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Existence and stability of solutions to fractional-order differential equations in a weighted space

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Abstract. We prove an existence and uniqueness result for fractional-order differential equations and a result regarding the Hyers-Ulam stability of this problem based on the use of weighted spaces.

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Key words and phrases: fractional-order; existence and uniqueness; Hyers-Ulam stability

1. Introduction

The more general form of the classical integer order differential equation, fractional-order differential equations have recently been utilized to model problems in a wide range of disciplines, including engineering, finance, astrophysics, thermodynamics, and mathematical physics [11, 16, 4, 15, 14]. Studies on the existence, uniqueness, and stability of fractional-order differential equations in the literature have also gained prominence as a result of the rapid rise in the importance of ordinary and partial fractional-order differential equations and developments in this field [7, 6, 2, 1].

Ulam was the first to introduce the idea of stability for functional equations at a conference in 1940. This type of stability concept came to be known as Hyers-Ulam stability after Hyers made his first contribution to Ulam's work in 1941. Obloza is the first author to study the this type of stability of linear differential equations [9]. Later, the concept of Hyers-Ulam stability is studied in many topics such as ordinary differential equations, partial differential equations, and delay differential equations [5, 8, 13, 19, 10]. As fractional-order differential equations widened traditional integer order differential equations, the stability problem gained even more significance. Many authors have examined this type of stability concerning the Caputo derivative for fractional-order differential equations [17, 18, 19, 2]. We refer the reader to papers [12, 13, 2] and the references therein for further information on the evolution of Hyers-Ulam stability.

In this article, we investigate the existence and uniqueness of solutions and Hyers-Ulam type stability for the following fractional-order differential equation in the sense of Caputo

$$\begin{cases} {}^{c}D^{\alpha}\upsilon(t) = f(t,\upsilon(t)) \qquad t \in [0,T] \\ \qquad \upsilon(0) = \upsilon_0. \end{cases} \tag{1.1}$$

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where ${}^cD^\alpha$ is the fractional derivative of order $\alpha \in (0, 1)$. The structure of the paper is as follows: The general concepts of Hyers-Ulam stability and the Caputo fractional-order derivative are discussed in Section 2. The useful inequalities and fundamental properties of the Caputo fractional-order derivative are described here. In Section 3, we investigate the existence and uniqueness of the solution to this problem. These type of theorems are usually proved under the condition of the continuity of the function f. The necessity to weaken the continuity assumptions on f contributes to the interest in such results. Here again, we extend the existence and uniqueness theorems on weighted spaces introduced on integer order differential equations in the paper [3] to fractional-order differential equations by modifying them. In Theorem 3.5, we obtain the existence and uniqueness of the solution in a weighted space under the hypothesis that the function on the righthand side of our problem satisfies Carathéodory and Lipschitz type conditions. In Section 4, we focus our attention on Hyers-Ulam stability for the problem.

2. Preliminaries

In this section, we present some notations, definitions, and preliminary facts used throughout this paper.

Definition 2.1. [11, 6] The Riemann–Liouville integral of order $\alpha > 0$ for the function υ is defined as

$$I^\alpha \upsilon(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \upsilon(s) ds, \quad t \in [0, T],$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2. [11, 6] The Caputo derivative of fractional-order α for the function υ is defined as

$$D^\alpha \upsilon(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \upsilon^{(n)}(s) ds, \quad t \in [0, T],$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3. The equation (1.1) is Hyers-Ulam stable if there exists a real number $c > 0$ such that for each $\epsilon > 0$ and for each solution $\vartheta \in C([0, T], \mathbb{R}^n)$ to the inequality

$$\|D^\alpha \vartheta(t) - f(t, \vartheta(t))\| \leq \epsilon \quad t \in [0, T], \tag{2.1}$$

there exists a solution $\upsilon \in C([0, T], \mathbb{R}^n)$ to the equation (1.1) with

$$\big\|\vartheta(t) - \upsilon(t)\big\| \leq c\epsilon \quad t \in [0, T].$$

Remark 2.4. A function $\vartheta \in C([0, T], \mathbb{R}^n)$ is a solution of inequality (2.1) if and only if there exists a function $\Psi \in C([0, T], \mathbb{R}^n)$ such that

i) $\|\Psi(t)\| \leq \epsilon$ for all $t \in [0,T]$,

ii) ${}^cD^\alpha \vartheta(t) = f(t,\vartheta(t)) + \Psi(t)$ for all $t \in [0,T]$.

Remark 2.5. If $\vartheta \in C([0,T],\mathbb{R}^n)$ is a solution of the inequality (2.1), then it is a solution to the following integral inequality:

$$\left\| \vartheta(t) - \vartheta(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,\vartheta(s))ds \right\| \leq \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)}$$

for all $t \in [0,T]$.

3. Existence and Uniqueness results

We begin by defining the weighted space to be studied and then give our main results.

Definition 3.1. Let $0 < k \leq 1$ be fixed and $\upsilon \in \mathbb{R}^n$. We will say that a function $w \in C([0,T],\mathbb{R}^n)$ belongs to $B_{\upsilon,k}([0,T],\mathbb{R}^n)$ if and only if

$$\|w\|_{\upsilon,k} := \sup\left\{ \frac{\|w(t)-\upsilon\|}{t^k} : t \in (0,T] \right\} \tag{3.1}$$

is finite.

For the sake of simplicity of notation, we frequently write $B_{\upsilon,k}$ instead of $B_{\upsilon,k}([0,T],\mathbb{R}^n)$. We would like to emphasize that the results we obtain about existence and uniqueness will be local, thus all results could be stated on all finite intervals $[0,T]$.

Remark 3.2. We note that $\|w\|_{\upsilon,k} < \infty$ implies that $w(0) = \upsilon$. Moreover, it is verified that $B_{\upsilon,k}$ is a vector space only if $\upsilon = 0$ and then $\|.\|_{0,k}$ is a norm. From here, for any υ we can obtain a metric defined on this space as follows

$$d(w,u) = \|w-u\|_{0,k}.$$

It is easily seen that any Cauchy sequence in this metric space is uniformly convergent. Thus, we have the following lemmas to be used for the fixed point theorem.

Lemma 3.3. *For $0 < k \leq 1$ and $\upsilon \in \mathbb{R}^n$, $B_{\upsilon,k}$ is a complete metric space.*

Lemma 3.4. *Assume that $k_1 < k_2$. Then we have the followings*

$$B_{\upsilon,k_2} \subset B_{\upsilon,k_1} \quad \text{and} \quad \|w\|_{\upsilon,k_1} \leq T^{k_2-k_1}\|w\|_{\upsilon,k_2}.$$

Proof. If we take any $w \in B_{\upsilon,k_2}$, then $\|w\|_{\upsilon,k_2} < \infty$ and

$$\frac{\|w(t)-\upsilon\|}{t^{k_1}} \frac{t^{k_2}}{t^{k_2}} \leq T\|w\|_{\upsilon,k_2} < \infty.$$

So, we have $B_{\upsilon,k_2} \subset B_{\upsilon,k_1}$. Now for $k_1 < k_2$,

$$\begin{aligned}\|w\|_{\upsilon,k_1} =& \frac{1}{T^{k_1}} \sup\left\{ \frac{\|w(t)-\upsilon\|}{(\frac{t}{T})^{k_1}} : t \in (0,T] \right\} \\ \leq& \frac{1}{T^{k_1}} \sup\left\{ \frac{\|w(t)-\upsilon\|}{(\frac{t}{T})^{k_2}} : t \in (0,T] \right\} = T^{k_2-k_1}\|w\|_{\upsilon,k_2}.\end{aligned}$$

□

Let us state the following hypotheses that will be used later:

(H1) $\upsilon \mapsto f(t,\upsilon)$ is continuous for a.e. t in $[0,T]$.

(H2) $t \mapsto f(t,\upsilon)$ is Lebesgue measurable for all υ.

(H3) There exist a locally integrable function m such that $\|f(t,\upsilon)\| \leq m(t)$.

(H4) There exist a function $\xi \in L^1((0,T),\mathbb{R})$ such that

$$\|f(t,\nu) - f(t,\tilde{\nu})\| \leq \frac{\xi(t)}{t}\|\nu - \tilde{\nu}\| \quad \text{for all } t \in (0,T]$$

(H5) Let the function ξ in (H4) satisfy the following quantity

$$\Lambda(\xi,T) := \sup\left\{ \frac{1}{t\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}\xi(s)ds : t \in (0,T] \right\} < 1$$

If f satisfies the conditions (H1),(H2) and (H3), it is called Carathéodory function. Since we consider the right-hand side of the problem (1.1) to be discontinuous, we need to explain the concept of solution. The necessary and sufficient condition for the $\upsilon \in C([0,T],\mathbb{R}^n)$ to be a solution to the problem (1.1) is that it is absolutely continuous function which satisfies (1.1) for almost every $t \in [0,T]$ and $\upsilon(0) = \upsilon_0$. Now we are ready to state our first result.

Theorem 3.5. *Let f be a Carathéodory function and hypotheses (H4) and (H5) are satisfied. If $\Lambda(f(\cdot,\upsilon_0),T)$ is finite, then the problem (1.1) has a unique solution belonging to $B_{\upsilon_0,1}$.*

Proof. Equivalently, finding a solution of the problem (1.1) is to find a solution of the following integral equation on $[0,T]$

$$\upsilon(t) = \upsilon_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,\upsilon(s))ds.$$

In order to prove our claim, we only prove the existence of solution for the integral equation which can be turned into a fixed point problem in $B_{\upsilon_0,1}$. Therefore, let's denote the right-hand side of the integral equation with $\mathcal{F}(\upsilon)$.

Step 1: Firslty we show that $\mathcal{F}(\upsilon) \in B_{\upsilon_0,1}$. So our initial task is to demonstrate that for any element $\upsilon \in B_{\upsilon_0,1}$, $\mathcal{F}$ is map from $B_{\upsilon_0,1}$ to $B_{\upsilon_0,1}$. Let us consider the following calculation.

$$\begin{aligned}
\left\| \frac{\mathcal{F}(\upsilon)(t) - \upsilon_0}{t} \right\| \le & \frac{1}{t\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,\upsilon(s))\| ds \\
= & \frac{1}{t\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big(\|f(s,\upsilon(s)) + f(s,\upsilon_0) - f(s,\upsilon_0)\| \Big) ds \\
\le & \frac{1}{t\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,\upsilon(s)) - f(s,\upsilon_0)\| ds \\
& + \frac{1}{t\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,\upsilon_0)\| ds \\
\le & \frac{1}{t\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\xi(s)}{s} \|\upsilon(s) - \upsilon_0\| ds + \Lambda(f(\cdot,\upsilon_0),T) \\
\le & \Lambda(\xi,T) \|\upsilon\|_{\upsilon_0,1} + \Lambda(f(\cdot,\upsilon_0),T)
\end{aligned}$$

Then we have $\mathcal{F}(\upsilon) \in B_{\upsilon_0,1}$.

Step 2: Now we prove that $\mathcal{F} : B_{\upsilon_0,1} \to B_{\upsilon_0,1}$ is a contraction mapping.

$$\begin{aligned}
\Big\| \mathcal{F}(\upsilon)(t) - \mathcal{F}(u)(t) \Big\| \le & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,\upsilon(s)) - f(s,u(s))\| ds \\
\le & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\xi(s)}{s} \|\upsilon(s) - u(s)\| ds \\
\le & \|\upsilon - u\|_{0,1} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi(s) ds.
\end{aligned}$$

After dividing both sides by t^k we get that

$$\begin{aligned}
\left\| \frac{\mathcal{F}(\upsilon)(t) - \mathcal{F}(u)(t)}{t^k} \right\| \le & \frac{\|\upsilon - u\|_{0,1}}{t^k \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi(s) ds. \\
= & \frac{\|\upsilon - u\|_{0,1}}{t^{k-1}} \frac{1}{t\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi(s) ds. \\
\le & \|\upsilon - u\|_{0,k} \Lambda(\xi,T).
\end{aligned}$$

Hence we have $d(\mathcal{F}(\upsilon), \mathcal{F}(u)) \le d(\upsilon,u)\Lambda(\xi,T)$. Since $\Lambda(\xi,T) < 1$, then the map $\mathcal{F}$ is a contraction mapping, and the Banach fixed point principle implies that exists a unique solution. Thus the proof is complete. $\square$

Corollary 3.6. *Let f be a Carathéodory function and hypotheses (H4) and (H5) are satisfied. Then there exists at most one solution to the problem (1.1) in $B_{\upsilon_0,1}$.*

Proof. Suppose that there are two solutions υ and u in the class $B_{\upsilon_0,1}$. It is easy to see that $\upsilon - u \in B_{0,1}$. In addition, we have the following

$$\begin{aligned}
{}^cD^\alpha \upsilon(t) - {}^c D^\alpha u(t) &= f(t,\upsilon(t)) - f(t,u(t)) \quad \text{a.e. } t \in [0,T] \\
\upsilon(0) - u(0) &= 0.
\end{aligned}$$

From the definition of derivative in the sense of Caputo, we have

$$v(t) - u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \big(f(s, v(s)) - f(s, u(s)) \big) ds.$$

Hence,

$$\begin{aligned} \|v(t) - u(t)\| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, v(s)) - f(s, u(s))\| ds \\ \leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \frac{\xi(s)}{s} \|v(s) - u(s)\| ds \\ \leq & \Big(\sup_{z \in (0,T]} \frac{\|v(z) - u(z)\|}{z} \Big) \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \xi(s) \end{aligned}$$

If we divide both sides by t, we have

$$\frac{\|v(t) - u(t)\|}{t} \leq \|v - u\|_{0,1} \Lambda(\xi, T).$$

Thus, using the $\Lambda(\xi, T) < 1$, we obtain that $\|v - u\|_{0,1} = 0$. Hence these solutions must be equal. □

4. Hyers-Ulam stability result

In this section we give a result on the Hyers-Ulam stability of the first equation in the problem (1.1).Based on the Definition 2.3, the concept of stability in the sense defined here will be adapted to the space we are studying on.

Theorem 4.1. *Let f be a Carathéodory function and assume that (H4) and (H5) are satisfied. If $\alpha \geq k$ then the first equation of the problem (1.1) is Hyers-Ulam stable.*

Proof. Let ϑ be a solution to (2.1). We indicate v as a unique solution to the following problem by Theorem 3.5,

$$\begin{cases} {}^cD^\alpha v(t) = f(t, v(t)) & t \in [0, T] \\ v(0) = \vartheta(0) \end{cases}$$

It follows we have

$$v(t) = \vartheta(0) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, v(s)) ds \quad t \in [0, T].$$

Now consider the following quantity using Remark 2.4 and (H4),

$$\begin{aligned}
\big\|\vartheta(t)-\upsilon(t)\big\| =& \Big\|\vartheta(t)-\vartheta(0)-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\upsilon(s))ds\Big\| \\
\leq& \Big\|\vartheta(t)-\vartheta(0)-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,\vartheta(s))ds\Big\| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big\|f(s,\vartheta(s))-f(s,\upsilon(s))\big\|ds \\
\leq& \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\frac{\xi(s)}{s}\big\|\vartheta(s)-\upsilon(s)\big\|ds \\
\leq& \frac{\epsilon t^\alpha}{\Gamma(\alpha+1)}+\|\vartheta(s)-\upsilon(s)\|_{0,1}\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\xi(s)ds.
\end{aligned}$$

If we divide both sides by t^k we have

$$\begin{aligned}
\Big\|\frac{\vartheta(t)-\upsilon(t)}{t^k}\Big\| \leq& \frac{\epsilon t^\alpha}{t^k\Gamma(\alpha+1)}+\frac{\|\vartheta(t)-\upsilon(t)\|_{0,1}}{t^k\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\xi(s)ds \\
\leq& \frac{\epsilon t^{\alpha-k}}{\Gamma(\alpha+1)}+\|\vartheta-\upsilon\|_{0,k}\Lambda(\xi,T).
\end{aligned}$$

Since $\Lambda(\xi,T)<1$, we obtain that

$$\|\vartheta-\upsilon\|_{0,k} \leq \frac{\epsilon T^{\alpha-k}}{\Gamma(\alpha+1)(1-\Lambda(\xi,T))} \quad \text{where} \quad c=\frac{T^{\alpha-k}}{\Gamma(\alpha+1)(1-\Lambda(\xi,T))}.$$

Hence, we say that the first equation of the problem (1.1) is Hyers-Ulam stable. □

References

[1] Develi, F. Existence and Ulam-Hyers stability results for nonlinear fractional Langevin equation with modified argument. *Math. Methods Appl. Sci. 45*, 7 (2022), 3417–3425.

[2] Duman, O., and Develi, F. Existence and Hyers-Ulam stability results for partial fractional-order delay differential equations. *Results Math. 77*, 3 (2022), Paper No. 97, 17.

[3] Górka, P. a., and Rybka, P. Existence and uniqueness of solutions to singular ODE's. *Arch. Math. (Basel) 94*, 3 (2010), 227–233.

[4] Hilfer, R., Ed. *Applications of fractional calculus in physics*. World Scientific Publishing Co., Inc., River Edge, NJ, 2000.

[5] Jung, S.-M. Hyers-Ulam stability of linear differential equations of first order. *Appl. Math. Lett. 17*, 10 (2004), 1135–1140.

[6] Kilbas, A. A., Srivastava, H. M., and Trujillo, J. J. *Theory and applications of fractional differential equations*, vol. 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2006.

[7] Lakshmikantham, V., and Vatsala, A. S. Basic theory of fractional differential equations. *Nonlinear Anal. 69*, 8 (2008), 2677–2682.

[8] Lungu, N., and Popa, D. On the Hyers-Ulam stability of a first order partial differential equation. *Carpathian J. Math. 28*, 1 (2012), 77–82.

[9] Obłoza, M. Hyers stability of the linear differential equation. *Rocznik NaukDydakt. Prace Mat. 13* (1993), 259–270.

[10] Otrocol, D., and Ilea, V. Ulam stability for a delay differential equation. *Cent. Eur. J. Math. 11*, 7 (2013), 1296–1303.

[11] Podlubny, I. *Fractional differential equations*, vol. 198 of *Mathematics in Science and Engineering*. Academic Press, Inc., San Diego, CA, 1999. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.

[12] Rassias, T. M. *Hyers-Ulam-Rassias stability of functional equations in nonlinear analysis* [book review of mr2790773]. *Eur. Math. Soc. Newsl.*, 82 (2011), 60–61.

[13] Rus, I. A. Ulam stability of ordinary differential equations. *Stud. Univ. Babeş-Bolyai Math. 54*, 4 (2009), 125–133.

[14] Sun, H., Zhang, Y., Baleanu, D., Chen, W., and Chen, Y. A new collection of real world applications of fractional calculus in science and engineering. *Commun. Nonlinear Sci. Numer. Simul. 64* (2018), 213–231.

[15] Tarasov, V. E. *Fractional dynamics.* Nonlinear Physical Science. Springer, Heidelberg; Higher Education Press, Beijing, 2010. Applications of fractional calculus to dynamics of particles, fields and media.

[16] Voyiadjis, G. Z., and Wojciech, S. Brain modelling in the framework of anisotropic hyperelasticity with time fractional damage evolution governed by the caputo-almeida fractional derivative. *J. Mech. Behav. Biomed. 89* (2019), 209–216.

[17] Wang, J., Lv, L., and Zhou, Y. Ulam stability and data dependence for fractional differential equations with Caputo derivative. *Electron. J. Qual. Theory Differ. Equ.* (2011), No. 63, 10.

[18] Wang, J., Lv, L., and Zhou, Y. New concepts and results in stability of fractional differential equations. *Commun. Nonlinear Sci. Numer. Simul. 17*, 6 (2012), 2530–2538.

[19] Wang, J., Zhou, Y., and Fečkan, M. Nonlinear impulsive problems for fractional differential equations and Ulam stability. *Comput. Math. Appl. 64*, 10 (2012), 3389–3405.

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