On generalized weakly (Ricci) $\phi$-symmetric Lorentzian Para Sasakian manifold

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Abstract. The present paper attempt to introduce the notion of generalized weakly $\phi$-symmetric and generalized weakly Ricci $\phi$-symmetric Lorentzian Para Sasakian manifold. Furthermore, we have studied generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetimes. In addition, the existence of generalized weakly $\phi$-symmetric Lorentzian Para Sasakian manifold is ensured by a suitable example.

AMS Mathematics Subject Classification (2010): 53C15; 53C25
Key words and phrases: Generalized weakly $\phi$-symmetric, Generalized weakly Ricci $\phi$-symmetric, $\eta$-Einstein, Lorentzian Para Sasakian manifold.

1. Introduction

Throughout the paper, we shall denote the Levi-Civita connection, Riemannian curvature tensor, Ricci tensor and Ricci operator by the symbols $\nabla$, $R$ (or $\bar{R}$), $S$ and $Q$ respectively. As a mild version of local symmetry [5], Takahashi [28] started the studies on locally $\phi$-symmetric manifold. Further research has been continued to weaken such notion by many authors. For details, we refer the reader to ([10], [11], [16], [12], [7], [23], [24], [21], [27], [8], [17], [18], [19] and the references therein).

Recently, the author in [2], has introduced the concept of generalized weakly symmetric manifold which is defined as follows:

A Riemannian (or semi-Riemannian) manifold of dimension $n$ is said to be a generalized weakly symmetric [2], if it admits the equation

$$
(\nabla_X \bar{R})(Y, U, V, Z) = A(X)\bar{R}(Y, U, V, Z) + B(Y)\bar{R}(X, U, V, Z) + B(U)\bar{R}(Y, X, V, Z) + D(V)\bar{R}(Y, U, X, Z) + \alpha(X)\bar{G}(Y, U, V, W) + \beta(Y)\bar{G}(X, U, V, Z) + \gamma(U)\bar{G}(Y, X, V, Z) + \gamma(Z)\bar{G}(Y, U, V, X),
$$

(1.1)

where

$$
\bar{G}(Y, U, V, W) = g(U, V)g(Y, W) - g(Y, V)g(U, W)
$$

(1.2)

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for any vector fields $X, Y$ and $U$ and the 1-forms $A(X) = g(X, \pi_1)$, $B(X) = g(X, \pi_2)$, $D(X) = g(X, \gamma)$, $\alpha(X) = g(X, \delta_1)$, $\beta(X) = g(X, \delta_2)$ and $\gamma(X) = g(X, \sigma)$. The relation (1.1) can also be written as

$$\nabla_X R(Y, U) V = A(X) R(Y, U) V + B(Y) R(Y, U) V + B(U) R(Y, X) V + D(V) R(Y, U) X$$

$$+ g(R(Y, U) V, X) g + \alpha(X) G(Y, U) V + \beta(Y) G(X, U) V$$

(1.3) $$+ \beta(U) G(Y, X) V + \gamma(V) G(Y, U) X + g(G(Y, U) V, X) \sigma.$$

Analogously, a semi-Riemannian (or Riemannian) manifold $(M^n, g)$ is said to be generalized weakly Ricci-symmetric, if it satisfies the condition


(1.4) $$+ \alpha^*(X) g(U, V) + \beta^*(U) g(V, X) + \gamma^*(V) g(U, X)$$

which can also be expressed as

$$(\nabla_X Q)(U) = A^*(X) Q U + B^*(U) Q X + S(U, X) g^*$$

(1.5) $$+ \alpha^*(X) U + \beta^*(U) X + g(U, \sigma) g^*$$

for any vector fields $X, U$ and $V$ and the 1-forms $A^*(X) = g(X, \pi_1^*)$, $B^*(X) = g(X, \pi_2^*)$, $D^*(X) = g(X, \gamma^*)$, $\alpha^*(X) = g(X, \delta_1^*)$, $\beta^*(X) = g(X, \delta_2^*)$ and $\gamma(X) = g(X, \sigma^*)$.

Recently, Hui [10] studied $\phi$-pseudo symmetric and $\phi$-pseudo Ricci symmetric Kenmotsu manifolds. In tune with [10], in this paper we would like to introduce the notion of generalized weakly $\phi$-symmetric manifold and generalized weakly Ricci $\phi$-symmetric manifold.

A Lorentzian Para-Sasakian manifold is named to be generalized weakly $\phi$-symmetric if $R$ admits the equation

$$\phi^2 ((\nabla_X R)(Y, U) V)$$

$$= A(X) R(Y, U) V + B(Y) R(Y, U) V + B(U) \bar{R}(Y, X) V + D(V) \bar{R}(Y, U) X$$

$$+ g(R(Y, U) V, X) g + \alpha(X) G(Y, U) V + \beta(Y) G(X, U) V$$

(1.6) $$+ \beta(U) G(Y, X) V + \gamma(V) G(Y, U) X + g(G(Y, U) V, X) \sigma.$$

The beauty of such generalized weakly $\phi$-symmetric manifold is that it has the flavour of

(i) locally $\phi$-symmetric space [28] (for $A = B = D = 0 = \alpha = \beta = \gamma$),

(ii) locally $\phi$-recurrent space [8] (for $A \neq 0, B = D = \alpha = \beta = \gamma = 0$),

(iii) generalized $\phi$-recurrent space in the sense of [9] (for $A \neq 0, \alpha \neq 0, B = D = \beta = \gamma = 0$),

(iv) quasi $\phi$-recurrent space in the sense [20] ( $A \neq 0, B = D = 0, \alpha \neq 0, \beta = \gamma = (\beta^* - \gamma^*) \alpha$),

(v) pseudo $\phi$-symmetric space in the sense of [10] (for $A \neq 0, B = D = H \neq 0, \alpha = \beta = \gamma = 0$),
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(vi) generalized pseudo $\phi$-symmetric space in the sense of [1] (for $\frac{A}{2} = B = D = H_1 \neq 0, \frac{\alpha}{2} = \beta = \gamma = H_2 \neq 0$),

(vii) semi-pseudo $\phi$-symmetric space in the sense of [30] ($A = \alpha = \beta = \gamma = 0, B = D \neq 0$),

(viii) generalized semi-pseudo $\phi$-symmetric space in the sense of [3] ($A = 0 = \alpha, B = D \neq 0, \beta = \gamma \neq 0$),

(ix) almost pseudo $\phi$-symmetric space in the sense of [6] ($A = H_1 + K_1, B = D = H_1 \neq 0$ and $\alpha = \beta = \gamma = 0$),

(x) almost generalized pseudo $\phi$-symmetric space in the sense of [3] ($A = H_1 + K_1, B = D = H_1 \neq 0, \alpha = H_2 + K_2, \beta = \gamma = H_2 \neq 0$),

(xi) weakly $\phi$-symmetric space in the sense of [29] (for $A, B, D \neq 0, \alpha = \beta = \gamma = 0$).

A Lorentzian Para-Sasakian manifold is said to be generalized weakly Ricci $\phi$-symmetric if $Q$ admits the following

$$
\phi^2((\nabla_X Q)(U)) = A^*(X)QU + B^*(U)QX + S(U, X)\varrho^* + \alpha^*(X)U + \beta^*(U)X + g(U, X)\sigma^*.
$$

(1.7)

In 1989, Matsumoto [13] started the studies on Lorentzian Para-Sasakian manifolds which is also independently defined by Mihai and Rosca [15]. Matsumoto, Mihai and Rosaca [14] gave a five dimensional example of Lorentzian Para-Sasakian manifold.

We represent our paper as follows: Section 2, is concerned with some known results of Lorentzian Para Sasakian manifold. In section 3, we have studied generalized weakly $\phi$-symmetric Lorentzian Para Sasakian manifold. We have observed that generalized weakly $\phi$-symmetric Lorentzian Para Sasakian manifold is $\eta$-Einstein whereas Lorentzian Para Sasakian manifold can not be an $\eta$-Einstein for each of the curvature restrictions (i) $\phi$-symmetric, (ii) $\phi$-recurrent, (iii) generalized $\phi$-recurrent, (iv) pseudo $\phi$-symmetric, (v) almost pseudo $\phi$-symmetric, (vi) generalized pseudo $\phi$-symmetric and (vii) generalized almost pseudo $\phi$-symmetric. It is also found that there does not exist a Lorentzian Para-Sasakian manifold which is (i) $\phi$-recurrent, (ii) generalized $\phi$-recurrent provided the 1-forms are colinear, (iii) pseudo $\phi$-symmetric, (iv) generalized pseudo $\phi$-symmetric provided the 1-forms are colinear, (v) generalized semi-pseudo $\phi$-symmetric provided the 1-forms are colinear, (v) generalized almost pseudo $\phi$-symmetric provided the 1-forms are colinear. Section 4 is dealt with generalized weakly Ricci $\phi$-symmetric Lorentzian Para Sasakian manifold. Keeping the tune of Shaikh, Yoon and Hui [25], in section 5, we have studied generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime and obtain some interesting results. Finally, we have constructed an example of generalized weakly $\phi$-symmetric Lorentzian Para Sasakian manifold.

2. Properties of Lorentzian Para Sasakian manifold

Let $M$ be an $n$-dimensional differential manifold endowed with a $(1, 1)$ tensor field $\phi$, a vector field $\xi$, a 1-form $\eta$, and a Lorentzian metric $g$ of type $(0, 2)$
such that for each point \(a \in M\), the tensor \(g_a : T_aM \times T_aM \to \mathbb{R}\) is a non-degenerate inner product of signature \((- opposed,+,...,+)\), where \(T_aM\) denotes the tangent space of \(M\) at \(a\) and \(\mathbb{R}\) is the real number space which satisfies

\[
\phi^2 = I + \eta \otimes \xi,
\]

(2.1)

\[
\eta(\xi) = -1,
\]

(2.2)

\[
g(X, \xi) = \eta(X),
\]

(2.3)

\[
g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),
\]

(2.4)

for all vector fields \(X, Y\) on \(M^n\). Then, the structure \((\phi, \xi, \eta, g)\) is called Lorentzian almost para contact structure and the manifold with the structure \((\phi, \xi, \eta, g)\) is called a Lorentzian almost para contact manifold. In the Lorentzian almost para contact manifold \(M\), the following relations hold \[13\]

\[
\phi \xi = 0, \quad \eta \circ \phi = 0,
\]

(2.5)

\[
g(\phi X, Y) = g(X, \phi Y).
\]

(2.6)

If we put

\[
\Omega(X, Y) = g(\phi X, Y) = g(X, \phi Y)
\]

(2.7)

for any vector fields \(X\) and \(Y\) then the tensor field \(\Omega(X, Y)\) is a symmetric \((0, 2)\) tensor field.

A Lorentzian almost para contact manifold \(M\) endowed with the structure \((\phi, \xi, \eta, g)\) is called an Lorentzian Para-Sasakian manifold if

\[
(\nabla_X \phi) Y - g(\phi X, \phi Y) \xi = \eta(Y) \phi^2 X,
\]

(2.8)

where \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\). In a Lorentzian Para-Sasakian manifold \(M\) with the structure \((\phi, \xi, \eta, g)\), it is easily seen that \([13, 4, 22]\)

\[
\nabla_X \xi = \phi X,
\]

(2.9)

\[
(\nabla_X \eta) Y = g(X, \phi Y) = \Omega(X, Y) = (\nabla_Y \eta) X,
\]

(2.10)

\[
S(X, \xi) = (n - 1) \eta(X), \quad Q \xi = (n - 1) \xi,
\]

(2.11)

\[
R(\xi, X) Y + \eta(Y) X = g(X, Y) \xi,
\]

(2.12)
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\begin{equation}
R(Y, U)\phi X = \phi R(Y, U)X + g(U, X)\phi Y - g(Y, X)\phi U + g(\phi Y, X)U
- g(\phi U, X)Y + 2[g(\phi Y, X)\eta(U) - g(\phi U, X)\eta(Y)]\xi
+ 2[\eta(U)\phi Y - \eta(Y)\phi U]\eta(X),
\end{equation}

\begin{equation}
R(X, Y)Z = \phi R(X, Y)Z + g(X, Z)Y - g(Y, Z)X
+ \Omega(Y, Z)\phi X - \Omega(X, Z)\phi Y + 2[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\xi
+ 2[\eta(X)Y - \eta(Y)X]\eta(Z),
\end{equation}

\begin{equation}
(\nabla_X R)(Y, U)\xi = 2[g(\phi U, X)Y - g(\phi Y, X)U] - \phi R(Y, U)X
+ g(Y, X)\phi U - g(U, X)\phi Y - 2[g(\phi Y, X)\eta(U)
- g(\phi U, X)\eta(Y)]\xi
- 2[\eta(U)\phi Y - \eta(Y)\phi U]\eta(X).
\end{equation}

for all vector fields $X, Y, U$ and $Z$ on $M^n$.

3. Generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian manifold

In this section we consider a generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian manifold. Then by the virtue of (2.1), the equation (1.6) takes the following form

\begin{equation}
(\nabla_X R)(Y, U)V + \eta(\nabla_X R)(Y, U)V)\xi
= A(X)R(Y, U)V + B(Y)R(X, U)V
+ B(U)R(Y, X)V + D(V)R(Y, U)X
+ g(R(Y, U)V, X)\phi + \alpha(X)G(Y, U)V
+ \beta(Y)G(X, U)V + \beta(U)G(Y, X)V
+ \gamma(V)G(Y, U)X + g(G(Y, U)V, X)\sigma.
\end{equation}

From which we get

\begin{equation}
g((\nabla_X R)(Y, U)V, W) + \eta((\nabla_X R)(Y, U)V)\eta(W)
= A(X)g(R(Y, U)V, W) + B(Y)g(R(X, U)V, W)
+ B(U)g(R(Y, X)V, W) + D(V)g(R(Y, U)V, W)
+ D(W)g(R(Y, U)V, X) + \alpha(X)g(G(Y, U)V, W)
+ \beta(Y)g(G(X, U)V, W) + \beta(U)g(G(Y, X)V, W)
+ \gamma(V)g(G(Y, U)V, X).
\end{equation}
Now, taking an orthonormal frame field and then contracting (3.2) over \(Y\) and \(W\) we obtain

\[
(\nabla_X S)(U, V) + (\nabla_X \bar{R})(\xi, U, V, \xi)
\]

\[
= A(X)S(U, V) + B(U)S(X, V) + D(V)S(U, X)
\]

\[+(n-1)\alpha(X)g(U, V) + \beta(G(X, U)V)
\]

\[
(3.3)
\]

\[+(n-1)\beta(U)g(X, V) + (n-1)\gamma(V)g(U, X) + \gamma(G(X, V)U).
\]

Using (2.12), (2.5) and (2.9), we obtain

\[
(3.4) (\nabla_X R)(\xi, U, V, \xi) = 0.
\]

In view of (3.3) and (3.4), we have

\[
(\nabla_X S)(U, V)
\]

\[
= A(X)S(U, V) + B(U)S(X, V) + D(V)S(U, X)
\]

\[+(n-1)\alpha(X)g(U, V) + \beta(U)g(X, V) + \gamma(V)g(U, X)\}

\[
(3.5)
\]

\[+B(R(X, U)V) + D(R(X, V)U) + \beta(G(X, U)V) + \gamma(G(X, V)U).
\]

**Theorem 3.1.** Every generalized weakly \(\phi\)-symmetric Lorentzian Para Sasakian manifold is a generalized weakly Ricci-symmetric provided that

\[
D(\xi) + \gamma(\xi) = 0.
\]

**Proof.** From, (3.5) it is obvious that a generalized weakly \(\phi\)-symmetric Lorentzian Para Sasakian manifold is a generalized weakly Ricci-symmetric if

\[
(3.6) B(R(X, U)V) + D(R(X, V)U) + \beta(G(X, U)V) + \gamma(G(X, V)U) = 0.
\]

From, (3.6), one can easily bring out that \(D(\xi) + \gamma(\xi) = 0.\)

Now, setting \(V = \xi\) in (3.5) we get

\[
(\nabla_X S)(U, \xi)
\]

\[
= (n-1)A(X)\eta(U) + (n-2)B(U)\eta(X) + D(\xi)S(U, X)
\]

\[+(n-1)\alpha(X)\eta(U) + (n-2)\beta(U)\eta(X) + (n-1)\gamma(\xi)g(U, X)
\]

\[+B(X)\eta(U) + D(X)\eta(U) - D(\xi)g(U, X)
\]

\[
(3.7)
\]

\[+\beta(X)\eta(U) + \gamma(X)\eta(U) - \gamma(\xi)g(U, X).
\]

Again, from the relation

\[
(\nabla_X S)(U, V) = \nabla_X S(U, V) - S(\nabla_X U, V) - S(U, \nabla_X V)
\]

we get

\[
(3.8) (\nabla_X S)(U, \xi) = (n-1)g(\phi X, U) - S(\phi X, U).
\]
Using (3.8) in (3.7) we have
\[(n - 1)g(\phi X, U) - S(\phi X, U)\]
\[= (n - 1)A(X)\eta(U) + (n - 2)B(U)\eta(X) + D(\xi)S(U, X)\]
\[+ (n - 1)\alpha(X)\eta(U) + (n - 2)\beta(U)\eta(X) + (n - 1)\gamma(\xi)g(U, X)\]
\[+ B(X)\eta(U) + D(X)\eta(U) - D(\xi)g(U, X)\]
\[(3.9)\]
\[+ \beta(X)\eta(U) + \gamma(X)\eta(U) - \gamma(\xi)g(U, X).\]

Plugging \(U = \xi, X = \xi\) and \(U = X = \xi\) in succession, we obtain from (3.9) that
\[(n - 1)\{A(X) + \alpha(X)\} + B(X) + D(X) + \beta(X) + \gamma(X)\]
\[= (n - 2)\eta(X)[B(\xi) + D(\xi) + \beta(\xi) + \gamma(\xi)],\]
\[(3.10)\]
\[= (n - 2)[B(U) + \beta(U)]\]
\[= \eta(U)[(n - 1)\{A(\xi) + \alpha(\xi) + D(\xi) + \gamma(\xi)\} + B(\xi) + \beta(\xi)],\]
\[(3.11)\]
and
\[A(\xi) + B(\xi) + D(\xi) + \alpha(\xi) + \beta(\xi) + \gamma(\xi) = 0\]
respectively. From, (3.12) we infer that

**Theorem 3.2.** In a generalized pseudo \(\phi\)-symmetric Lorentzian Para-Sasakian manifold the vector field associate to the 1-form is perpendicular to the characteristic vector field \(\xi\).

Replacing \(U\) by \(X\) in (3.11), we get
\[(n - 2)[B(X) + \beta(X)]\]
\[= \eta(X)[(n - 1)\{A(X) + \alpha(X) + D(X) + \gamma(X)\} + B(X) + \beta(X)].\]
\[(3.13)\]

Now, by virtue of (3.10), (3.13) and (3.12) we obtain
\[(n - 1)\{A(X) + B(X) + \alpha(X) + \beta(X)\} + D(X) + \gamma(X)\]
\[= (n - 2)[D(\xi) + \gamma(\xi)]\eta(X).\]
\[(3.14)\]

Now using (3.10), (3.11) and (3.12) in (3.9) we get
\[(n - 1)g(\phi X, U) - S(\phi X, U)\]
\[= D(\xi)S(U, X) + \{(n - 2) - D(\xi)\}g(U, X)\]
\[+ (n - 2)[D(\xi) + \gamma(\xi)]\eta(U)\eta(X).\]
\[(3.15)\]
Replacing \(X\) by \(\phi X\) in (3.15) we have
\[(n - 1)g(X, U) - S(X, U)\]
\[= D(\xi)S(\phi X, U) + \{(n - 2) - D(\xi)\}g(\phi X, U).\]}
Using (3.16) in (3.15) we obtain
\[
[(n-1) + D(\xi) \{ (n-2) - D(\xi) \}] g(X, U) = [1 - D(\xi) D(\xi)] S(X, U) + \{(n-2) + (n-2) D(\xi)\} g(\phi X, U) \\
- (n-2) D(\xi) [D(\xi) + \gamma(\xi)] \eta(U) \eta(X).
\]
(3.17)

This leads to the following:

**Theorem 3.3.** In a generalized weakly \(\phi\)-symmetric Lorentzian Para-Sasakian manifold the Ricci tensor \(S\) satisfies the relation (3.17).

Again, setting \(U = \xi\) in (3.5) we get
\[
(\nabla_X S)(V, \xi) = (n-1) A(X) \eta(V) + B(\xi) S(V, X) + (n-2) D(\eta) \eta(X) \\
+ (n-1) \alpha(X) \eta(V) + (n-2) \gamma(V) \eta(X) + (n-1) \beta(\xi) g(V, X) \\
+ D(\xi) \eta(V) + B(X) \eta(V) - B(\xi) g(V, X) \\
+ \gamma(X) \eta(V) + \beta(\xi) \eta(V) - \beta(\xi) g(V, X).
\]
(3.18)

Replacing \(U\) by \(V\) in (3.7) then (3.18) yields
\[
[B(\xi) - D(\xi)] S(V, X) = \{(n-2) \gamma(\xi) - D(\xi) - (n-2) \beta(\xi) + B(\xi)\} g(V, X) \\
+ (n-2) [B(V) + \beta(V) - D(V) - \gamma(V)] \eta(X).
\]
(3.19)

Now, putting \(X = \xi\) in (3.19) we get
\[
B(V) + \beta(V) - D(V) - \gamma(V) = [D(\xi) - B(\xi) + \gamma(\xi) - \beta(\xi)] \eta(V).
\]
(3.20)

Using (3.20) in (3.19) we get
\[
S(V, X) = L_1 g(V, X) + L_2 \eta(V) \eta(X)
\]
(3.21)

where \(L_1 = \left(\frac{(n-2) \gamma(\xi) - B(\xi) - D(\xi) + B(\xi)}{B(\xi) - D(\xi)}\right)\) and \(L_2 = \left(\frac{(n-2) [D(\xi) - B(\xi) + \gamma(\xi) - \beta(\xi)]}{B(\xi) - D(\xi)}\right)\).

This leads to the following:

**Theorem 3.4.** Every generalized weakly \(\phi\)-symmetric Lorentzian Para-Sasakian manifold is an \(\eta\)-Einstein provided that \(B(\xi) \neq D(\xi)\).

**Theorem 3.5.** Let \((M^n, g)\) be a Lorentzian Para-Sasakian manifold. Then \(M\) cannot be an \(\eta\)-Einstein for each of the curvature restriction (i) \(\phi\)-symmetric, (ii) \(\phi\)-recurrent, (iii) generalized \(\phi\)-recurrent, (iv) pseudo \(\phi\)-symmetric, (v) almost pseudo \(\phi\)-symmetric, (vi) generalized pseudo \(\phi\)-symmetric and (vii) generalized almost pseudo \(\phi\)-symmetric.
Using (3.8) in (3.18) we get
\[
(n - 1)g(\phi X, V) - S(\phi X, V) = (n - 1)A(X)\eta(V) + B(\xi)S(V, X) + (n - 2)D(V)\eta(X) + (n - 1)\alpha(\xi)g(V, X) + D(\xi)\eta(V) + B(\xi)\eta(V) - B(\xi)g(V, X) + \gamma(\xi)\eta(V) + \beta(\xi)\eta(V) - \beta(\xi)g(V, X).
\]
(3.22)

Now, setting \(X = \xi\) in (3.22) and using (3.12), we get
\[
[D(\xi) + \gamma(\xi)]\eta(V) = -[D(V) + \gamma(V)].
\]
(3.23)

Using (3.21) in (3.14), we obtain
\[
A(X) + B(X) + D(X) + \alpha(X) + \beta(X) + \gamma(X) = 0
\]
(3.24)

Next, in view of \(\alpha = \beta = \gamma = 0\), the relation (3.24) yields
\[
A(X) + B(X) + D(X) = 0.
\]
(3.25)

This motivates us to state

**Theorem 3.6.** In a weakly \(\phi\)-symmetric Lorentzian Para-Sasakian manifold \((M^n, g)(n > 2)\), the sum of the associated 1-forms is given by (3.25).

**Theorem 3.7.** There does not exist an Lorentzian Para-Sasakian manifold which is

\(i\) \(\phi\)-recurrent,
\(ii\) generalized \(\phi\)-recurrent provided the 1-forms are colinear,
\(iii\) pseudo \(\phi\)-symmetric,
\(iv\) generalized semi-pseudo \(\phi\)-symmetric provided the 1-forms are colinear,
\(v\) generalized almost pseudo \(\phi\)-symmetric provided the 1-forms are colinear.

**4. Generalized weakly Ricci \(\phi\)-symmetric Lorentzian Para-Sasakian manifold**

In this section we consider a generalized weakly Ricci \(\phi\)-symmetric Lorentzian Para-Sasakian manifold. Then by the virtue of (2.1), (1.7) yields
\[
(\nabla_X Q)(U) + \eta((\nabla_X Q)(U))\xi = A^*(X)QU + B^*(U)QX + S(U, X)g^* + \alpha^*(X)U + \beta^*(U)X + g(U, X)\sigma^*.
\]
(4.1)

from which it follows that
\[
g(\nabla_X Q(U), V) - S(\nabla_X U, V) + \eta((\nabla_X Q)(U))\eta(V) = A^*(X)S(U, V) + B^*(U)S(V, X) + D^*(V)S(U, X) + \alpha^*(X)g(U, V) + \beta^*(U)g(V, X) + \gamma^*(V)g(U, X).
\]
(4.2)
Putting $U = \xi$ in (4.2) and using (2.9), (2.11) we get
\[(n - 1)g(\phi X, V) - S(\phi X, V) = A^*(X)(n - 1)\eta(V) + B^*(\xi)S(V, X) + D^*(V)(n - 1)\eta(X) + \alpha^*(X)\eta(V) + \beta^*(\xi)g(V, X) + \gamma^*(V)\eta(X).\]

(4.3)

Setting $X = V = \xi$, $X = \xi$ and $V = \xi$ successively in (4.3) we get
\[(n - 1)\{A^*(\xi) + B^*(\xi) + D^*(\xi)\} + \alpha^*(\xi) + \beta^*(\xi) + \gamma^*(\xi) = 0,\]

(4.4)

\[(n - 1)D^*(V) + \gamma^*(V) = (n - 1)\{A^*(\xi) + B^*(\xi)\} + \alpha^*(\xi) + \beta^*(\xi)\]

and
\[(n - 1)A^*(X) + \alpha^*(X) = (n - 1)\{B^*(\xi) + D^*(\xi)\} + \beta^*(\xi) + \gamma^*(\xi)\]

(4.5) respectively.

Next using (4.4), (4.5) and (4.6) in (4.3), we get
\[(n - 1)g(\phi X, V) - S(\phi X, V) = B^*(\xi)S(V, X) + \beta^*(\xi)g(V, X) + [(n - 1)B^*(\xi) + \beta^*(\xi)]\eta(X)\eta(V).\]

(4.7)

Replacing $X$ by $\phi X$ in (4.7) and then using (4.7) we obtain
\[\{B^*(\xi)B^*(\xi) - 1\}S(X, V) = \{(n - 1)B^*(\xi) + \beta^*(\xi)\}g(\phi X, V) - \{B^*(\xi)\beta^*(\xi) + 1\}g(X, V)\]

(4.8) $-B^*(\xi)[(n - 1)B^*(\xi) + \beta^*(\xi)]\eta(X)\eta(V)$.

Thus we infer

**Theorem 4.1.** In a generalized weakly Ricci $\phi$-symmetric Lorentzian Para-Sasakian manifold the Ricci tensor $S$ takes the form (4.8).

If we put $(n - 1)B^*(\xi) = -\beta^*(\xi)$ in (4.8) then we get
\[S(X, V) = \left(\frac{B^*(\xi)\beta^*(\xi) + 1}{1 - B^*(\xi)B^*(\xi)}\right)g(X, V)\]

which implies that the manifold under consideration is Einstein.

**Corollary 4.2.** Every generalized weakly Ricci $\phi$-symmetric Lorentzian Para-Sasakian manifold is an Einstein manifold if the vector fields provided $(n - 1)B^*(\xi) = -\beta^*(\xi)$. 
5. Generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime

We now consider that the matter distribution of a generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime is perfect fluid. Then the Einstein’s field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) = kT(X,Y)$$

for all vector fields $X, Y$, where $S$ is the Ricci tensor of type $(0,2)$, $r$ is the scalar curvature, $k$ is the gravitational constant and $T$ is the energy-momentum tensor of type $(0, 2)$. In a perfect fluid spacetime, the energy-momentum tensor admits the following

$$T(X,Y) = pg(x, y) + (\sigma + p)\eta(X)\eta(Y);$$

where $\sigma, p$ are respectively the energy density, isotropic pressure and $\xi$ is considered as flow vector field of the fluid.

**Definition 5.1.** Ricci tensor $S$ of a Lorentzian Para-Sasakian manifold $(M^n, g)$ is named cyclic parallel if

$$(\nabla_X S)(U, V) + (\nabla_U S)(V, X) + (\nabla_V S)(X, U) = 0$$

In analogous with the work of [26] and the equation (3.21) we infer,

**Theorem 5.2.** ([26], Theorem 2.1, Page-310) In a generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime with cyclic parallel Ricci tensor, the associated scalars $L_1$ and $L_2$ are constants.

**Theorem 5.3.** ([26], Theorem 2.2, Page-311) In a generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime with cyclic parallel Ricci tensor, the energy-momentum tensor is cyclic parallel.

**Theorem 5.4.** ([26], Theorem 2.3, Page-311) In a generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime with constant associated scalars $L_1$ and $L_2$, the energy-momentum tensor is cyclic parallel, then the Ricci tensor is cyclic parallel.

**Theorem 5.5.** ([26], Theorem 3.1, Page-314) If a perfect fluid generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime obeys Einstein equation without cosmological constant and the square of the length of the Ricci operator is $\frac{1}{3} \left( \frac{L_1}{L_2^2} \right)$, then the spacetime can not contain pure matter. Also in such a spacetime without pure matter the pressure of the fluid is positive or negative according as $L_1 < \frac{L_2^4}{4}$ or $L_1 > \frac{L_2^4}{4}$.

**Definition 5.6.** Ricci tensor of a Lorentzian Para-Sasakian spacetime is said to be Codazzi type if

Theorem 5.7. ([26], Theorem 4.1., Page-315) In a generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime, the energy-momentum tensor is of Codazzi type if and only if its Ricci tensor is of Codazzi type.

Theorem 5.8. ([26], Theorem 4.2., Page-315) If the energy-momentum tensor of a perfect fluid a generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime is of Codazzi type, then the integral curves of the flow vector field are geodesics.

Theorem 5.9. ([26], Theorem 4.3., Page-318) If the energy-momentum tensor of a perfect fluid generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime is of Codazzi type, then both the energy density and isotropic pressure of the fluid are constants over a hypersurface orthogonal to $\xi$.

Theorem 5.10. ([26], Theorem 4.4., Page-318) In a perfect fluid generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime with Codazzi type energy-momentum tensor, the fluid has vanishing vorticity and vanishing shear.

Theorem 5.11. ([26], Theorem 4.5., Page-319) If the energy-momentum tensor of a perfect fluid generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime is of Codazzi type, then the possible local cosmological structure of the spacetime are of Petrov type I, D or O.

Theorem 5.12. ([26], Theorem 4.6., Page-319) If a perfect fluid generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian spacetime with Codazzi type energy-momentum tensor admits a conformal Killing vector field, then the spacetime is either conformally flat or of Petrov type N.

6. Example of a generalized weakly $\phi$-symmetric Lorentzian Para-Sasakian manifold

(see [4], p-286-287) Let $M^3(\phi, \xi, \eta, g)$ be a Lorentzian Para-Sasakian manifold with a $\phi$-basis
\[
e_1 = e^z \frac{\partial}{\partial x}, \quad e_2 = \phi e_1 = e^{z-\alpha x} \frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial z},
\]
where $\alpha$ is a non-zero constant. Then from Koszul’s formula for Lorentzian metric $g$, we can obtain the Levi-Civita connection as follows
\[
\nabla_{e_1} e_3 = e_2, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_1 = -e_3,
\]
\[
\nabla_{e_2} e_3 = e_1, \quad \nabla_{e_2} e_2 = \alpha e^z e_3, \quad \nabla_{e_2} e_1 = \alpha e^z e_2,
\]
\[
\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_1 = 0.
\]

Using the above relations, one can easily calculate the non-vanishing components of the curvature tensor $R$ (up to symmetry and skew-symmetry)
\[
R(e_1, e_2)e_1 = -(1 - \alpha^2 e^{2z}) e_2,
\]
\[
R(e_1, e_2)e_2 = (1 - \alpha^2 e^{2z}) e_1,
\]
\[
R(e_1, e_3)e_1 = -e_3, \quad R(e_1, e_3)e_3 = e_1,
\]
\[
R(e_2, e_3)e_2 = -e_3, \quad R(e_2, e_3)e_3 = e_2.
\]
Since \( \{e_1, e_2, e_3\} \) forms a basis, any vector field \( X, Y, U, V \in \chi(M) \) can be written as

\[
X = \sum_{i=1}^{3} a_i e_i, \quad Y = \sum_{i=1}^{3} b_i e_i, \quad Z = \sum_{i=1}^{3} c_i e_i.
\]

This implies that

\[ R(X, Y)Z = le_1 + me_2 + ne_3, \]

where

\[
l = (a_1 b_3 - a_3 b_1)c_3 + c_2 (a_1 b_2 - a_2 b_1)(1 - \alpha^2 e^{2z}),
\]

\[
m = (a_2 b_3 - a_3 b_2)c_3 - c_1 (a_1 b_2 - a_2 b_1)(1 - \alpha^2 e^{2z}),
\]

\[
n = -(a_1 b_3 - a_3 b_1)c_1 - c_2 (a_2 b_3 - a_3 b_2).
\]

Also,

\[
R(e_1, Y)Z = \{(1 - \alpha^2 e^{2z})b_2 c_2 + b_3 c_3\} e_1 - b_2 c_1 (1 - \alpha^2 e^{2z})e_2 - b_3 c_3 e_3,
\]

\[
R(e_2, Y)Z = -b_1 c_2 (1 - \alpha^2 e^{2z})e_1 + \{(1 - \alpha^2 e^{2z})b_1 c_1 - b_3 c_3\} e_2 - b_3 c_2 e_3,
\]

\[
R(e_3, Y)Z = -b_1 c_3 e_1 - b_2 c_3 e_2 + (b_1 c_1 + b_2 c_2)e_3,
\]

\[
R(X, e_1)Z = -(a_3 c_3 + (1 - \alpha^2 e^{2z})a_2 c_2) e_1 + a_2 c_1 (1 - \alpha^2 e^{2z})e_2 + a_3 c_1 e_3,
\]

\[
R(X, e_2)Z = a_1 c_2 (1 - \alpha^2 e^{2z})e_1 + \{a_3 c_3 - (1 - \alpha^2 e^{2z})a_1 c_1\} e_2 + a_3 c_2 e_3,
\]

\[
R(X, e_3)Z = a_1 c_3 e_1 + a_2 c_3 e_2 - (a_1 c_1 + a_2 c_2)e_3,
\]

\[
R(X, Y)e_1 = -(a_1 b_2 - a_2 b_1)(1 - \alpha^2 e^{2z})e_2 - (a_1 b_3 - a_3 b_1)e_3,
\]

\[
R(X, Y)e_2 = (a_1 b_2 - a_2 b_1)(1 - \alpha^2 e^{2z})e_1 - (a_2 b_3 - a_3 b_2)e_3,
\]

\[
R(X, Y)e_3 = (a_3 b_1 - a_1 b_3)e_1 + (a_3 b_2 - a_2 b_3)e_2,
\]

Again,

\[
G(X, Y)Z = pe_1 + qe_2 + re_3,
\]

where

\[
p = (a_1 b_2 - a_2 b_1)c_2 + (a_3 b_1 - a_1 b_3)c_3,
\]

\[
q = (a_2 b_1 - a_1 b_2)c_1 + (a_3 b_2 - a_2 b_3)c_3,
\]

\[
r = (a_3 b_1 - a_1 b_3)c_1 + (a_3 b_2 - a_2 b_3)c_2.
\]

Also, we have

\[
G(e_1, Y)Z = (b_2 c_2 - b_3 c_3)e_1 - b_2 c_1 e_2 - b_3 c_1 e_3,
\]

\[
G(e_2, Y)Z = -b_1 c_2 e_1 + (b_1 c_1 - b_3 c_3)e_2 - b_3 c_2 e_3,
\]

\[
G(e_3, Y)Z = b_1 c_3 e_1 + b_2 c_3 e_2 - (b_1 c_1 + b_2 c_2)e_3,
\]

\[
G(X, e_1)Z = -(a_2 c_2 - a_3 c_3)e_1 + a_2 c_1 e_2 + a_3 c_1 e_3,
\]

\[
G(X, e_2)Z = a_1 c_2 e_1 - (a_1 c_1 - a_3 c_3)e_2 + a_3 c_2 e_3,
\]

\[
G(X, e_3)Z = a_1 c_3 e_1 + a_2 c_3 e_2 - (a_1 c_1 + a_2 c_2)e_3,
\]

\[
G(X, Y)e_1 = -(a_1 b_2 - a_2 b_1)e_2 + (a_3 b_1 - a_1 b_3)e_3,
\]

\[
G(X, Y)e_2 = (a_1 b_2 - a_2 b_1)e_1 + (a_3 b_2 - a_2 b_3)e_3,
\]

\[
G(X, Y)e_3 = (a_3 b_1 - a_1 b_3)e_1 + (a_3 b_2 - a_2 b_3)e_2.
\]
and the components which can be obtained from these by the symmetry properties. Now, we calculate the covariant derivatives of the non-vanishing components of the curvature tensor as follows

\[(\nabla_{e_1} R)(X, Y)Z = -le_3 + ne_2 + a_1 R(e_3, Y)Z - a_3 R(e_2, Y)Z + b_1 R(X, e_3)Z - b_3 R(X, e_2)Z + c_1 R(X, Y)e_3 - c_3 R(X, Y)e_2\]

\[= -le_3 + ne_2 + a_1\{b_1 c_3 e_1 + b_2 c_3 e_2 - (b_1 c_1 + b_2 c_2)e_3\} - a_3\{-b_1 c_2(1 - \alpha^2 e^{2z})e_1 + \{(1 - \alpha^2 e^{2z})b_1 c_1 - b_3 c_3\}e_2 - b_3 c_2 e_3\} + b_1\{-a_1 c_3 e_1 - a_2 c_3 e_2 + (a_1 c_1 + a_2 c_2)e_3\} - b_3\{a_1 c_2(1 - \alpha^2 e^{2z})e_1 + \{a_3 c_3 - (1 - \alpha^2 e^{2z})a_1 c_1\}e_2 + a_3 c_2 e_3\} + c_1\{(a_3 b_1 - a_1 b_3)e_1 + (a_3 b_2 - a_2 b_3)e_2\} - c_3\{(a_1 b_2 - a_2 b_1)(1 - \alpha^2 e^{2z})e_1 - (a_2 b_3 - a_3 b_2)e_3\},\]

\[(\nabla_{e_2} \tilde{R})(X, Y)Z = \alpha e^z(\{e_2 + me_3\} + ne_1 - \alpha e^z\{a_1 R(e_2, Y)Z + a_2 R(e_3, Y)Z\} - a_3 R(e_1, Y)Z - \alpha e^z\{c_1 R(e_2, Y)Z + c_2 R(e_3, Y)Z\} - c_3 R(e_1, Y)Z,\]

\[= \alpha e^z(\{e_2 + me_3\} + ne_1 - \alpha e^z\{a_1 + b_1 + c_1\}\{-b_1 c_2(1 - \alpha^2 e^{2z})e_1 + \{(1 - \alpha^2 e^{2z})b_1 c_1 - b_3 c_3\}e_2 - b_3 c_2 e_3\} - \alpha e^z\{a_2 + b_2 + c_2\}\{b_1 c_3 e_1 + b_2 c_3 e_2 - (b_1 c_1 + b_2 c_2)e_3\}\{((1 - \alpha^2 e^{2z})b_2 c_2 - b_3 c_3\}e_1 - b_2 c_1(1 - \alpha^2 e^{2z})e_2 - b_3 c_1 e_3\} - \alpha e^z b_1 c_2\{(a_1 + b_1 + c_1)(1 - \alpha^2 e^{2z}) - \alpha e^z b_1 c_3(a_2 + b_2 + c_2)\}
- \alpha e^z\{a_3 + b_3 + c_3\}\{(1 - \alpha^2 e^{2z})b_2 c_2 - b_3 c_3\}e_1 + \{\alpha e^z l - \alpha e^z\{a_1 + b_1 + c_1\}\{(1 - \alpha^2 e^{2z})b_1 c_1 - b_3 c_3\} - \alpha e^z b_2 c_3\{(a_2 + b_2 + c_2) - b_2 c_1 a e^z\{(a_3 + b_3 + c_3)(1 - \alpha^2 e^{2z})\}e_2 - \alpha e^z\{(a_2 + b_2 + c_2)(b_1 c_1 + b_2 c_2) - \alpha e^z b_3 c_1\{(a_3 + b_3 + c_3)\}e_3,\}

and \((\nabla_{e_3} \tilde{R})(X, Y)Z = 0.\)

With the help of the above relations one can easily bring out the followings

\[\phi^2 ((\nabla_{e_1} R)(X, Y)Z)\]

\[= \{(1 - \alpha^2 e^{2z})\{(a_3 b_1 - a_1 b_3)e_2 + c_1(a_3 b_1 - a_1 b_3)
- c_3(a_1 b_2 - a_2 b_1)(1 - \alpha^2 e^{2z})\}e_1
+\{n + a_1 b_2 c_3 - (1 - \alpha^2 e^{2z})a_3 b_1 c_1 - a_2 b_1 c_3
+(1 - \alpha^2 e^{2z})a_1 b_3 c_1 + c_1(a_3 b_2 - a_2 b_3)\}e_2,\]
For the following choice of the the one forms

\[ \phi^2 ((\nabla_{e_2} \bar{R})(X,Y)Z) \]

\[ = [n + \alpha e^z b_1 c_2(a_1 + b_1 + c_1)(1 - \alpha^2 e^{2z}) - \alpha e^z b_1 c_3(a_2 + b_2 + c_2) + b_3 c_3]e_1 + [\alpha e^z l - \alpha e^z(a_1 + b_1 + c_1)\{ 1 - \alpha^2 e^{2z}\}b_1 c_1 - b_3 c_3 - \alpha e^z b_2 c_3(a_2 + b_2 + c_2) - b_2 c_1 \alpha e^z(a_3 + b_3 + c_3)](1 - \alpha^2 e^{2z})|e_2, \]

\[ \phi^2 ((\nabla_{e_3} \bar{R})(X,Y)Z) = 0. \]

For the following choice of the the one forms

\[ A(e_1) = \frac{1}{b_1 + c_1}, \quad A(e_2) = \frac{1}{b_2 + c_2}, \quad A(e_3) = \frac{1}{b_3 + c_3}, \]

\[ \alpha(e_1) = -\frac{1}{b_1 + c_1}, \quad \alpha(e_2) = -\frac{1}{b_2 + c_2}, \quad \alpha(e_3) = \frac{1}{b_3 + c_3}, \]

\[ B(e_1) = \frac{1}{a_1 + b_1}, \quad B(e_2) = \frac{1}{a_2 + b_2}, \quad B(e_3) = \frac{1}{a_3 + b_3}, \]

\[ \beta(e_1) = -\frac{1}{a_1 + b_1}, \quad \beta(e_2) = -\frac{1}{a_2 + b_2}, \quad \beta(e_3) = -\frac{1}{a_3 + b_3}, \]

\[ D(e_1) = \gamma(e_1) = 0 = D(e_2) = \gamma(e_2) = D(e_3) = \gamma(e_3), \]

one can easily verify the relations

\[ \phi^2 ((\nabla_{e_i} R)(X,Y)Z) = A(e_i)R(X,Y)Z + B(X)R(e_i,Y)Z + B(Y)R(X,e_i)Z + D(Z)R(X,Y)e_i + g(R(X,Y)Z,e_i)\rho + \alpha(e_i)G(X,Y)Z + \beta(X)G(e_i,Y)Z + \beta(Y)G(X,e_i)Z + \gamma(U)G(X,Y,e_i) + g(G(X,Y)Z,e_i)\sigma \]

provided \( a_i = kb_i = -2kc_i \) for \( i = 1, 2, 3 \). From the above, we can state that

**Theorem 6.1.** There exist a Lorentzian Para-Sasakian manifold (\( M^3, g \)) which is a generalized weakly \( \phi \)-symmetric.

**Acknowledgement**

The second named author gratefully acknowledges to UGC, F.No. 16-6(DEC.2018)/2019(NET/CSIR) and UGC-Ref.No. 1147/(CSIR-UGC NET DEC. 2018) for financial assistance.

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On generalized weakly (Ricci) $\phi$-symmetric LP-Sasakian manifold


Received by the editors April 26, 2020
First published online October 23, 2020