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On graded A-2-absorbing submodules of graded modules over graded commutative rings

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Abstract. Let G be an abelian group with identity e. Let R be a G-graded commutative ring, M a graded R-module and $A \subseteq h(R)$ a multiplicatively closed subset of R. In this paper, we introduce the concept of graded A-2-absorbing submodules of M as a generalization of graded 2-absorbing submodules and graded A-prime submodules of M. We investigate some properties of this class of graded submodules.

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1. Introduction and Preliminaries

Throughout this paper all rings are commutative with identity and all modules are unitary. Badawi in [16] introduced the concept of 2-absorbing ideals of commutative rings. The notion of 2-absorbing ideals was extended to 2-absorbing submodules in [27] and [24]. The concept of A-2-absorbing submodules, as a generalization of 2-absorbing submodules, was introduced in [26] and studied in [18].

In [25], Refai and Al-Zoubi introduced the concept of graded primary ideal. The concept of graded 2-absorbing ideals, as a generalization of graded prime ideals, was introduced and studied by Al-Zoubi, Abu-Dawwas and Ceken in [4]. The concept of graded prime submodules was introduced and studied by many authors, see for example [1, 3, 10, 11, 12, 13, 14, 23]. The concept of graded 2-absorbing submodules, as a generalization of graded prime submodules, was introduced by Al-Zoubi and Abu-Dawwas in [2] and studied in [8, 5]. Then many generalizations of graded 2-absorbing submodules were studied such as graded 2-absorbing primary (see [17]), graded weakly 2-absorbing primary (see [7]) and graded classical 2-absorbing submodule (see [6]).

Recently, Al-Zoubi and Al-Azaizeh in [9] introduced the concept of graded A-prime submodule over a commutative graded ring as a new generalization of graded prime submodule. The main purpose of this paper is to introduce the notion of graded A-2-absorbing submodules over a commutative graded ring as a new generalization of graded 2-absorbing submodules and graded

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A-prime submodules. A number of results concerning these classes of graded submodules and their homogeneous components are given.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [19], [20], [21] and [22] for these basic properties and more information on graded rings and modules.

Let G be an abelian group with identity element e. A ring R is called *a graded ring (or G-graded ring)* if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are said to be homogeneous of degree g and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{g\in G} R_g$. If $r \in R$, then r can be written uniquely as $\sum_{g\in G} r_g$, where r_g is called *a homogeneous component of r in R_g*. Moreover, R_e is a subring of R and $1 \in R_e$ (see [22]).

Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. An ideal I of R is said to be a *graded ideal* if $I = \oplus_{g\in G} I_g$ where $I_g = I \cap R_g$ for all $g \in G$ (see [22]).

Let $R = \oplus_{h\in G} R_h$ be a G-graded ring. A left R-module M is said to be *a graded R-module (or G-graded R-module)* if there exists a family of additive subgroups $\{M_h\}_{h\in G}$ of M such that $M = \oplus_{h\in G} M_h$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Also if an element of M belongs to $\cup_{h\in G} M_h =: h(M)$, then it is called *homogeneous*. Note that M_h is an R_e-module for every $h \in G$. Let $M = \oplus_{h\in G} M_h$ be a G-graded R-module. A submodule C of M is said to be a *graded submodule of M* if $C = \oplus_{h\in G} C_h$ where $C_h = C \cap M_h$ for all $h \in G$. In this case, C_h is called the *h-component of C* (see [22]).

Let R be a G-graded ring, M a graded R-module, C a graded submodule of M and I a graded ideal of R. Then $(C :_R M)$ is defined as $(C :_R M) = \{r \in R : rM \subseteq C\}$. It is shown in [14] that if C is a graded submodule of M, then $(C :_R M)$ is a graded ideal of R. The graded submodule $\{m \in M : mI \subseteq C\}$ will be denoted by $(C :_M I)$. Also we use $(C :_M s)$ to denoted $(C :_M Rs)$ for each $s \in R$.

2. RESULTS

Definition 2.1. Let R be a G-graded ring, M a graded R-module, $A \subseteq h(R)$ a multiplicatively closed subset of R and C a graded submodule of M such that $(C :_R M) \bigcap A = \emptyset$. We say that C is *a graded A-2-absorbing submodule of M* if there exists a fixed $a_\alpha \in A$ and whenever $r_g s_h m_\lambda \in C$, where r_g, $s_h \in h(R)$ and $m_\lambda \in h(M)$, implies that $a_\alpha r_g s_h \in (C :_R M)$ or $a_\alpha r_g m_\lambda \in C$ or $a_\alpha s_h m_\lambda \in C$. In particular, a graded ideal J of R is said to be *a graded A-2-absorbing ideal* if J is a graded A-2-absorbing submodule of the graded R-module R.

Recall from [2] that a proper graded submodule C of a G-graded R-module M is said to be *a graded 2-absorbing submodule of M* if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_g s_h m_\lambda \in C$, then either $r_g m_\lambda \in C$ or $s_h m_\lambda \in C$ or $r_g s_h \in (C :_R M)$.

Let $A \subseteq h(R)$ be a multiplicatively closed subset of R. It is easy to see that every graded 2-absorbing submodule C of M with $(C :_R M) \bigcap A = \emptyset$ is a graded A-2-absorbing submodule. The following example shows that the converse is not true in general.

Example 2.2. Let $G = (\mathbb{Z}, +)$ and $R = (\mathbb{Z}, +, .)$. Define $R_g = \left\{ \begin{array}{cc} \mathbb{Z} & \text{if } g = 0 \\ 0 & \text{otherwise} \end{array} \right\}$. Then R is a G-graded ring. Let $M = \mathbb{Z} \times \mathbb{Z}_6$. Then M is a G-graded R-module with

$$M_g = \left\{ \begin{array}{cc} \mathbb{Z} \times \{\bar{0}\} & \text{if } g = 0 \\ \{0\} \times \mathbb{Z}_6 & \text{if } g = 1 \\ \{0\} \times \{\bar{0}\} & \text{otherwise} \end{array} \right\}.$$

Now, consider the zero graded submodule $C = \{0\} \times \{\bar{0}\}$ of M. Then C is not a graded 2-absorbing submodule of M since $2 \cdot 3 \cdot (0, \bar{1}) = (0, \bar{0}) \in C$, but $2 \cdot 3 \notin (C :_{\mathbb{Z}} M) = \{0\}$, $2 \cdot (0, \bar{1}) = (0, \bar{2}) \notin C$ and $3 \cdot (0, \bar{1}) = (0, \bar{3}) \notin C$. Let $A = \mathbb{Z} - \{0\} \subseteq R_0 \subseteq h(R)$ be a multiplicatively closed subset of R and put $a_\alpha = 6 \in A$. An easy computation shows that C is a graded A-2-absorbing submodule of M.

Let R be a G-graded ring, M a graded R-module and $A \subseteq h(R)$ be a multiplicatively closed subset of R. Then, $A^* = \{a_g \in h(R) : \frac{a_g}{1}$ is a unit of $A^{-1}R\}$ is a multiplicatively closed subset of R containing A.

Theorem 2.3. *Let R be a G-graded ring, M a graded R-module and $A \subseteq h(R)$ a multiplicatively closed subset of R. Then the following statements hold:*

(i) *Suppose that $A_1 \subseteq A_2 \subseteq h(R)$ are two multiplicatively closed subsets of R. If C is a graded A_1-2-absorbing submodule and $(C :_R M) \bigcap A_2 = \emptyset$, then C is a graded A_2-2-absorbing submodule of M.*

(ii) *A graded submodule C of M is a graded A-2-absorbing submodule of M if and only if it is a graded A^*-2-absorbing submodule of M.*

Proof. (i) It is clear.

(ii) Assume that C is a graded A-2-absorbing submodule of M. First, we want to show that $(C :_R M) \bigcap A^* = \emptyset$. Suppose there exists $r_g \in (C :_R M) \bigcap A^*$. Then $\frac{r_g}{1}$ is a unit of $A^{-1}R$, it follows that there exist $s_h \in h(R)$ and $a_i \in A$ such that $\frac{r_g}{1}\frac{s_h}{a_i} = 1$. Hence $b_j a_i = b_j r_g s_h$ for some $b_j \in A$. So $b_j a_i = b_j r_g s_h \in (C :_R M) \bigcap A$, a contradiction. Therefore $(C :_R M) \bigcap A^* = \emptyset$. Then by (i), we get C is a graded A^*-2-absorbing submodule of M since $A \subseteq A^*$. Conversely, suppose that C is a graded A^*-2-absorbing submodule of M. Let $r_g s_k m_h \in C$ for some r_g, $s_k \in h(R)$ and $m_h \in h(M)$. Then there is a fixed $a_i^* \in A^*$ such that $a_i^* r_g s_k \in (C :_R M)$ or $a_i^* r_g m_h \in C$ or $a_i^* s_k m_h \in C$. Since $\frac{a_i^*}{1}$ is a unit of $A^{-1}R$, there exist $t_j \in h(R)$ and $a_k, b_l \in A$ such that $b_l a_k = b_l a_i^* t_j$. It follows that either $(b_l a_k) r_g s_k = b_l a_i^* t_j r_g s_k \in (C :_R M)$ or $(b_l a_k) r_g m_h = b_l a_i^* t_j r_g m_h \in C$ or $(b_l a_k) s_k m_h = b_l a_i^* t_j s_k m_h \in C$. Therefore, C is a graded A-2-absorbing submodule of M. □

Theorem 2.4. *Let R be a G-graded ring, M a graded R-module and $A \subseteq h(R)$ a multiplicatively closed subset of R. If C is a graded A-2-absorbing submodule of M, then $A^{-1}C$ is a graded 2-absorbing submodule of $A^{-1}M$.*

Proof. Assume that C is a graded A-2-absorbing submodule of M. Let $\frac{r_{g_1}}{a_{h_1}}\frac{s_{g_2}}{a_{h_2}} \in h(A^{-1}R)$ and $\frac{m_{g_3}}{a_{h_3}} \in h(A^{-1}M)$ such that $\frac{r_{g_1}}{a_{h_1}}\frac{s_{g_2}}{a_{h_2}}\frac{m_{g_3}}{a_{h_3}} \in A^{-1}C$. Then, there exists $a_{h_4} \in A$ such that $(a_{h_4}r_{g_1})s_{g_2}m_{g_3} \in C$. As C is a graded A-2-absorbing submodule of M, there is a fixed $a_{h_5} \in A$ such that $a_{h_5}(a_{h_4}r_{g_1})s_{g_2} \in (C :_R M)$ or $a_{h_5}(a_{h_4}r_{g_1})m_{g_3} \in C$ or $a_{h_5}s_{g_2}m_{g_3} \in C$. Hence, we get either $\frac{r_{g_1}}{a_{h_1}}\frac{s_{g_2}}{a_{h_2}} = \frac{a_{h_5}a_{h_4}}{a_{h_5}a_{h_4}}\frac{r_{g_1}s_{g_2}}{a_{h_1}a_{h_2}} \in A^{-1}(C :_R M) \subseteq (A^{-1}C :_{A^{-1}R} A^{-1}M)$ or $\frac{r_{g_1}}{a_{h_1}}\frac{m_{g_3}}{a_{h_3}} = \frac{a_{h_5}r_{g_1}m_{g_3}}{a_{h_5}a_{h_1}a_{h_3}} \in A^{-1}C$ or $\frac{s_{g_2}}{a_{h_2}}\frac{m_{g_3}}{a_{h_3}} = \frac{a_{h_4}s_{g_2}m_{g_3}}{a_{h_4}a_{h_2}a_{h_3}} \in A^{-1}C$. Therefore, $A^{-1}C$ is a graded 2-absorbing submodule of $A^{-1}M$. □

Lemma 2.5. *Let R be a G-graded ring, M a graded R-module, $A \subseteq h(R)$ be a multiplicatively closed subset of R and C a graded A-2-absorbing submodule of M. Let $K = \bigoplus_{\lambda\in G} K_\lambda$ be a graded submodule of M. Then there exists a fixed $a_\alpha \in A$ and whenever r_g, $s_h \in h(R)$ and $\lambda \in G$ such that $r_g s_h K_\lambda \subseteq C$, then $a_\alpha r_g K_\lambda \subseteq C$ or $a_\alpha s_h K_\lambda \subseteq C$ or $a_\alpha r_g s_h \in (C :_R M)$.*

Proof. Let r_g, $s_h \in h(R)$, and $\lambda \in G$ such that $r_g s_h K_\lambda \subseteq C$. Since C is a graded A-2-absorbing submodule of M, there exists $a_\alpha \in A$ so that $r_g s_h m_\lambda \in C$ implies $a_\alpha r_g s_h \in (C :_R M)$ or $a_\alpha r_g m_\lambda \in C$ or $a_\alpha s_h m_\lambda \in C$ for each $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$. Now, we will show that $a_\alpha r_g K_\lambda \subseteq C$ or $a_\alpha s_h K_\lambda \subseteq C$ or $a_\alpha r_g s_h \in (C :_R M)$. Assume on the contrary that $a_\alpha r_g K_\lambda \nsubseteq C$, $a_\alpha s_h K_\lambda \nsubseteq C$ and $a_\alpha r_g s_h \notin (C :_R M)$. Then there exist k_λ , $k'_\lambda \in K$ such that $a_\alpha r_g k_\lambda \notin C$ and $a_\alpha s_h k'_\lambda \notin C$. Since C is a graded A-2-absorbing submodule of M, $r_g s_h k_\lambda \in C$, $a_\alpha r_g k_\lambda \notin C$ and $a_\alpha r_g s_h \notin (C :_R M)$, we get $a_\alpha s_h k_\lambda \in C$. In a similar manner, we get $a_\alpha r_g k'_\lambda \in C$. By $k_\lambda + k'_\lambda \in K_\lambda \subseteq h(M)$ it follows that $r_g s_h(k_\lambda + k'_\lambda) \in C$. Since C is a graded A-2-absorbing submodule of M and $a_\alpha r_g s_h \notin (C :_R M)$, we have either $a_\alpha r_g(k_\lambda + k'_\lambda) \in C$ or $a_\alpha s_h(k_\lambda + k'_\lambda) \in C$. If $a_\alpha r_g(k_\lambda + k'_\lambda) = a_\alpha r_g k_\lambda + a_\alpha r_g k'_\lambda \in C$, then we get $a_\alpha r_g k_\lambda \in C$ since $a_\alpha r_g k'_\lambda \in C$, a contradiction. If $a_\alpha s_h(k_\lambda + k'_\lambda) = a_\alpha s_h k_\lambda + a_\alpha s_h k'_\lambda \in C$, then we get $a_\alpha s_h k'_\lambda \in C$ since $a_\alpha s_h k_\lambda \in C$, a contradiction. Therefore $a_\alpha r_g K_\lambda \subseteq C$ or $a_\alpha s_h K_\lambda \subseteq C$ or $a_\alpha r_g s_h \in (C :_R M)$.

□

Theorem 2.6. *Let R be a G-graded ring, M a graded R-module, C a graded submodule of M and $A \subseteq h(R)$ be a multiplicatively closed subset of R with $(C :_R M) \bigcap A = \emptyset$. Let $I = \bigoplus_{g\in G} I_g$ and $J = \bigoplus_{h\in G} J_h$ be graded ideals of R and $K = \bigoplus_{\lambda\in G} K_\lambda$ a graded submodule of M. Then the following statements are equivalent:*

(i) *C is a graded A-2-absorbing submodule of M;*

(ii) *There exists a fixed $a_\alpha \in A$ such that $I_g J_h K_\lambda \subseteq C$ for some $g, h, \lambda \in G$ implies either $a_\alpha I_g K_\lambda \subseteq C$ or $a_\alpha J_h K_\lambda \subseteq C$ or $a_\alpha I_g J_h \subseteq (C :_R M)$.*

Proof. $(i) \Rightarrow (ii)$ Assume that C is a graded A-2-absorbing submodule of M and $g, h, \lambda \in G$ such that $I_g J_h K_\lambda \subseteq C$. Since C is a graded A-2-absorbing submodule of M, there exists a fixed $a_\alpha \in A$ so that $r_g s_h m_\lambda \in C$ implies $a_\alpha r_g s_h \in (C :_R M)$ or $a_\alpha r_g m_\lambda \in C$ or $a_\alpha s_h m_\lambda \in C$ for each $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$. Now, we will show that $a_\alpha I_g K_\lambda \subseteq C$ or $a_\alpha J_h K_\lambda \subseteq C$ or $a_\alpha I_g J_h \subseteq (C :_R M)$. Assume on the contrary that $a_\alpha I_g K_\lambda \nsubseteq C$, $a_\alpha J_h K_\lambda \nsubseteq C$ and $a_\alpha I_g J_h \nsubseteq (C :_R M)$. Then there exist $x_g \in I_g$ and $y_h \in J_h$ such that $a_\alpha x_g K_\lambda \nsubseteq C$ and $a_\alpha y_h K_\lambda \nsubseteq C$. Since $x_g y_h K_\lambda \subseteq C$, by Lemma 2.5, we get $a_\alpha x_g y_h \in (C :_R M)$. Since $a_\alpha I_g J_h \nsubseteq (C :_R M)$, there exist $r_g \in I_g$ and $s_h \in J_h$ such that $a_\alpha r_g s_h \notin (C :_R M)$. Then by Lemma 2.5, we have $a_\alpha r_g K_\lambda \subseteq C$ or $a_\alpha s_h K_\lambda \subseteq C$ since $r_g s_h K_\lambda \subseteq C$. Consider the following three cases:

Case1: $a_\alpha r_g K_\lambda \subseteq C$ and $a_\alpha s_h K_\lambda \nsubseteq C$. Since $x_g s_h K_\lambda \subseteq C$, $a_\alpha s_h K_\lambda \nsubseteq C$ and $a_\alpha x_g K_\lambda \nsubseteq C$, by Lemma 2.5, we get $a_\alpha x_g s_h \in (C :_R M)$. As $a_\alpha x_g K_\lambda \nsubseteq C$ and $a_\alpha r_g K_\lambda \subseteq C$, we have $a_\alpha (x_g + r_g) K_\lambda \nsubseteq C$. By $(x_g + r_g) \in I_g$ it follows that $(x_g + r_g) s_h K_\lambda \subseteq C$. Since $(x_g + r_g) s_h K_\lambda \subseteq C$, $a_\alpha (x_g + r_g) K_\lambda \nsubseteq C$ and $a_\alpha s_h K_\lambda \nsubseteq C$, by Lemma 2.5, we get $a_\alpha (x_g + r_g) s_h \in (C :_R M)$. By $a_\alpha (x_g + r_g) s_h \in (C :_R M)$ and $a_\alpha x_g s_h \in (C :_R M)$ it follows that $a_\alpha r_g s_h \in (C :_R M)$ which is a contradiction.

Case2: $a_\alpha r_g K_\lambda \nsubseteq C$ and $a_\alpha s_h K_\lambda \subseteq C$. The proof is similar to that of Case 1.

Case 3: $a_\alpha r_g K_\lambda \subseteq C$ and $a_\alpha s_h K_\lambda \subseteq C$. Since $a_\alpha y_h K_\lambda \nsubseteq C$ and $a_\alpha s_h K_\lambda \subseteq C$, we get $a_\alpha (s_h + y_h) K_\lambda \nsubseteq C$. By $(s_h + y_h) \in J_h$ it follows that $x_g (s_h + y_h) K_\lambda \subseteq C$. Since $x_g (s_h + y_h) K_\lambda \subseteq C$, $a_\alpha (s_h + y_h) K_\lambda \nsubseteq C$ and $a_\alpha x_g K_\lambda \nsubseteq C$, by Lemma 2.5, we get $a_\alpha x_g (s_h + y_h) \in (C :_R M)$. Then we get $a_\alpha x_g s_h \in (C :_R M)$ since $a_\alpha x_g (s_h + y_h) \in (C :_R M)$ and $a_\alpha x_g y_h \in (C :_R M)$. As $a_\alpha x_g K_\lambda \nsubseteq C$ and $a_\alpha r_g K_\lambda \subseteq C$, we have $a_\alpha (r_g + x_g) K_\lambda \nsubseteq C$. Then by Lemma 2.5, $a_\alpha (r_g + x_g) y_h \in (C :_R M)$ since $(r_g + x_g) y_h K_\lambda \subseteq C$, $a_\alpha (r_g + x_g) K_\lambda \nsubseteq C$ and $a_\alpha y_h K_\lambda \nsubseteq C$. Since $a_\alpha (r_g + x_g) y_h \in (C :_R M)$ and $a_\alpha x_g y_h \in (C :_R M)$, we get $a_\alpha r_g y_h \in (C :_R M)$. Thus by Lemma 2.5, we get $a_\alpha (r_g + x_g)(s_h + y_h) \in (C :_R M)$ since $(r_g + x_g)(s_h + y_h) K_\lambda \subseteq C$, $a_\alpha (r_g + x_g) K_\lambda \nsubseteq C$ and $a_\alpha (s_h + y_h) K_\lambda \nsubseteq C$. As $a_\alpha (r_g + x_g)(s_h + y_h) = a_\alpha r_g s_h + a_\alpha r_g y_h + a_\alpha x_g s_h + a_\alpha x_g y_h \in (C :_R M)$ and $a_\alpha r_g y_h$, $a_\alpha x_g s_h$, $a_\alpha x_g y_h \in (C :_R M)$, we have $a_\alpha r_g s_h \in (C :_R M)$, a contradiction.

$(ii) \Rightarrow (i)$ Assume that (ii) holds. Let r_g, $s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $r_g s_h m_\lambda \in C$. Let $I = r_g R$ and $J = s_h R$ be a graded ideals of R generated by r_g and s_h, respectively and $K = m_\lambda R$ a graded submodule of M generated by m_λ. Then $I_g J_h K_\lambda \subseteq C$. By our assumption, there exists $a_\alpha \in A$ such that either $a_\alpha I_g K_\lambda \subseteq C$ or $a_\alpha J_h K_\lambda \subseteq C$ or $a_\alpha I_g J_h \subseteq (C :_R M)$. This yields that either $a_\alpha r_g m_\lambda \in C$ or $a_\alpha s_h m_\lambda \in C$ or $a_\alpha r_g s_h \in (C :_R M)$. Therefore, C is a graded A-2-absorbing submodule of M. $\square$

Corollary 2.7. *Let R be a G-graded ring, P a graded ideal of R and $A \subseteq h(R)$ be a multiplicatively closed subset of R with $P \bigcap A = \emptyset$. Let $I = \bigoplus_{g \in G} I_g$, $J =$*

$\bigoplus_{h\in G} J_h$ *and* $L = \bigoplus_{\lambda\in G} L_\lambda$ *be graded ideals of* R*. Then the following statements are equivalent:*

(i) P *is a graded* A*-2-absorbing ideal of* R*;*

(ii) *There exists* $a_\alpha \in A$ *such that* $I_g J_h L_\lambda \subseteq P$ *for some* $g, h, \lambda \in G$ *implies either* $a_\alpha I_g L_\lambda \subseteq P$ *or* $a_\alpha J_h L_\lambda \subseteq P$ *or* $a_\alpha I_g J_h \subseteq P$*.*

Lemma 2.8. *[15, Lemma 2.2] Let* R *be a* G*-graded ring and* N*,* K*,* L *graded* R*-submodules of a graded* R*-module* M *with* $N \subseteq K \cup L$*. Then* $N \subseteq K$ *or* $N \subseteq L$*.*

Theorem 2.9. *Let* R *be a* G*-graded ring,* M *a graded* R*-module,* C *a graded submodule of* M *and* $A \subseteq h(R)$ *be a multiplicatively closed subset of* R *with* $(C :_R M) \bigcap A = \emptyset$*. Then the following statements are equivalent:*

(i) C *is a graded* A*-2-absorbing submodule of* M*;*

(ii) *There is a fixed* $a_\alpha \in A$ *such that for every* r_g*,* $s_g \in h(R)$*, we have either* $(C :_M a_\alpha^2 r_g s_h) = (C :_M a_\alpha^2 r_g)$ *or* $(C :_M a_\alpha^2 r_g s_h) = (C :_M a_\alpha^2 s_h)$ *or* $(C :_M a_\alpha^3 r_g s_h) = M$*.*

Proof. $(i) \Rightarrow (ii)$Assume that C is a graded A-2-absorbing submodule of M. Then there exists a fixed $a_\alpha \in A$ such that whenever $r_g s_h m_\lambda \in C$, where r_g, $s_g \in h(R)$ and $m_\lambda \in h(M)$, then either $a_\alpha r_g s_h \in (C :_R M)$ or $a_\alpha r_g m_\lambda \in C$ or $a_\alpha s_h m_\lambda \in C$. Now let $m_\lambda \in (C :_M a_\alpha^2 r_g s_h) \cap h(M)$. Hence $(a_\alpha r_g)(a_\alpha s_h) m_\lambda \in C$. Then either $a_\alpha^2 r_g m_\lambda \in C$ or $a_\alpha^2 s_h m_\lambda \in C$ or $a_\alpha^3 r_g s_h \in (C :_R M)$ as C is a graded A-2-absorbing submodule of M. If for every $m_\lambda \in (C :_M a_\alpha^2 r_g s_h) \cap h(M)$, it holds that $a_\alpha^2 r_g m_\lambda \in C$ or $a_\alpha^2 s_h m_\lambda \in C$, then $(C :_M a_\alpha^2 r_g s_h) \subseteq (C :_M a_\alpha^2 r_g) \cup (C :_M a_\alpha^2 s_h)$. Clearly $(C :_M a_\alpha^2 r_g) \cup (C :_M a_\alpha^2 s_h) \subseteq (C :_M a_\alpha^2 r_g s_h)$. So $(C :_M a_\alpha^2 r_g) \cup (C :_M a_\alpha^2 s_h) = (C :_M a_\alpha^2 r_g s_h)$. By Lemma 2.8, $(C :_M a_\alpha^2 r_g) = (C :_M a_\alpha^2 r_g s_h)$ or $(C :_M a_\alpha^2 s_h) = (C :_M a_\alpha^2 r_g s_h)$. If there exists $m_\lambda \in (C :_M a_\alpha^2 r_g s_h) \cap h(M)$ such that $a_\alpha^2 r_g m_\lambda \notin C$ and $a_\alpha^2 s_h m_\lambda \notin C$, then $a_\alpha^3 r_g s_h \in (C :_R M)$, and therefore $(C :_M a_\alpha^3 r_g s_h) = M$.

$(ii) \Rightarrow (i)$ Let $r_g s_h m_\lambda \in C$, where r_g, $s_g \in h(R)$ and $m_\lambda \in h(M)$. Thus $m_\lambda \in (C :_M a_\alpha^2 r_g s_h)$. By given hypothesis, we have $(C :_M a_\alpha^2 r_g s_h) = (C :_M a_\alpha^2 r_g)$ or $(C :_M a_\alpha^2 r_g s_h) = (C :_M a_\alpha^2 s_h)$ or $(C :_M a_\alpha^3 r_g s_h) = M$. Then $a_\alpha^2 r_g m_\lambda \in C$ or $a_\alpha^2 s_h m_\lambda \in C$ or $a_\alpha^3 r_g s_h \in (C :_R M)$. This yields that either $a_\alpha^3 r_g m_\lambda \in C$ or $a_\alpha^3 s_h m_\lambda \in C$ or $a_\alpha^3 r_g s_h \in (C :_R M)$. By setting $s^* = a_\alpha^3$, C is a graded A-2-absorbing submodule of M. $\square$

Lemma 2.10. *Let* R *be a* G*-graded ring,* M *a graded* R*-module,* $A \subseteq h(R)$ *be a multiplicatively closed subset of* R *and* C *a graded* A*-2-absorbing submodule of* M*. Then the following statements hold:*

(i) *There exists a fixed* $a_\alpha \in A$ *such that* $(C :_M a_\alpha^3) = (C :_M a_\alpha^n)$ *for all* $n \geqslant 3$*.*

(ii) *There exists a fixed* $a_\alpha \in A$ *such that* $(C :_R a_\alpha^3 M) = (C :_R a_\alpha^n M)$ *for all* $n \geqslant 3$.

Proof. (i) Since C a graded A-2-absorbing submodule of M, there exists a fixed $a_\alpha \in A$ such that whenever $r_g s_h m_\lambda \in C$, where $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$, then either $a_\alpha r_g s_h \in (C :_R M)$ or $a_\alpha r_g m_\lambda \in C$ or $a_\alpha s_h m_\lambda \in C$. Let $m_\lambda \in (C :_M a_\alpha^4) \cap h(M)$, it follows that $a_\alpha^4 m_\lambda = a_\alpha^2(a_\alpha^2 m_\lambda) \in C$. Since C is a graded A-2-absorbing submodule of M, then either $a_\alpha^3 m_\lambda \in C$, or $a_\alpha^5 \in (C :_R M)$. But $a_\alpha^5 \in (C :_R M)$ is impossible, since $a_\alpha^5 \in A$, therefore $a_\alpha^3 m_\lambda \in C$ must hold, it follows that $m_\lambda \in (C :_M a_\alpha^3)$. Hence $(C :_M a_\alpha^4) \subseteq (C :_M a_\alpha^3)$. Since the other inclusion is always satisfied, we get $(C :_M a_\alpha^4) = (C :_M a_\alpha^3)$. Assume that $(C :_M a_\alpha^3) = (C :_M a_\alpha^k)$ for all $k < n$. We will show that $(C :_M a_\alpha^3) = (C :_M a_\alpha^n)$. Let $m'_\lambda \in (C :_M a_\alpha^n) \cap h(M)$, it follows that $a_\alpha^n m'_\lambda = a_\alpha^2(a_\alpha^{n-2} m'_\lambda) \in C$. Since C is a graded A-2-absorbing submodule of M, then either $a_\alpha^3 m'_\lambda \in C$ or $a_\alpha^{n-1} m'_\lambda \in C$ or $a_\alpha^{n+1} \in (C :_R M)$. But $a_\alpha^{n+1} \in (C :_R M)$ is impossible, since $a_\alpha^{n+1} \in A$, it follows that $m'_\lambda \in (C :_M a_\alpha^3) \cup (C :_M a_\alpha^{n-1}) = (C :_M a_\alpha^3)$ by induction hypothesis. Therefore $(C :_M a_\alpha^3) = (C :_M a_\alpha^n)$ for every $n \geqslant 3$.

(ii) Follows directly from (i). □

Theorem 2.11. *Let* R *be a* G*-graded ring,* M *a graded* R*-module,* $A \subseteq h(R)$ *be a multiplicatively closed subset of* R *and* C *a graded submodule of* M *with* $(C :_R M) \cap A = \emptyset$. *Then the following statements are equivalent:*

(i) C *is a graded* A*-2-absorbing submodule.*

(ii) $(C :_M a_\alpha)$ *is a graded 2-absorbing submodule of* M *for some* $a_\alpha \in A$.

Proof. $(i) \Rightarrow (ii)$ Assume that C is a graded A-2-absorbing submodule. Then there exists a fixed $a_\alpha \in A$ such that whenever $r_g s_h m_\lambda \in C$, where $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$, then either $a_\alpha r_g s_h \in (C :_R M)$ or $a_\alpha r_g m_\lambda \in C$ or $a_\alpha s_h m_\lambda \in C$. By Lemma 2.10, we have $(C :_M a_\alpha^3) = (C :_M a_\alpha^n)$ and $(C :_R a_\alpha^3 M) = (C :_R a_\alpha^n M)$ for all $n \geqslant 3$. We show that $(C :_M a_\alpha^6) = (C :_M a_\alpha^3)$ is a graded 2-absorbing submodule of M. Let $r_g s_h m_\lambda \in (C :_M a_\alpha^6)$ for some $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$. It follows that, $a_\alpha^6(r_g s_h m_\lambda) = (a_\alpha^2 r_g)(a_\alpha^2 s_h)(a_\alpha^2 m_\lambda) \in C$. Then either $a_\alpha(a_\alpha^2 r_g)(a_\alpha^2 s_h) = a_\alpha^5 r_g s_h \in (C :_R M)$ or $a_\alpha(a_\alpha^2 r_g)(a_\alpha^2 m_\lambda) = a_\alpha^5 r_g m_\lambda \in C$ or $a_\alpha(a_\alpha^2 s_h)(a_\alpha^2 m_\lambda) = a_\alpha^5 s_h m_\lambda \in C$ as C is a graded A-2-absorbing submodule of M. It follows that either $r_g s_h \in (C :_R a_\alpha^5 M) = (C :_R a_\alpha^6 M) = ((C :_M a_\alpha^6) :_R M)$ or $r_g m_\lambda \in (C :_M a_\alpha^5) = (C :_M a_\alpha^6)$ or $s_h m_\lambda \in (C :_M a_\alpha^5) = (C :_M a_\alpha^6)$. Thus $(C :_M a_\alpha^6)$ is a graded 2-absorbing submodule of M.

$(ii) \Rightarrow (i)$ Assume that $(C :_M a_\alpha)$ is a graded 2-absorbing submodule of M for some $a_\alpha \in A$. Let $r_g s_h m_\lambda \in C \subseteq (C :_M a_\alpha)$, where r_g, $s_g \in h(R)$ and $m_\lambda \in h(M)$. Since $(C :_M a_\alpha)$ is a graded 2-absorbing submodule of M, we get either $r_g s_h \in ((C :_M a_\alpha) :_R M)$ or $r_g m_\lambda \in (C :_M a_\alpha)$ or $s_h m_\lambda \in (C :_M a_\alpha)$. Thus $a_\alpha r_g s_h \in (C :_R M)$ or $a_\alpha r_g m_\lambda \in C$ or $a_\alpha s_h m_\lambda \in C$. Therefore, C is a graded A-2-absorbing submodule. □

Let M and M' be two graded R-modules. A graded homomorphism of graded R-modules $f : M \to M'$ is a homomorphism of R-modules verifying $f(M_g) \subseteq M'_g$ for every $g \in G$, (see [22]).

The following result studies the behavior of graded A-2-absorbing submodules under graded homomorphism.

Theorem 2.12. *Let R be a G-graded ring and M, M' be two graded R-modules and $f : M \to M'$ be a graded homorphism. Let $A \subseteq h(R)$ be a multiplicatively closed subset of R.*

(i) *If C' is a graded A-2-absorbing submodule of M' and $(f^{-1}(C') :_R M) \cap A = \emptyset$, then $f^{-1}(C')$ is a graded A-2-Absorbing submodule of M.*

(ii) *If moreover, f is surjective and C is a graded A-2-absorbing submodule of M with $Kerf \subseteq C$, then $f(C)$ is a graded A-2-absorbing submodule of M'.*

Proof. (i) Assume that C' is a graded A-2-absorbing submodule of M'. Now, let $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $r_g s_h m_\lambda \in f^{-1}(C')$. Hence $f(r_g s_h m_\lambda) = r_g s_h f(m_\lambda) \in C'$. Since C' is a graded A-2-absorbing submodule, there exists $a_\alpha \in A$ such that either $a_\alpha r_g s_h \in (C' :_R M')$ or $a_\alpha r_g f(m_\lambda) = f(a_\alpha r_g m_\lambda) \in C'$ or $a_\alpha s_h f(m_\lambda) = f(a_\alpha s_h m_\lambda) \in C'$. It follows that either $a_\alpha r_g s_h \in (C' :_R M') \subseteq (f^{-1}(C') :_R M)$ or $a_\alpha r_g m_\lambda \in f^{-1}(C')$ or $a_\alpha s_h m_\lambda \in f^{-1}(C')$. Therefore, $f^{-1}(C')$ is a graded A-2-absorbing submodule of M.

(ii) Assume that C is a graded A-2-absorbing submodule of M containing $Kerf$. First, we want to show that $(f(C) :_R M') \bigcap A = \emptyset$. Suppose on the contrary that there exists $a_g \in (f(C) :_R M') \bigcap A$. Hence $a_g M' \subseteq f(C)$, this implies that $f(a_g M) = a_g f(M) \subseteq a_g M' \subseteq f(C)$. It follows that, $a_g M \subseteq a_g M + Kerf \subseteq C + Kerf = C$. Hence $a_g M \subseteq C$ and so, $a_g \in (C :_R M)$, which is a contradiction since $(C :_R M) \bigcap A = \emptyset$. Now, let $r_g s_h m'_\lambda \in f(C)$ for some r_g, $s_h \in h(R)$ and $m'_\lambda \in h(M')$. Then, there exists $c_\beta \in C \bigcap h(M)$ such that $r_g s_h m'_\lambda = f(c_\beta)$. Since f is a graded epimorphism and $m'_\lambda \in h(M')$, there exists $m_\lambda \in h(M)$ such that $m'_\lambda = f(m_\lambda)$. Then $f(c_\beta) = r_g s_h m'_\lambda = r_g s_h f(m_\lambda) = f(r_g s_h m_\lambda)$, and so $c_\beta - r_g s_h m_\lambda \in Kerf \subseteq C$, it follows that $r_g s_h m_\lambda \in C$. Since C is a graded A-2-absorbing submodule of M, there exists $a_\alpha \in A$ such that $a_\alpha r_g s_h \in (C :_R M)$ or $a_\alpha r_g m_\lambda \in C$ or $a_\alpha s_h m_\lambda \in C$. Then we have either $a_\alpha r_g s_h \in (C :_R M) \subseteq (f(C) :_R M')$ or $a_\alpha r_g m'_\lambda = a_\alpha r_g f(m_\lambda) = f(a_\alpha r_g m_\lambda) \in f(C)$ or $a_\alpha s_h m'_\lambda = a_\alpha s_h f(m_\lambda) = f(a_\alpha s_h m_\lambda) \in f(C)$. Thus $f(C)$ is a graded A-2-absorbing submodule of M'. □

Let R be a G-graded ring, M a graded R-module, $A \subseteq h(R)$ a multiplicatively closed subset of R and C a graded submodule of M with $(C :_R M) \bigcap A = \emptyset$. We say that C is *a graded A-prime submodule of M* if there exists a fixed $a_\alpha \in A$ and whenever $r_g m_\lambda \in C$ where $r_g \in h(R)$ and $m_\lambda \in h(M)$, implies that either $a_\alpha r_g \in (C :_R M)$ or $a_\alpha m_\lambda \in C$ (see [9]).

It is easy to see that every graded A-prime submodule of M is a graded A-2-absorbing submodule. The following example shows that the converse is not true in general.

Example 2.13. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_6$ be a graded R-module with $M_0 = \mathbb{Z}_6$ and $M_1 = \{\bar{0}\}$. Now, consider the graded submodule $C = \{\bar{0}\}$ of M, then C is not a graded prime submodule since $2 \cdot \bar{3} \in C$ where $2 \in R_0$ and $\bar{3} \in M_0$ but $\bar{3} \notin C$ and $2 \notin (C :_{\mathbb{Z}} M) = 6\mathbb{Z}$. However an easy computation shows that C is a graded 2-absorbing submodule of M. Now let A be the set of units in R. Then C is a graded A-2-absorbing of M but not a graded A-prime submodule.

Theorem 2.14. *Let R be a G-graded ring, M a graded R-module and $A \subseteq h(R)$ be a multiplicatively closed subset of R. Then the intersection of two graded A-prime submodules of M is a graded A-2-absorbing submodule of M.*

Proof. Let C_1 and C_2 be two graded A-prime submodules of M and $C = C_1 \cap C_2$. Let $r_g s_h m_\lambda \in C$ for some $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$. Since C_1 is a graded A-prime submodule of M and $r_g(s_h m_\lambda) \in C_1$, there exists $a_{1_\alpha} \in A$ such that $a_{1_\alpha} r_g \in (C_1 :_R M)$ or $a_{1_\alpha} s_h m_\lambda \in C_1$. If $a_{1_\alpha} s_h m_\lambda = s_h(a_{1_\alpha} m_\lambda) \in C_1$, then either $a_{1_\alpha} s_h \in (C_1 :_R M)$ or $a_{1_\alpha}^2 m_\lambda \in C_1$ since C_1 is a graded A-prime submodule of M and hence either $a_{1_\alpha} s_h \in (C_1 :_R M)$ or $a_{1_\alpha} m_\lambda \in C_1$. In a similar manner, since C_2 is a graded A-prime submodule of M and $r_g s_h m_\lambda \in C_2$, there exists $a_{2_\alpha} \in A$ such that $a_{2_\alpha} r_g \in (C_2 :_R M)$ or $a_{2_\alpha} s_h \in (C_2 :_R M)$ or $a_{2_\alpha} m_\lambda \in C_2$. Now put $a_\beta = a_{1_\alpha} a_{2_\alpha} \in A$. Then either $a_\beta r_g s_h \in (C :_R M)$ or $a_\beta r_g m_\lambda \in C$ or $a_\beta s_h m_\lambda \in C$. Therefore, C is a graded A-2-absorbing submodule of M. □

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