\leq \frac{1}{p^-}\Gamma(u) + \frac{b_2}{q^-}\int_{\partial\Omega}|u|^{q(x)}\,d\sigma - \int_\Omega F(x,u)dx - \frac{\lambda}{\gamma^-}\int_\Omega |u|^{\gamma(x)}\,dx \\ &\quad -\int_{\partial\Omega} G(x,u)d\sigma \\ &\leq \frac{\xi_1}{p^-}||u||^{p^+} + \frac{b_2}{q^-}\max\{|u|^{q^-}_{L^{q(x)}(\partial\Omega)}, |u|^{q^+}_{L^{q(x)}(\partial\Omega)}\} \\ &\quad -m_1\int_\Omega |u|^{\theta_1}\,dx - m_2\int_{\partial\Omega}|u|^{\theta_2}\,d\sigma.\end{aligned}$$

If $\max\{|u|^{q^-}_{L^{q(x)}(\partial\Omega)}, |u|^{q^+}_{L^{q(x)}(\partial\Omega)}\} = |u|^{q^+}_{L^{q(x)}(\partial\Omega)}$, then we have

$$\Psi(u) \leq \frac{\xi_1}{p^-}||u||^{p^+} + \frac{b_2}{q^-}|u|^{q^+}_{L^{q(x)}(\partial\Omega)} - m_1\int_\Omega |u|^{\theta_1}\,dx - m_2\int_{\partial\Omega}|u|^{\theta_2}\,d\sigma.$$

Since $\dim Y_k < \infty$, all norms are equivalent in Y_k. So, we get

$$\Psi(u) \leq \frac{\xi_1}{p^-}||u||^{p^+} + \frac{b_2}{q^-}c_2||u||^{q^+} - c_3||u||^{\theta_1} - c_4||u||^{\theta_2}. \tag{4.13}$$

also, since $\max(q^+, p^+) < \min(\theta_1, \theta_2)$, then we get $\Psi(u) \to -\infty$ as $||u|| \to \infty$. For the other case, the proof is similar, so, H_3 holds. Thus, we complete the proof.

□

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