Starcompact and related spaces on Pixley-Roy hyperspaces

Huynh Thi Oanh Trieu¹, Luong Quoc Tuyen² and Ong Van Tuyen³

Abstract. In this paper, we study the relation between a space $X$ satisfying certain generalized metric properties and the Pixley-Roy hyperspace $\text{PR}[X]$ over $X$ satisfying the same properties. We prove that if $\text{PR}[X]$ is starcompact (resp., star-Lindelöf), then $X$ is compact (resp., Lindelöf). However, there exists a compact space $X$ such that $|X| = \omega$, but $\text{PR}_n[X]$ for all $n \in \mathbb{N}$ and $\text{PR}[X]$ are not starcompact spaces. Moreover, we show that $\text{PR}[X]$ is strongly starcompact (resp., strongly star-Lindelöf) if and only if $X$ is finite (resp., countable). By these results, we obtain that $\text{PR}[X]$ is set strongly starcompact (resp., cosmic, set strongly star-Lindelöf) if and only if $X$ is finite (resp., countable).

AMS Mathematics Subject Classification (2010): 54B20; 54D20

Key words and phrases: Pixley-Roy; hyperspace; set strongly starcompact; strongly starcompact; starcompact; set strongly star-Lindelöf; strongly star-Lindelöf; star-Lindelöf

1. Introduction

The generalized metric properties on Pixley-Roy hyperspaces have been studied by many authors (2, 4–9, 12–15, for example). They considered several generalized metric properties and studied the relation between a space $X$ satisfying such property and its Pixley-Roy hyperspaces satisfying the same property.

In this paper, we study concepts such as compact, set strongly starcompact, strongly starcompact, cosmic, Lindelöf, set strongly star-Lindelöf, strongly star-Lindelöf, star-Lindelöf on Pixley-Roy hyperspaces. We obtain some new results about Pixley-Roy hyperspaces.

Throughout this paper, all spaces are assumed to be at least $T_1$, $\mathbb{N}$ denotes the set of all positive integers, the first infinite ordinal denoted by $\omega$. Furthermore, if $\mathcal{P}$ is a family of subsets of a space $X$ and $A \subset X$, then

$$\text{St}(A, \mathcal{P}) = \bigcup \{ P \in \mathcal{P} : P \cap A \neq \emptyset \}.$$
2. Definitions

The *Pixley-Roy hyperspace* $\mathcal{PR}[X]$ over a space $X$, defined by C. Pixley and P. Roy in [7], is the set of all non-empty finite subsets of $X$ with the topology generated by the sets of the form

$$[F, V] = \{G \in \mathcal{PR}[X] : F \subset G \subset V\},$$

where $F \in \mathcal{PR}[X]$ and $V$ is an open subset in $X$ containing $F$. For any space $X$, $\mathcal{PR}[X]$ is zero-dimensional, completely regular and hereditarily metacompact (see [15]).

For each $n \in \mathbb{N}$, let $\mathcal{PR}_n[X] = \{F \in \mathcal{PR}[X] : |F| \leq n\}$.

**Remark 2.1** ( [12], p. 305). For each $n \in \mathbb{N}$, $\mathcal{PR}_n[X]$ is a closed subspace of $\mathcal{PR}[X]$ and in particular, $\mathcal{PR}_1[X]$ is a closed discrete subspace of $\mathcal{PR}[X]$.

**Remark 2.2** ( [4], Remark 1.2). Every $\mathcal{PR}_m[X]$ is a closed subspace of $\mathcal{PR}_n[X]$ for each $m, n \in \mathbb{N}, m < n$.

For each $F \in \mathcal{PR}[X]$ and $A \subset X$, denote

$$[F, A] = \{H \in \mathcal{PR}[X] : F \subset H \subset A\}.$$

**Definition 2.3.** Let $X$ be a space.

1. $X$ is said to be *starcompact* (resp., *star-Lindelöf*) [3], if for every open cover $\mathcal{U}$ of $X$, there exists a finite (resp., countable) $\mathcal{V} \subset \mathcal{U}$ such that $\text{St}(\bigcup \mathcal{V}, \mathcal{U}) = X$.

2. $X$ is said to be *strongly starcompact* (resp., *strongly star-Lindelöf*) [3], if for every open cover $\mathcal{U}$ of $X$, there is a finite (resp., countable) subset $A$ of $X$ such that $\text{St}(A, \mathcal{U}) = X$.

3. $X$ is said to be *set starcompact* (resp., *set star-Lindelöf*) [11], if for each nonempty subset $A$ of $X$ and each collection $\mathcal{U}$ of open sets in $X$ such that $A \subset \bigcup \mathcal{U}$, there is a finite (resp., countable) subset $\mathcal{V}$ of $\mathcal{U}$ such that $A \subset \text{St}(\bigcup \mathcal{V}, \mathcal{U})$.

4. $X$ is said to be *set strongly starcompact* (resp., *set strongly star-Lindelöf*) [11], if for each nonempty subset $A$ of $X$ and each collection $\mathcal{U}$ of open sets in $X$ such that $A \subset \bigcup \mathcal{U}$, there is a finite (resp., countable) subset $F$ of $\mathcal{A}$ such that $A \subset \text{St}(F, \mathcal{U})$.

5. $X$ is said to be *cosmic* [1], if $X$ has a countable network.

compact $\rightarrow$ countable $\rightarrow$ set strongly starcompact $\rightarrow$ set starcompact $\rightarrow$ strongly starcompact $\rightarrow$ starcompact $\rightarrow$ cosmic $\rightarrow$ Lindelöf $\rightarrow$ set strongly star-Lindelöf $\rightarrow$ set star-Lindelöf $\rightarrow$ strongly star-Lindelöf $\rightarrow$ star-Lindelöf

**Remark 2.4.** Definition starcompact (resp., strongly starcompact) is also 1-starcompact (resp., strongly 1-starcompact) in [10].
3. Main results

**Theorem 3.1.** Let $X$ be a space. If $\text{PR}[X]$ is starcompact, then $X$ is compact.

**Proof.** Suppose that $\mathcal{U}$ is an open cover of $X$. Then, for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. If we put

$$\mathcal{U} = \left\{ [F, \bigcup_{x \in F} U_x] : F \in \text{PR}[X] \right\},$$

then it is clear that $\mathcal{U}$ is an open cover of $\text{PR}[X]$. Because $\text{PR}[X]$ is starcompact, there exists a finite subfamily $\mathfrak{V}$ of $\mathcal{U}$ such that

$$\text{St}(\bigcup \mathfrak{V}, \mathcal{U}) = \text{PR}[X].$$

Put

$$\mathfrak{V} = \left\{ [F_i, \bigcup_{x \in F_i} U_x] : i \leq n \right\};$$

$$\mathcal{V} = \left\{ U_x : x \in F_i, i \leq n \right\}.$$

Then, it is clear that $\mathcal{V}$ is a finite subfamily of $\mathcal{U}$. Next, we need only to prove that $\mathcal{V}$ is a cover of $X$. Indeed, for each $y \in X$, since $\{y\} \in \text{PR}[X], \{y\} \in \text{St}(\bigcup \mathfrak{V}, \mathcal{U})$. Hence, there exist $F \in \text{PR}[X]$ and $i \leq n$ such that

$$\{y\} \in [F, \bigcup_{x \in F} U_x] \text{ and } [F, \bigcup_{x \in F} U_x] \cap [F_i, \bigcup_{x \in F_i} U_x] \neq \emptyset.$$

This implies that $F = \{y\}$ and there exists $H \in \text{PR}[X]$ such that

$$F \subset H \subset \bigcup_{x \in F} U_x \text{ and } F_i \subset H \subset \bigcup_{x \in F_i} U_x.$$ 

Hence, $F \subset \bigcup_{x \in F} U_x$. This shows that there exists $U_x \in \mathcal{V}$ such that $y \in U_x$. Therefore, $y \in \bigcup \mathcal{V}$. This implies that $X \subset \bigcup \mathcal{V}$. Thus, $\mathcal{V}$ is a cover of $X$.  

**Example 3.2.** There exists a compact space $X$ such that $|X| = \omega$, but $\text{PR}_n[X]$ for all $n \in \mathbb{N}$ and $\text{PR}[X]$ are not starcompact spaces.

**Proof.** Assume that

$$X = \{x_0\} \cup \{x_k : k \in \mathbb{N}\},$$

where every $x_k$ and $x_0$ are different from each other. The set $X$ endowed with the following topology: each $x_k$ is isolated; a basic neighborhood of $x_0$ has the form $\{x_0\} \cup \{x_k : k \geq m\}$ for some $m \in \mathbb{N}$.

1. It is obvious that $X$ is compact.

2. $\text{PR}[X]$ is not starcompact. Indeed, we take an open cover of $\text{PR}[X]$ as follows:

$$\mathcal{U} = \left\{ \{x_0\}, X \right\} \cup \left\{ \{x_k\}, \{x_k\} : k \in \mathbb{N} \right\} \cup \left\{ [A, X] : A \in \text{PR}[X] \setminus \text{PR}_1[X] \right\}$$

$$= \left\{ \{x_0\}, X \right\} \cup \left\{ \{x_k\} : k \in \mathbb{N} \right\} \cup \left\{ [A, X] : A \in \text{PR}[X] \setminus \text{PR}_1[X] \right\}.$$ 

Obviously, $\{x_k\} \notin \{x_0\}, X$ and $\{x_k\} \notin [A, X]$ for each $k \in \mathbb{N}$ and for each $A \in \text{PR}[X] \setminus \text{PR}_1[X]$. Now, suppose that $\mathfrak{W}$ is a finite subfamily of $\mathcal{U}$. Then, 

$$|\{k \in \mathbb{N} : \{x_k\} \in \bigcup \mathfrak{W}\}| < \omega.$$
This implies that there exists $k_0 \in \mathbb{N}$ such that $\{x_{k_0}\} \notin \bigcup \mathcal{V}$. Let $\{x_{k_0}\} \in \text{St}(\bigcup \mathcal{V}, \mathcal{U})$. Then, there exists $\mathcal{A} \in \mathcal{U}$ such that

$$\bigcup \mathcal{V} \cap \mathcal{A} \neq \emptyset, \{x_{k_0}\} \in \mathcal{A}.$$ 

Since $\{x_{k_0}\} \notin \{\{x_0\}, X\}$ and $\{x_{k_0}\} \notin \{A, X\}$ for each $A \in \text{PR}[X] \setminus \text{PR}_1[X], \mathcal{A} = \{\{x_{k_0}\}\}$. Thus, $\{x_{k_0}\} \in \bigcup \mathcal{V}$, which is a contradiction. Therefore, $\text{St}(\bigcup \mathcal{V}, \mathcal{U}) \neq \text{PR}[X]$. Hence, $\text{PR}[X]$ is not starcompact.

(3) $\text{PR}_n[X]$ is not starcompact for all $n \in \mathbb{N}$.

In deed, for each $n \in \mathbb{N}$, we take an open cover $\text{PR}_n[X]$ as follows:

$$\mathcal{U} = \big\{\{x_0\}, X \cap \text{PR}_n[X]\big\} \cup \big\{\{x_k\}, \{x_k\} \cap \text{PR}_n[X] : k \in \mathbb{N}\big\}$$

$$\cup \big\{[A, X] \cap \text{PR}_n[X] : A \in \text{PR}_n[X] \setminus \text{PR}_1[X]\big\}$$

$$= \big\{\{x_0\}, X \cap \text{PR}_n[X]\big\} \cup \big\{\{x_k\} : k \in \mathbb{N}\big\}$$

$$\cup \big\{[A, X] \cap \text{PR}_n[X] : A \in \text{PR}_n[X] \setminus \text{PR}_1[X]\big\}.$$ 

Then, it is obvious that $\{x_k\} \notin \{\{x_0, X\} \cap \text{PR}_n[X]$ and $\{x_k\} \notin \{A, V\} \cap \text{PR}_n[X]$ for each $k \in \mathbb{N}$ and for each $A \in \text{PR}_n[X] \setminus \text{PR}_1[X]$. Now, if $\mathcal{V}$ is a finite subfamily of $\mathcal{U}$, then by the proof of (2), there exists $k_0 \in \mathbb{N}$ such that $\{x_{k_0}\} \notin \text{St}(\bigcup \mathcal{V}, \mathcal{U})$. Hence, $\text{St}(\bigcup \mathcal{V}, \mathcal{U}) \neq \text{PR}_n[X]$. This shows that $\text{PR}_n[X]$ is not starcompact.  

**Theorem 3.3.** Let $X$ be a space. Then, the following statements are equivalent:

1. $\text{PR}[X]$ is compact;
2. $\text{PR}[X]$ is countably compact;
3. $\text{PR}[X]$ is set strongly starcompact;
4. $\text{PR}[X]$ is strongly starcompact;
5. $X$ is finite.

**Proof.** (1) $\leftrightarrow$ (2) $\leftrightarrow$ (3) $\leftrightarrow$ (4) by [10] Theorem 2.8 and $\text{PR}[X]$ is a metacompact space. On the other hand, (1) $\leftrightarrow$ (5) by [13] Remark 3.5.  

**Theorem 3.4.** Let $X$ be a space. If $\text{PR}[X]$ is star-Lindelöf, then $X$ is Lindelöf.

**Proof.** Let $\mathcal{U}$ be an open cover of $X$. Then, for each $x \in X$, there exists $U_x \in \mathcal{U}$ such that $x \in U_x$. If we put

$$\mathcal{U} = \big\{[F, \bigcup_{x \in F} U_x] : F \in \text{PR}[X]\big\},$$

then it is obvious that $\mathcal{U}$ is an open cover of $\text{PR}[X]$. Since $\text{PR}[X]$ is star-Lindelöf, there exists a countable subfamily $\mathcal{V}$ of $\mathcal{U}$ such that

$$\text{St}(\bigcup \mathcal{V}, \mathcal{U}) = \text{PR}[X].$$
Put
\[ V = \left\{ \bigcup_{x \in F_i} U_x : i \in \mathbb{N} \right\}; \]
\[ V = \{ U_x : x \in F_i, i \in \mathbb{N} \}. \]
Obviously, \( V \) is a countable subfamily of \( U \). On the other hand, similar to the proof of Theorem 3.1, we claim that \( V \) is a cover of \( X \). Therefore, \( X \) is Lindelöf.

Since set starcompact (resp., set star-Lindelöf) \( \Rightarrow \) starcompact (resp., star-Lindelöf) and by Theorems 3.1 and 3.4 we obtain the following corollary.

**Corollary 3.5.** Let \( X \) is a space. If \( PR[X] \) is set starcompact (resp., set star-Lindelöf), then \( X \) is compact (resp., Lindelöf).

**Question 1.** If \( X \) is Lindelöf, then are \( PR[X] \) and \( PR_n[X] \) for some \( n \in \mathbb{N} \) star-Lindelöf?

**Theorem 3.6.** Let \( X \) be a space. Then, the following statements are equivalent:

1. \( PR[X] \) is cosmic;
2. \( PR[X] \) is Lindelöf;
3. \( PR[X] \) is set strongly star-Lindelöf;
4. \( PR[X] \) is strongly star-Lindelöf;
5. \( X \) is countable.

**Proof.** (1) \( \Rightarrow \) (2) \( \iff \) (5) \( \Rightarrow \) (1) is obvious. Moreover, since \( PR[X] \) is a metacom pact space, \( PR[X] \) is a metaLindelöf space. It follows from [11, Theorem 2.9] that (2) \( \iff \) (3) \( \iff \) (4). \( \square \)

**Acknowledgement**

The authors would like to express their thanks to referee for his/her helpful comments and valuable suggestions.

**References**


Received by the editors June 7, 2023
First published online September 25, 2023