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Zalcman and generalized Zalcman conjecture for a subclass of univalent functions

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Abstract. Function $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$, normalized, analytic and univalent in the unit disk $\mathbb{D}=\{z:|z|<1\}$, belongs to the class $\mathcal{U}$. if, and only if,

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<1 \qquad (z\in\mathbb{D}).$$

In this paper we prove the Zalcman and the generalized Zalcman conjecture for the class $\mathcal{U}$ and some values of parameters in the conjectures.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions f which are analytic in the open unit disc $\mathbb{D}=\{z:|z|<1\}$ of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots, \tag{1.1}$$

and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathbb{D}$.

Important milestone in study of univalent functions was the proof of the famous Bieberbach conjecture $|a_n|\le n$ for $n\ge 2$ by Lewis de Branges in 1985 [1]. This ended an era, but a great many other problems concerning the coefficients a_n remain open. One such is the Zalcman conjecture,

$$|a_n^2-a_{2n-1}|\le (n-1)^2 \qquad (n\in\mathbb{N}, n\ge 2),$$

posed in the early 1970's. Remarkable work along these lines is done by Krushkal ([2]) using complex geometry of the universal Teichmüller space. In 1999, Ma ([3]) proposed a generalized Zalcman conjecture,

$$|a_ma_n-a_{m+n-1}|\le (m-1)(n-1) \qquad (m,n\in\mathbb{N}, m\ge 2, n\ge 2),$$

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which is still an open problem, closed by Ma for the class of starlike functions and for the class of univalent functions with real coefficients. Ravichandran and Verma in [6] closed it for the classes of starlike and convex functions of given order and for the class of functions with bounded turning.

In this paper we study the generalized Zalcman conjecture for the class

$$\mathcal{U}=\left\{f\in\mathcal{A}:\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|<1,\, z\in\mathbb{D}\right\}.$$

Functions from this class are proven to be univalent (Ozaki and Nunokawa, [5]) but do not follow the traditional patterns of other univalent functions. For example, they are not starlike which makes them interesting since the class of starlike functions is very wide. So, class $\mathcal{U}$ attracts significant attention in the past decades and an overview of the most valuable results is given in Chapter 12 from [7].

Here we will prove of the generalized Zalcman conjecture for the class $\mathcal{U}$ and for the cases $m=2$, $n=3$; $m=2$, $n=4$; and $m=n=3$.

We also give direct proof and sharpness of the inequality

$$|a_n^p-a_2^{p(n-1)}|\le 2^{p(n-1)}-n^p$$

over the class $\mathcal{U}$ for the cases $n=4$, $p=1$ and $n=5$, $p=1$. This inequality introduced by Krushkal and proven for the whole class of univalent functions in [2].

For the study and the proofs we will use the following useful property of functions in $\mathcal{U}$.

Lemma 1.1 ([4]). *For each function f in $\mathcal{U}$, there exists function ω_1, analytic in the unit disk, such that $|\omega_1(z)|\le|z|<1$ and $|\omega_1'(z)|\le 1$ for all $z\in\mathbb{D}$, with*

$$\frac{z}{f(z)}=1-a_2z-z\omega_1(z). \tag{1.2}$$

Additionally, for $\omega_1(z)=c_1z+c_2z^2+\cdots$,

$$|c_1|\le 1,\qquad |c_2|\le\frac12(1-|c_1|^2)\quad\text{and}\quad |c_3|\le\frac13\left(1-|c_1|^2-\frac{4|c_2|^2}{1+|c_1|}\right).$$

Let note that for functions f from $\mathcal{U}$, of form (1.1), from Lemma 1.1 we have

$$z=[1-a_2z-z\omega_1(z)]\cdot f(z),$$

and after equating the coefficients,

$$\begin{aligned}a_3&=c_1+a_2^2,\\ a_4&=c_2+2a_2c_1+a_2^3,\\ a_5&=c_3+2a_2c_2+c_1^2+3a_2^2c_1+a_2^4.\end{aligned}$$

2. Zalcman conjecture for the class $\mathcal{U}$

We now give direct proof of the Zalcman conjecture for the class $\mathcal{U}$ for the cases when $n = 2$ and $n = 3$.

Theorem 2.1. *Let $f \in \mathcal{U}$ be of the form* (1.1). *Then*

(*i*) $|a_2^2 - a_3| \le 1$;

(*ii*) $|a_3^2 - a_5| \le 4$.

These inequalities are sharp with equality for the Koebe function $k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2} nz^n$ and its rotations.

Proof.

(*i*) From $a_3 = c_1 + a_2^2$ we have $|a_2^2 - a_3| = |-c_1| \le 1$.

(*ii*) From the Bieberbach conjecture, $|a_3| = |c_1 + a_2^2| \le 3$, and further calculations show that

$$\begin{aligned} \left|a_3^2 - a_5\right| &= \left|c_3 + 2a_2c_2 + a_2^2c_1\right| \\ &= \left|c_3 + 2a_2c_2 - c_1^2 + c_1(c_1 + a_2^2)\right| \\ &\le |c_3| + 2|a_2||c_2| + |c_1|^2 + |c_1||c_1 + a_2^2| \\ &\le |c_3| + 2|a_2||c_2| + |c_1|^2 + 3|c_1| \\ &\le \frac{1}{3}\left(1 - |c_1|^2 - \frac{4|c_2|^2}{1 + |c_1|}\right) + 4|c_2| + |c_1|^2 + 3|c_1| \\ &:= f_1(|c_1|, |c_2|), \end{aligned}$$

where

$$f_1(x, y) = \frac{1}{3}\left(1 - x^2 - \frac{4y^2}{1 + x}\right) + 4y + x^2 + 3x,$$

$0 \le x = |c_1| \le 1$ and $0 \le y = |c_2| \le \frac{1}{2}(1 - x^2)$, i.e., $(x, y) \in G := [0, 1] \times [0, (1 - x^2)/2]$.

Since, $\frac{\partial f_1}{\partial x}(x, y) = \frac{4}{3}\left(\frac{y}{1+x}\right)^2 + \frac{4}{3}x + 3 > 0$ for all $(x, y) \in G$, we have that there are no singular points in the interior of G and f_1 attains its maximum on the boundary of G.

Further, for $x = 0$ we have $0 \le y \le \frac{1}{2}$ and $f_1(0, y) = \frac{1}{3}(1 - 4y^2) + 4y \le 2$.

Also, for $0 \le x \le 1$ and $y = 0$ we have $f_1(x, 0) = \frac{1}{3}(1 - x^2) + x^2 + 3x \le 4$.

Finally, for $0 \le x \le 1$ and $y = \frac{1}{2}(1 - x^2)$ we have $f_1(x, \frac{1}{2}(1 - x^2)) = 2 + \frac{10}{3}x - x^2 - \frac{1}{3}x^3 \le 4$, since the last function is an increasing one on $[0, 1]$.

$\square$

3. Generalized Zalcman conjecture for the class $\mathcal{U}$

In this section we give direct proof of the generalized Zalcman conjecture for the class $\mathcal{U}$ for the cases $m=2$, $n=3$; and $m=2$, $n=4$.

Theorem 3.1. *Let $f\in\mathcal{U}$ be of the form* (1.1). *Then*

(*i*) $|a_2a_3-a_4|\le 2$;

(*ii*) $|a_2a_4-a_5|\le 3$.

These inequalities are sharp with equality for the Koebe function $k(z)=\frac{z}{(1-z)^2}=z+\sum_{n=2}nz^n$ and its rotations.

Proof.

(*i*) In this case we have

$$\begin{aligned}|a_2a_3-a_4| &= |c_2+a_2c_1|\le|c_2|+|a_2||c_1|\le|c_2|+2|c_1|\\ &\le\frac{1}{2}(1-|c_1|^2)+2|c_1|\le\frac{1}{2}(1-|c_1|^2+4|c_1|)\le 2.\end{aligned}$$

(*ii*) In a similar way as in the proof of Theorem 2.1(*ii*),

$$\begin{aligned}|a_4a_2-a_5| &= \left|c_3+a_2c_2+a_2^2c_1+c_1^2\right|\\ &\le|c_3|+|a_2||c_2|+|c_1||a_2^2+c_1|\\ &\le|c_3|+|a_2||c_2|+3|c_1|\\ &\le\frac{1}{3}\left(1-|c_1|^2-\frac{4|c_2|^2}{1+|c_1|}\right)+2|c_2|+3|c_1|\\ &:=f_2(|c_1|,|c_2|),\end{aligned}$$

where

$$f_2(x,y)=\frac{1}{3}\left(1-x^2-\frac{4y^2}{1+x}\right)+2y+3x,$$

$0\le x=|c_1|\le 1$ and $0\le y=|c_2|\le\frac{1}{2}(1-x^2)$, i.e., $(x,y)\in G:=[0,1]\times[0,(1-x^2)/2]$.

Again, $\frac{\partial f_2}{\partial x}(x,y)=\frac{4}{3}\left(\frac{y}{1+x}\right)^2-\frac{2}{3}x+3>0$ for all $(x,y)\in G$, so f_2 attains its maximum on the boundary of G.

The conclusion follows since on the edges of G we have:

- $x=0$, $0\le y\le\frac{1}{2}$ and $f_2(0,y)=\frac{1}{3}(1-4y^2)+2y\le 1$;
- $y=0$, $0\le x\le 1$ and $f_2(x,0)=\frac{1}{3}(1-x^2)+3x\le 3$;
- $y=\frac{1}{2}(1-x^2)$, $0\le x\le 1$ and $f_2(x,\frac{1}{2}(1-x^2))=1+\frac{10}{3}x-x^2-\frac{1}{3}x^3\le 3$.

$\square$

4. Krushkal inequality for the class $\mathcal{U}$

In this section we give direct proof of the Krushkal inequality for the class $\mathcal{U}$ in the cases when $n=4$, $p=1$ and $n=5$, $p=1$.

Theorem 4.1. *Let $f \in \mathcal{U}$ be of the form* (1.1). *Then*

(*i*) $|a_4 - a_2^3| \leq 4$;

(*ii*) $|a_5 - a_2^4| \leq 11$.

These inequalities are sharp with equality for the Koebe function $k(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2} nz^n$ and its rotations.

Proof.

(*i*) It is easy to verify that

$$\begin{aligned} |a_4 - a_2^3| &= |c_2 + 2a_2c_1| \leq |c_2| + 2|a_2||c_1| \\ &\leq \frac{1}{2}(1-|c_1|^2) + 4|c_1| = \frac{1}{2}\left(1 + 8|c_1| - |c_1|^2\right) \leq 4. \end{aligned}$$

(*ii*) We will again use that $|a_3| = |c_1 + a_2^2| \leq 3$ and receive

$$\begin{aligned} |a_5 - a_2^4| &= |c_3 + 2a_2c_2 + c_1^2 + 3a_2^2c_1| \\ &= |c_3 + 2a_2c_2 - 2c_1^2 + 3c_1(c_1 + a_2^2)| \\ &\leq |c_3| + 2|a_2||c_2| + 2|c_1|^2 + 9|c_1| \\ &\leq \frac{1}{3}\left(1 - |c_1|^2 - \frac{4|c_2|^2}{1+|c_1|}\right) + 4|c_2| + 2|c_1|^2 + 9|c_1| \\ &:= g(|c_1|, |c_2|), \end{aligned}$$

where

$$g(x,y) = \frac{1}{3}\left(1 - x^2 - \frac{4y^2}{1+x}\right) + 4y + 2x^2 + 9x,$$

$0 \leq x = |c_1| \leq 1$ and $0 \leq y = |c_2| \leq \frac{1}{2}(1-x^2)$, i.e., $(x,y) \in G := [0,1] \times [0, (1-x^2)/2]$.

Since, $\frac{\partial g}{\partial x}(x,y) = \frac{10}{3}x + \frac{4}{3}\left(\frac{y}{1+x}\right)^2 + 9 > 0$ for all $(x,y) \in G$, so g has no critical points in the interior of G and attains its maximum on the boundary:

- $x = 0$, $0 \leq y \leq \frac{1}{2}$ and $g(0,y) = \frac{1}{3}(1 + 12y - 4y^2) \leq 2$;
- $y = 0$, $0 \leq x \leq 1$ and $g(x,0) = \frac{5}{3}x^2 + 9x + \frac{1}{3} \leq 11$;
- $y = \frac{1}{2}(1-x^2)$, $0 \leq x \leq 1$ and $g(x, \frac{1}{2}(1-x^2)) = 2 + \frac{28}{3}x - \frac{1}{3}x^3 \leq 11$.

The statement (*ii*) follows directly.

$\square$

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