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Ring isomorphic to a structural matrix ring over a division ring

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Abstract. In this paper, we introduce conditions under which a ring is isomorphic to a structural matrix ring over a division ring.

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1. Introduction and notations

Structural matrix rings (a set of matrices in which certain entries are zero) have been investigated since they provide examples and counterexamples in Ring Theory, and for their connection to **PI** algebra. Structural matrix rings include the rings of triangular matrices and the rings of blocked triangular matrices, as well as the complete matrix-rings and is studied in [2] and [11]. Matrix rings play a fundamental role in mathematics and its applications. Triangular matrix rings appear naturally in Lie theory of both nilpotent and solvable Lie algebras. Since then, they have become an important ring construction; indeed a main tool in the description of semiprimary hereditary rings (for example, [3]). For example, the automorphisms of structural matrix rings over certain rings were described in [1] and [4], and a left Artinian CI-prime ring was characterized as a complete blocked triangular matrix ring over a division ring in [12].

A difficult question is to decide whether a given ring is isomorphic to a matrix ring or one of its variants. Several "hidden matrix-rings" have been shown in the literature (see [10] for the definition and more). These rings did not appear to be matrix-rings at the first sight, nevertheless they turned out to be isomorphic to matrix-rings.

In this paper, for any set S of subgroups of an additive group we set $\Sigma(S) = \sum_{I\in S} I$ and S is called independent if $\sum_{I\in S} I$ is a direct sum.

For a class $\mathcal{C}$ of subgroups and a subgroup L of an additive group, the sum of $\mathcal{C}$-subgroups properly contained in L is denoted by $\mathrm{Tp}_{\mathcal{C}}(L)$. If the group is a module, and $\mathcal{C}$ is the class of submodules, then we use the notation $\mathrm{Tp}(L)$ instead. Also, a group M is called **$\mathcal{C}$-Artinian** if M satisfies the descending chain condition on $\mathcal{C}$-subgroups, in other words, if whose $\mathcal{C}$-subgroups satisfy

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DCC and M is called **$\mathcal{C}$-ind.finite** if every independent set of $\mathcal{C}$-subgroups is finite. Finally, the class of submodules is denoted by $\mathbb{M}$. Through this paper we heavily use the notations, the terminologies, and the results in [7].

2. Basic Facts

Let $\mathcal{A}$ be a nonempty set and for every $i,j \in \mathcal{A}$, D_{ij} be an additive group. The set of maps $A : \mathcal{A}\times\mathcal{A} \longrightarrow \bigcup_{i,j\in\mathcal{A}} D_{ij}$ that for every $i,j \in \mathcal{A}$, $A(i,j) \in D_{ij}$ and for every $j \in \mathcal{A}$, there exist only finitely many $i \in \mathcal{A}$ with $A(i,j) \neq 0$, is denoted by $\mathrm{M}_{\mathcal{A}}([D])$, also for every $i,j \in \mathcal{A}$, $A(i,j)$ will be denoted by A_{ij}. If for every $i,j,k \in \mathcal{A}$, $D_{ij} \times D_{jk} \longrightarrow D_{ik}$ is a distributive multiplication map such that for every $i,j,k,l \in \mathcal{A}$,

$$\begin{array}{ccc} D_{ij}\times D_{jk}\times D_{kl} & \longrightarrow & D_{ij}\times D_{jl} \\ \downarrow & & \downarrow \\ D_{ik}\times D_{kl} & \longrightarrow & D_{il} \end{array}$$

in other words, for every $i,j,k,l \in \mathcal{A}$, $x \in D_{ij}$, $y \in D_{jk}$ and $z \in D_{kl}$, $x(yz) = (xy)z$, then we say that $[D]$ is a ring. It is clear that using the multiplication, $\mathrm{M}_{\mathcal{A}}([D]) \times \mathrm{M}_{\mathcal{A}}([D]) \longrightarrow \mathrm{M}_{\mathcal{A}}([D])$ given by, $(AB)_{ik} = \sum_{j\in\mathcal{A}} A_{ij}B_{jk}$. $\mathrm{M}_{\mathcal{A}}([D])$ is a ring (the ring of row finite matrices).

Let $\mathcal{H} \subseteq \mathcal{A}\times\mathcal{A}$ be a quasi-ordering and for every $(i,j) \notin \mathcal{H}$, $D_{ij} = 0$. Also let for every $(i,j),(j,k) \in \mathcal{H}$, $D_{ij}\times D_{jk} \longrightarrow D_{ik}$ be a distributive multiplication map such that for every $(i,j),(j,k),(k,l) \in \mathcal{H}$, $x \in D_{ij}$, $y \in D_{jk}$ and $z \in D_{kl}$, $x(yz) = (xy)z$. Considering the zero multiplication $D_{ij} \times D_{jk} \longrightarrow D_{ik}$ for every $(i,j) \notin \mathcal{H}$ or $(j,k) \notin \mathcal{H}$, then for every $i,j,k,l \in \mathcal{A}$, $x \in D_{ij}$, $y \in D_{jk}$ and $z \in D_{kl}$, $x(yz) = (xy)z$ and so $[D]$ is a ring.

Let $[D]$ and $[T]$ be rings. If for every $i,j \in \mathcal{A}$, $\Delta_{ij} : D_{ij} \longrightarrow T_{ij}$ is a group homomorphism such that for every $i,j,k \in \mathcal{A}$, $x \in D_{ij}$ and $y \in D_{jk}$, $\Delta_{ij}(x)\Delta_{jk}(y) = \Delta_{ik}(xy)$, then it is clear that the map $\Delta : \mathrm{M}_{\mathcal{A}}([D]) \longrightarrow \mathrm{M}_{\mathcal{A}}([T])$ given by $\Delta(A)_{ij} = \Delta_{ij}(A_{ij})$ is a ring homomorphism. Moreover, if for every $i,j \in \mathcal{A}$, Δ_{ij} is one to one (onto) then so is Δ and conversely.
Let $\mathcal{H} \subseteq \mathcal{A}\times\mathcal{A}$ be a quasi-ordering and for every $(i,j) \notin \mathcal{H}$, $D_{ij} = 0 = T_{ij}$. Also let for every $(i,j) \in \mathcal{H}$, $\Delta_{ij} : D_{ij} \longrightarrow T_{ij}$ be a group homomorphism such that for every $(i,j),(j,k) \in \mathcal{H}$, $x \in D_{ij}$ and $y \in D_{jk}$, $\Delta_{ij}(x)\Delta_{jk}(y) = \Delta_{ik}(xy)$. It is easy to see that considering the zero map $\Delta_{ij} : D_{ij} \longrightarrow T_{ij}$ for every $(i,j) \notin \mathcal{H}$, then for every $i,j,k \in \mathcal{A}$, $x \in D_{ij}$ and $y \in D_{jk}$, $\Delta_{ij}(x)\Delta_{jk}(y) = \Delta_{ik}(xy)$. Thus, the map $\Delta : \mathrm{M}_{\mathcal{A}}([D]) \longrightarrow \mathrm{M}_{\mathcal{A}}([T])$ given by $\Delta(A)_{ij} = \Delta_{ij}(A_{ij})$ for every $(i,j) \in \mathcal{H}$ and $\Delta(A)_{ij} = 0$ for every $(i,j) \notin \mathcal{H}$, is a ring homomorphism because. Moreover, if for every $(i,j) \in \mathcal{H}$, Δ_{ij} is one to one (onto) then so is Δ and conversely.
Finally, let R be a ring and for every $i \in \mathcal{A}$, M_i be an R-module. For every

$i, j \in \mathcal{A}$, setting $D_{ij} = \mathrm{Hom}_R(M_i, M_j)$ and using the multiplication $D_{ij} \times D_{jk} \longrightarrow D_{ik}$, given by $f \cdot g = g \circ f$, it is clear that $[D]$ is a ring.

Let D be a ring, $\mathcal{A}$ be a nonempty set and $\mathcal{H} \subseteq \mathcal{A} \times \mathcal{A}$ be a quasi-ordering on $\mathcal{A}$. We set $\mathrm{M}_{\mathcal{H}}(D) = \{A \in \mathrm{M}_{\mathcal{A}\times\mathcal{A}}(D) \mid \forall (i,j) \notin \mathcal{H}, A_{ij} = 0\}$. We know that it is called a structural matrix ring.

Lemma 2.1. *Let $\mathcal{A}$ be a nonempty set, R be a ring and for every $i \in \mathcal{A}$, M_i and N_i be R-modules. Also, let for every $i \in \mathcal{A}$, $\alpha_i : M_i \longrightarrow N_i$ and $\beta_i : N_i \longrightarrow M_i$ be homomorphisms such that $\beta_i \circ \alpha_i = 1$. For every $i, j \in \mathcal{A}$ we set $D_{ij} = \mathrm{Hom}_R(M_i, M_j)$ and $T_{ij} = \mathrm{Hom}_R(N_i, N_j)$. Then, the map $\Delta : \mathrm{M}_{\mathcal{A}}([D]) \longrightarrow \mathrm{M}_{\mathcal{A}}([T])$ given by $\Delta(A)_{ij} = \beta_i \cdot A_{ij} \cdot \alpha_j$ is a monomorphism. Moreover, if for every $i \in \mathcal{A}$, $\alpha_i \circ \beta_i = 1$, then Δ is an isomorphism.*

Proof. For each $i, j \in \mathcal{A}$, consider the group homomorphism $\Delta_{ij} : D_{ij} \longrightarrow T_{ij}$ given by $\Delta_{ij}(f) = \beta_i \cdot f \cdot \alpha_j$. For each $i, j, k \in \mathcal{A}$, $f \in D_{ij}$ and $g \in D_{jk}$ we have

$$\Delta_{ij}(f) \cdot \Delta_{jk}(g) = (\beta_i \cdot f \cdot \alpha_j) \cdot (\beta_j \cdot g \cdot \alpha_k) = \beta_i \cdot f \cdot g \cdot \alpha_k = \beta_i \cdot (f \cdot g) \cdot \alpha_k = \Delta_{ik}(f \cdot g)$$

On the other hand, $\Delta_{ij}(A_{ij}) = \beta_i \cdot A_{ij} \cdot \alpha_j = \Delta(A)_{ij}$, consequently Δ is a homomorphism. Now let $f \in D_{ij}$ and $\Delta_{ij}(f) = 0$. Then, $f = \alpha_i \cdot (\beta_i \cdot f \cdot \alpha_j) \cdot \beta_j = 0$, thus Δ_{ij} is one to one. Thus, Δ is a monomorphism. Finally, let $g \in T_{ij}$. We have $\Delta_{ij}(\alpha_i \cdot g \cdot \beta_j) = \beta_i \cdot (\alpha_i \cdot g \cdot \beta_j) \cdot \alpha_j = g$. Thus, Δ_{ij} is onto. Consequently Δ is an isomorphism. □

Lemma 2.2. *Let $\mathcal{A}$ be a nonempty set, R be a ring and $\{M_i \mid i \in \mathcal{A}\}$ be a set of R-modules. For each $i, j \in \mathcal{A}$ we set $D_{ij} = \mathrm{Hom}_R(M_i, M_j)$. The set of R-modules $F : \sum_{k\in\mathcal{A}} M_k \longrightarrow \sum_{k\in\mathcal{A}} M_k$ such that for each $i \in \mathcal{A}$ there exist only finitely many $j \in \mathcal{A}$ with $\pi_j \circ F \circ \iota_i \neq 0$ is denoted by $\mathrm{End}^{\infty}({}_R(\sum_{k\in\mathcal{A}} M_k))$ or $\mathrm{End}^{\infty}(\sum_{k\in\mathcal{A}} M_k)$ (π_i is the canonical projection and ι_i is the canonical injection). The map $\Lambda : \mathrm{End}^{\infty}(\sum_{k\in\mathcal{A}} M_k)^{\mathrm{op}} \longrightarrow \mathrm{M}_{\mathcal{A}}([D])$ (the ring of column finite matrices) given by $\Lambda(F)_{ij} = \iota_i \cdot F \cdot \pi_j$ is an isomorphism.*

Proof. Let $F, G \in \mathrm{End}^{\infty}(\sum_{k\in\mathcal{A}} M_k)^{\mathrm{op}}$. For every $i, k \in \mathcal{A}$ we have

$$(\Lambda(F)\Lambda(G))_{ik} = \sum_{j\in\mathcal{A}} \Lambda(F)_{ij} \cdot \Lambda(G)_{jk} = \sum_{j\in\mathcal{A}} (\iota_i \cdot F \cdot \pi_j) \cdot (\iota_j \cdot G \cdot \pi_k) =$$

$$\iota_i \cdot F \cdot (\sum_{j\in\mathcal{A}} \pi_j \cdot \iota_j) \cdot G \cdot \pi_k = \iota_i \cdot F \cdot G \cdot \pi_k = \Lambda(F \cdot G)_{ik}$$

Notice that $\sum_{j\in\mathcal{A}} \pi_j \cdot \iota_j = 1$ element wise. Thus, $\Lambda(F)\Lambda(G) = \Lambda(F \cdot G)$. Consequently Λ is a homomorphism.
Now we show that the map is onto. Let for each $i, j \in \mathcal{A}$, $f_{ij} : M_i \longrightarrow M_j$ be a homomorphism such that for each $k \in \mathcal{A}$ there exist only finitely many $l \in \mathcal{A}$ with $f_{kl} \neq 0$. Consider $i \in \mathcal{A}$. There exists a homomorphism $\theta_i : M_i \longrightarrow \prod_{k\in\mathcal{A}} M_k$ such that for every $j \in \mathcal{A}$, $\pi_j \circ \theta_i = f_{ij}$. It is clear that $\theta_i(M_i) \subseteq \sum_{k\in\mathcal{A}} M_k$, so $\theta_i : M_i \longrightarrow \sum_{k\in\mathcal{A}} M_k$. Thus, there exists a homomorphism $F : \sum_{k\in\mathcal{A}} M_k \longrightarrow \sum_{k\in\mathcal{A}} M_k$ such that for every $i \in \mathcal{A}$, $F \circ \iota_i = \theta_i$, consequently $\pi_j \circ F \circ \iota_i = f_{ij}$. It is clear that $F \in \mathrm{End}^{\infty}(\sum_{k\in\mathcal{A}} M_k)$ and $\Lambda(F)_{ij} = \pi_j \circ F \circ \iota_i$, also it is clear that Λ is an injection. □

In the following, a set $\{N_i \mid i \in \mathcal{A}\}$ of fully invariant submodules of a module N is called a $\square$-set if
- For each $i \in \mathcal{A}$, every homomorphism $N_i \longrightarrow N_i$ can be extended to a homomorphism $N \longrightarrow N$.
- For every nonzero homomorphism $f : N \longrightarrow N$ and every $i \in \mathcal{A}$, $f(N_i) \neq 0$.
- For every $i, j \in \mathcal{A}$, either $\mathrm{Hom}_R(N_i, N_j) = 0$ or $N_i \subseteq N_j$ and for every homomorphism $f : N_i \longrightarrow N_j$, $f(N_i) \subseteq N_i$.

Lemma 2.3. *Let R be a ring, N_R be a module and $\{N_i \mid i \in \mathcal{A}\}$ be a $\square$-set. For every $i, j \in \mathcal{A}$ we set $T_{ij} = \mathrm{Hom}_R(N_i, N_j)$. Then, $\mathrm{M}_{\mathcal{A}}([T]) \cong \mathrm{M}_{\mathcal{H}}(\mathrm{End}(N_R)^{\mathrm{op}})$ for a quasi-ordering $\mathcal{H}$ on $\mathcal{A}$.*

Proof. It is easy to see that $\mathcal{H} = \{(i, j) \in \mathcal{A} \times \mathcal{A} \mid N_i \subseteq N_j\}$ is a quasi-ordering. For every $(i, j) \in \mathcal{H}$, we set $Q_{ij} = \mathrm{End}({}_R N)^{\mathrm{op}}$ and consider the map $\Delta_{ij} : Q_{ij} \longrightarrow T_{ij}$ given by $\Delta_{ij}(f) = f\,|_{N_i}$ and for every $(i, j) \notin \mathcal{H}$, we set $Q_{ij} = 0$. It is obvious that for every $(i, j) \in \mathcal{H}$, Δ_{ij} is a group isomorphism. Let $(i, j), (j, k) \in \mathcal{H}$, $f \in Q_{ij}$ and $g \in Q_{jk}$. Then, $\Delta_{ik}(f \cdot g) = \Delta_{ij}(f) \cdot \Delta_{jk}(g)$. Consequently, $\Delta : \mathrm{M}_{\mathcal{A}}([Q]) \longrightarrow \mathrm{M}_{\mathcal{A}}([T])$ is an isomorphism. On the other hand, $\mathrm{M}_{\mathcal{A}}([Q]) = \mathrm{M}_{\mathcal{H}}(\mathrm{End}({}_R N)^{\mathrm{op}})$. $\square$

Lemma 2.4. *Let R be a ring, M_R be a module and $\{M_i \mid i \in \mathcal{A}\}$ be an independent set of submodules of M such that $M = \sum_{k \in \mathcal{A}} M_k$ and for every homomorphism $F : M \longrightarrow M$ and every $i \in \mathcal{A}$, there exist only finitely many $j \in \mathcal{A}$ with $\pi_j \circ F \circ \iota_i \neq 0$. Also, let N_R be a module such that for every $i \in \mathcal{A}$, M_i has an isomorphic copy N_i in N. If $\{N_i \mid i \in \mathcal{A}\}$ is a $\square$-set, then $\mathrm{End}(M_R)^{\mathrm{op}} \cong \mathrm{M}_{\mathcal{H}}(\mathrm{End}(N_R)^{\mathrm{op}})$ for a quasi-ordering $\mathcal{H}$ on $\mathcal{A}$.*

Proof. For every $i \in \mathcal{A}$, there exists an isomorphism $\alpha_i : M_i \longrightarrow N_i$. For every $i, j \in \mathcal{A}$ we set $D_{ij} = \mathrm{Hom}_R(M_i, M_j)$, $\beta_j = \alpha_j^{-1}$ and $T_{ij} = \mathrm{Hom}_R(N_i, N_j)$. Then, $\mathrm{End}({}_R M)^{\mathrm{op}} \cong \mathrm{M}_{\mathcal{A}}([D])$ by Lemma 2.2 and $\mathrm{M}_{\mathcal{A}}([D]) \cong \mathrm{M}_{\mathcal{A}}([T])$ by Lemma 2.1. On the other hand, $\mathrm{M}_{\mathcal{A}}([T]) \cong \mathrm{M}_{\mathcal{H}}(\mathrm{End}(N_R)^{\mathrm{op}})$ for a quasi-ordering $\mathcal{H}$ on $\mathcal{A}$ by Lemma 2.3. $\square$

Lemma 2.5. *Let R be a ring with identity and $\{J_i \mid 1 \leq i \leq n\}$ be an independent set of right ideals with $R = \bigoplus_{i=1}^n J_i$. Also, let N_R be a module such that for every $i \in \mathcal{A} = \{1, 2, \cdots, n\}$, J_i has an isomorphic copy N_i in N. If $\{N_i \mid i \in \mathcal{A}\}$ is a $\square$-set, then $R \cong \mathrm{M}_{\mathcal{H}}(\mathrm{End}(N_R))$ for a quasi-ordering $\mathcal{H}$ on $\mathcal{A}$.*

Proof. Set $D = \mathrm{End}(N_R)$. Applying Lemma 2.4 for $M = R$, we have $\mathrm{End}(R_R)^{\mathrm{op}} \cong \mathrm{M}_{\mathcal{H}}(D^{\mathrm{op}})$. On the other hand, $\phi : R \longrightarrow \mathrm{End}(R_R)$ given by $\phi(r)(x) = rx$ is a ring isomorphism, so $R \cong \mathrm{End}(R_R)$. Also, it is easy to see that $\mathrm{M}_{\mathcal{H}^{-1}}(D)^{\mathrm{op}} \cong \mathrm{M}_{\mathcal{H}}(D^{\mathrm{op}})$. Thus, $R \cong \mathrm{M}_{\mathcal{H}^{-1}}(D)$. $\square$

3. Sufficiency of The Conditions

Definition 3.1. A module M is called **summand-form** if for every submodule L, the kernel of any homomorphism $L \longrightarrow M$ is a direct summand of L

and it is called **well behaved** if isomorphic submodules are identical. Thus, M is well behaved and summand-form iff for every submodule L and every homomorphism $\theta : L \longrightarrow M$, $L = \mathrm{Ker}(\theta) \oplus \mathrm{Img}(\theta)$. It is easy to see that for any indecomposable well behaved and summand-form module M, $\mathrm{End}(M)$ is a division ring. Finally, every well behaved and summand-form module is a Duo module.

Definition 3.2. For any set B, a set A is called B**-within** if $A \subseteq B$.

Definition 3.3. For a class $\mathcal{C}$ of subgroups and a subgroup K,

1. K is called $\mathcal{C}$**-minimal-like** if for every $\mathcal{C}$-subgroups J, either $K \cap J = 0$ or $K \subseteq J$.
2. K is called $\mathcal{C}$**-hole** if $\mathrm{Tp}_{\mathcal{C}}(K) \neq K$.
3. We say that K is $\mathcal{C}$**-summand** if there is a $\mathcal{C}$-subgroup J such that $M = K \oplus J$.
4. We say that K is $\mathcal{C}$**-indecomposable** if for any $\mathcal{C}$-subgroups I and J, $K = I \oplus J$ implies $I = 0$ or $J = 0$.

If the group is a module, and $\mathcal{C}$ is the class of submodules, then we use **minimal-like**, **hole** (or **local module** if L is a submodule) and **co-hole** and we drop the $\mathcal{C}$ sign in $\mathrm{Tp}_{\mathcal{C}}()$. Furthermore, the class of $\mathcal{C}$-hole subgroups is denoted by $\mathcal{C}^{\mathrm{hs}}$ and the class of $\mathcal{C}$-summand subgroups is denoted by $\mathcal{C}^{\oplus}$. Thus, according to notations in [6] and [8], the class of $\mathcal{C}$-hole $\mathcal{C}$-subgroups is denoted by $\mathcal{C} \cap \mathcal{C}^{\mathrm{hs}}$, the class of $\mathcal{C}$-summand $\mathcal{C}$-subgroups is denoted by $\mathcal{C} \cap \mathcal{C}^{\oplus}$, and the class of minimal $\mathcal{C}$-summand $\mathcal{C}$-subgroups is denoted by $(\mathcal{C} \cap \mathcal{C}^{\oplus})^{\mathrm{mn}}$, also the class of hole subgroups is denoted by $\mathbb{M}^{\mathrm{hs}}$ and the class of hole submodules is denoted by $\mathbb{M} \cap \mathbb{M}^{\mathrm{hs}}$.

Definition 3.4. Let R be a ring.

1. $I \subseteq R$ is called **left component**, if $I = Re$ for an idempotent $e \in R$.
2. $I \subseteq R$ is called **left annihilator**, if $\mathrm{ann}_{\mathrm{l}}(\mathrm{ann}_{\mathrm{r}}(I)) = I$.
3. $I \subseteq R$ is called **left inner-faithful**, if $I \cap \mathrm{ann}_{\mathrm{l}}(I) = 0$.

Furthermore, the class of left annihilators is denoted by $\mathrm{l}\mathbb{A}$ and the class of right annihilators is denoted by $\mathrm{r}\mathbb{A}$. Thus, according to notations in [6] and [8], the class of $\mathrm{l}\mathbb{A}$-summand left annihilators is denoted by $\mathrm{l}\mathbb{A} \cap \mathrm{l}\mathbb{A}^{\oplus}$.

Let R be a ring and M be a left R-module. In the following, a hole submodule K for which $\mathrm{Tp}(K) = \mathrm{P}(R)K$, is called $*$-hole and $\langle \mathrm{l}\mathbb{I}^{\cong} \colon M \rangle$ is the set of submodules which are isomorphic to a left ideal of R. In the following, $\mathrm{P}(R)$ is the prime radical of R.

Lemma 3.5. *Let R be a ring with identity.*

1. *For any left ideals I and J with $R = I \oplus J$, there exists an idempotent $e \in R$ such that $I = Re$ and $J = R(1-e)$.*

2. *for any idempotent $e \in R$ we have $R = Re \oplus R(1-e)$, $R = eR \oplus (1-e)R$, $\mathrm{ann_l}(e) = R(1-e)$ and $\mathrm{ann_r}(e) = (1-e)R$.*

Proof. Straightforward. □

Lemma 3.6. *Let R be a ring. For every $\mathrm{l}\mathbb{A}$-minimal-like left component I and left ideal J, every nonzero left R-module homomorphism $I \longrightarrow J$ is one to one and can be extended to a module homomorphism $R \longrightarrow J$.*

Proof. There exists an idempotent $e \in R$ with $I = Re$. Then, $Ie = I$. Now let $f : I \longrightarrow J$ be a nonzero module homomorphism. Set $v = f(e)$. Then, for every $x \in I$, $f(x) = f(xe) = xf(e) = xv$. Also, $I \cap \mathrm{ann_l}(v) = 0$ because otherwise, $I \subseteq \mathrm{ann_l}(v)$, then $f(I) = f(Ie) = If(e) = Iv = 0$ which is a contradiction. Thus, if $x \in I$ and $f(x) = 0$, then $xv = 0$, so $x \in I \cap \mathrm{ann_l}(v)$, implying $x = 0$. □

Lemma 3.7. *Let R be a ring with identity.*

1. *Every nonzero $\mathrm{l}\mathbb{I}$-summand left ideal contained in a $\mathrm{l}\mathbb{I}$-indecomposable left ideal I is equal to I.*

2. *Every minimal $\mathrm{l}\mathbb{I}$-summand left ideal is $\mathrm{l}\mathbb{I}$-indecomposable.*

3. *Every left component is an idempotent, left annihilator and left inner-faithful left ideal.*

4. *For every left component I and every ideal P, $P \cap I = PI$.*

Proof. (1) Let K be a nonzero $\mathrm{l}\mathbb{I}$-summand left ideal contained in I. There exists a left ideal L such that $R = K \oplus L$, then $I = K \oplus (I \cap L)$, implying $I = K$.
(2) Let I be a minimal $\mathrm{l}\mathbb{I}$-summand left ideal. Now let K and L be left ideals with $I = K \oplus L$ and $K \neq 0$. There exists an ideal J with $R = I \oplus J$. Then $R = K \oplus (L \oplus J)$. Thus, K is a nonzero $\mathrm{l}\mathbb{I}$-summand left ideal contained in I, implying $I = K$.
(3) Let I be a left component. There exists an idempotent $e \in R$ with $I = Re$. Then, $e \in I^2$, implying $I^2 = I$. Also, $I \cap \mathrm{ann_l}(e) = 0$ by Lemma 3.5. On the other hand, $\mathrm{ann_l}(I) \subseteq \mathrm{ann_l}(e)$. Thus, $I \cap \mathrm{ann_l}(I) = 0$. Furthermore, $I = \mathrm{ann_l}(1-e)$ by Lemma 3.5, so I is a left annihilator.
(4) There exists an idempotent $e \in R$ with $I = Re$. Clearly $PI \subseteq P \cap I$. Now let $x \in P \cap I$. Then, $x = xe \in Pe \subseteq PI$. Thus, $P \cap I \subseteq PI$. □

Lemma 3.8. *Let R be a ring with identity and I be a left ideal. The following are equivalent.*

1. *I is a $\mathrm{l}\mathbb{I}$-summand.*

2. *I is a $\mathrm{l}\mathbb{A}$-summand.*

3. *I is a left component.*

Proof. (1⇒3) There exists a left ideal J with $R = I \oplus J$. Thus, $I = Re$ for an idempotent $e \in R$ by Lemma 3.6.
(3⇒2) There exists an idempotent $e \in R$ with $I = Re$. Then, $R = I \oplus \mathrm{ann}_{\mathrm{l}}(e)$ by Lemma 3.6. Thus, I is a $\mathrm{l}\mathbb{A}$-summand.
(2⇒1) It is obvious. □

Lemma 3.9. *Let R be a ring. A nonzero left ideal is $\mathrm{l}\mathbb{I}$-summand and $\mathrm{l}\mathbb{I}$-indecomposable iff it is a minimal $\mathrm{l}\mathbb{I}$-summand left ideal.*

Proof. By Lemma 3.7. □

Lemma 3.10. *Let R be a ring. For a proper left ideal P, a left ideal J is a minimal non P-within left ideal iff J is $\mathrm{l}\mathbb{I}$-hole and $\mathrm{Tp}_{\mathrm{l}\mathbb{I}}(J) = J \cap P$.*

Proof. Straightforward. □

Lemma 3.11. *Let R be a $\mathrm{l}\mathbb{I}$-Artinian ring with identity. For every left ideal I, the following are equivalent.*

1. *I is a minimal non $\mathrm{P}(R)$-within left ideal.*

2. *I is a minimal non $\mathrm{P}(R)$-within left annihilator.*

3. *I is a minimal left inner-faithful left annihilator.*

4. *I is a minimal left component.*

Proof. (1⇒4) I contains a nonzero idempotent e by [9, (21.29)]. Re is a non $\mathrm{P}(R)$-within left ideal contained in I, so $I = Re$. Thus, I is a nonzero left component. Now let J be a nonzero left component contained in I. Then, $J \not\subseteq \mathrm{P}(R)$, implying $J = I$.
(4⇒1) I contains a minimal non $\mathrm{P}(R)$-within left ideal N because $I \not\subseteq \mathrm{P}(R)$. Then, N is a minimal left component by the above argument, implying $N = I$. Thus, I is a minimal non $\mathrm{P}(R)$-within left ideal.
(1⇒2) $I = Re$ for a nonzero idempotent e by the above argument. Thus, I is a non $\mathrm{P}(R)$-within left annihilator by Lemma 3.7. Now let J be a non $\mathrm{P}(R)$-within left annihilator contained in I. Then, $J = I$.
(2⇒1) I contains a minimal non $\mathrm{P}(R)$-within left ideal N. Then, N is a minimal non $\mathrm{P}(R)$-within left annihilator by the above argument, implying $N = I$. Thus, I is a minimal non $\mathrm{P}(R)$-within left ideal.
(1⇒3) $I = Re$ for a nonzero idempotent e by the above argument. Thus, I is a nonzero left inner-faithful left annihilator by Lemma 3.7. Now let J be a nonzero left inner-faithful left annihilator contained in I. Then, $J \not\subseteq \mathrm{P}(R)$, implying $J = I$.
(3⇒1) I contains a minimal non $\mathrm{P}(R)$-within left ideal N because $I \not\subseteq \mathrm{P}(R)$. Then, N is a minimal left inner-faithful left annihilator by the above argument, implying $N = I$. Thus, I is a minimal non $\mathrm{P}(R)$-within left ideal. □

Lemma 3.12. *Let R be a ring.*

1. *Every nilpotent* $\mathrm{l}\mathbb{A}$*-minimal-like left ideal is with zero square.*

2. *If* $\mathrm{P}(R)$ *is nilpotent, then every nonzero square minimal left annihilator is a minimal non* $\mathrm{P}(R)$*-within left annihilator.*

Proof. (1) Let I be a nonzero nilpotent $\mathrm{l}\mathbb{A}$-minimal-like left ideal and n be the smallest integer that $I^n = 0$. Then, $n \geq 2$ and $0 \neq I^{n-1} \subseteq I \cap \mathrm{ann}_{\mathrm{l}}(I)$, implying $I \subseteq \mathrm{ann}_{\mathrm{l}}(I)$. Thus, $I^2 = 0$.
(2) By (1). □

Lemma 3.13. *Let R be a ring. The following are equivalent.*

1. *Zero is the only zero square left annihilator (left annihilator ideal).*

2. *Zero is the only nilpotent left annihilator (left annihilator ideal).*

3. *Every left annihilator (left annihilator ideal) is left inner-faithful.*

In this case R is called $\mathrm{l}\mathbb{A}$**-semiprime** *(*$\mathrm{l}\mathbb{A}\mathbb{I}$**-semiprime***). Also, R is called* **left healthy** *if every minimal non* $\mathrm{P}(R)$*-within left annihilator is a minimal left annihilator. Clearly semiprime* $\Rightarrow$ $\mathrm{l}\mathbb{A}$*-semiprime* $\Rightarrow$ $\mathrm{l}\mathbb{A}\mathbb{I}$*-semiprime. Also, right nonsingular* $\Rightarrow$ $\mathrm{l}\mathbb{A}\mathbb{I}$*-semiprime by [5, Theorem 2.8]. Moreover, in a* $\mathrm{r}\mathbb{I}$*-Artinian ring,* $\mathrm{l}\mathbb{A}\mathbb{I}$*-semiprime* $\Rightarrow$ *right nonsingular by [5, Theorem 2.8]. Furthermore,* $\mathrm{l}\mathbb{A}$*-semiprime* $\Rightarrow$ *there is no zero square minimal left annihilator and if the ring is* $\mathrm{l}\mathbb{A}$*-Artinian, the converse also holds. Structural matrix rings over a division ring are left healthy, right healthy,* $\mathrm{l}\mathbb{A}\mathbb{I}$*-semiprime and* $\mathrm{r}\mathbb{A}\mathbb{I}$*-semiprime but not necessarily* $\mathrm{l}\mathbb{A}$*-semiprime or* $\mathrm{r}\mathbb{A}$*-semiprime*

Proof. Straightforward. □

Lemma 3.14. *Let R be a* $\mathrm{l}\mathbb{A}$*-Artinian ring. If* $\mathrm{P}(R)$ *is nilpotent and R has no zero square minimal left annihilator, then R is left healthy (The converse does not hold).*

Proof. By Let I be a minimal non $\mathrm{P}(R)$-within left annihilator. I contains a minimal left annihilator J. J is nonzero square, so $J \nsubseteq \mathrm{P}(R)$ by Lemma 3.12, implying $J = I$. Thus, I is a minimal left annihilator. □

Lemma 3.15. *Let R be a nonzero ring with identity. If R is either* $\mathrm{l}\mathbb{A} \cap \mathrm{l}\mathbb{A}^{\oplus}$*-Artinian or* $\mathrm{l}\mathbb{A} \cap \mathrm{l}\mathbb{A}^{\oplus}$*-ind.finite, then there exists a finite independent set T of minimal left components such that $R = \Sigma(T)$.*

Proof. By Lemma 3.8, $\langle \mathrm{l}\mathbb{I} \cap \mathrm{l}\mathbb{I}^{\oplus} : R\rangle = \langle \mathrm{l}\mathbb{A} \cap \mathrm{l}\mathbb{A}^{\oplus} : R\rangle$, so by [9, (19.20)], ${}_RR$ can be decomposed into a finite direct sum of indecomposable submodules. On the other hand, every indecomposable summand submodule is a minimal $\mathrm{l}\mathbb{A}$-summand left annihilator and conversely by Lemma 3.9. □

Lemma 3.16. *Let ${}_RN$ be a module isomorphic to a left ideal of R. For every* $\mathrm{l}\mathbb{A}$*-minimal-like left ideal of R and $n \in N$, either $In = 0$ or $In \cong I$.*

Proof. $In \neq 0$, implies $I \nsubseteq \mathrm{ann}_R(n)$. On the other hand, $\mathrm{ann}_R(n)$ is a left annihilator, so $I \cap \mathrm{ann}_R(n) = 0$. Thus $I \cong In$. □

Lemma 3.17. *Let ${}_RM$ be a module and $S \subseteq \langle \mathrm{l}\mathbb{I}^{\cong} \!:\! M \rangle$. If $\Sigma(S)$ is faithful, then every nonzero* $\mathrm{l}\mathbb{A}$*-minimal-like left ideal has a copy in a $N \in S$.*

Proof. Let I be a nonzero $\mathrm{l}\mathbb{A}$-minimal-like left ideal. There exists $N \in S$ with $IN \neq 0$. Then, there exists $n \in N$ with $In \neq 0$, implying $I \cong In$ by Lemma 3.16. □

Lemma 3.18. *Let ${}_RM$ be a module and N and L be submodules. If N is isomorphic to a* $\mathrm{l}\mathbb{A}$*-minimal-like left component and L is isomorphic to a left ideal, then every nonzero homomorphism $N \longrightarrow L$ is one to one.*

Proof. Let $f : N \longrightarrow L$ be a nonzero homomorphism. There exists a $\mathrm{l}\mathbb{A}$-minimal-like left component N_0 and an isomorphism $p : N_0 \longrightarrow N$. Also, there exists a left ideal L_0 and an isomorphism $p : L_0 \longrightarrow L$. Then, the nonzero homomorphism $q^{-1} \circ f \circ p : N_0 \longrightarrow L_0$ is one to one by Lemma 3.6. Thus, f is one to one. □

Lemma 3.19. *Let R be a nonzero ring with identity and $\{J_i \mid 1 \leq i \leq n\}$ be a set of minimal left annihilators with $R = \bigoplus_{i=1}^{n} J_i$. Also, let M be a left module such that for each $1 \leq i \leq n$, N_i is the unique submodule of M which is an isomorphic copy of J_i.*

1. *If M is indecomposable, then M is unitary.*
2. *If M is unitary, then every submodule N which is isomorphic to a left ideal is a sum of a set of members of $\{N_i \mid 1 \leq i \leq n\}$.*
3. *If M is unitary and $\Sigma\langle \mathrm{l}\mathbb{I}^{\cong} \!:\! M \rangle = M$, then $M = \sum_{i=1}^{n} N_i$.*
4. *If M is unitary and $\Sigma\langle \mathrm{l}\mathbb{I}^{\cong} \!:\! M \rangle = M$, and R is* $\mathrm{l}\mathbb{I}$*-Artinian, then M is Artinian.*
5. *If M is unitary, distributive, and $\Sigma\langle \mathrm{l}\mathbb{I}^{\cong} : M \rangle = M$,then for every left module P which is isomorphic to a left ideal, for every homomorphism $f : M \longrightarrow P$,* $\mathrm{Ker}(f)$ *is direct summand.*

Proof. (1) Because $M = \{m \in M \mid 1m = 0\} \oplus \{m \in M \mid 1m = m\}$.
(2) For $m \in N$, $m \in Rm = \sum_{i=1}^{n} J_i m$, on the other hand $J_i m = 0$ or $J_i m \cong J_i$ Lemma 3.16, implying $J_i m = N_i$.
(3) By (2).
(4) By (3).
(5) For each $1 \leq i \leq n$, J_i is a left component by Lemma 3.7, thus either $N_i \subseteq \mathrm{Ker}(f)$ or $N_i \cap \mathrm{Ker}(f) = 0$ by Lemma 3.18. Thus, setting L as the sum of N_i's with $N_i \cap \mathrm{Ker}(f) = 0$ we have $\mathrm{Ker}(f) \oplus L = M$. □

Theorem 3.20. *Let R be a nonzero right healthy* $\mathrm{r}\mathbb{I}$*-Artinian ring with identity. If R admits a faithful right module M with $\Sigma\langle \mathrm{l}\mathbb{I}^{\cong} \!:\! M \rangle = M$ such that* $\mathrm{End}(M_R)$ *is a division ring, isomorphic *-hole submodules are identical and for every *-hole submodule N, every homomorphism $N \longrightarrow N$ can be extended to a homomorphism $M \longrightarrow M$, then R is isomorphic to a structural matrix ring over a division ring.*

Proof. There exists an independent set $\{J_i \mid 1 \leq i \leq n\}$ of minimal right component with $R = \bigoplus_{i=1}^{n} J_i$ by Lemma 3.15. On the other hand, for every $i \in \mathcal{A} = \{1, 2, \cdots, n\}$, J_i is a minimal right annihilator by Lemma 3.11, so has an isomorphic copy N_i in M by Lemma 3.17. For every $i \in \mathcal{A}$, N_i is a *-hole submodule by Lemma 3.7, Lemma 3.10 and Lemma 3.11, so $\{N_i \mid i \in \mathcal{A}\}$ is a $\square$-set by Lemma 3.18. Thus, $R \cong \mathrm{M}_{\mathcal{H}}(\mathrm{End}(M_R))$ for a quasi-ordering $\mathcal{H}$ on $\mathcal{A}$ by Lemma 2.5. $\square$

Theorem 3.21. *Let R be a nonzero right healthy* r$\mathbb{I}$-*Artinian ring with identity. If R admits a faithful indecomposable well behaved and summand-form right module M with $\Sigma\langle \mathrm{l}\mathbb{I}^{\cong}\colon M\rangle = M$ such that for every hole submodule N, every homomorphism $N \longrightarrow N$ can be extended to a homomorphism $M \longrightarrow M$, then R is isomorphic to a structural matrix ring over a division ring.*

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