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# On the Zariski topology over the primary-like spectrum

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**Abstract.** Let $R$ be a commutative ring with identity and $M$ be a unital $R$-module. The primary-like spectrum $\mathcal{PS}(M)$ has a topology which is a generalization of the Zariski topology on the prime spectrum Spec($R$). We get several topological properties of $\mathcal{PS}(M)$, mostly for the case when the continuous mapping $\phi : \mathcal{PS}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\phi(Q) = \sqrt{(Q : M)}/\mathrm{Ann}(M)$ is surjective or injective. For example, if $\phi$ is surjective, then $\mathcal{PS}(M)$ is a connected space if and only if $\mathrm{Spec}(R/\mathrm{Ann}(M))$ is a connected space. It is shown that if $\phi$ is surjective, then a subset $Y$ of $\mathcal{PS}(M)$ is irreducible if and only if $Y$ is the closure of a singleton set. It is also proved that if the image of $\phi$ is a closed subset of $\mathrm{Spec}(R/\mathrm{Ann}(M))$, then $\mathcal{PS}(M)$ is a spectral space if and only if $\phi$ is injective.

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## 1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unital. Let $R$ be a ring and $M$ be an $R$-module. For a submodule $N$ of $M$, we let $(N : M) = \{r \in R \mid rM \subseteq N\}$. As usual, we denote the annihilator ideal $((0) : M)$ by $\mathrm{Ann}(M)$. A proper submodule $P$ of $M$ with $p = (P : M)$ is called a *prime submodule* (or a *p-prime submodule*) of $M$, if for $r \in R$ and $m \in M$, $rm \in P$ implies that either $r \in p$ or $m \in P$. The *prime spectrum* of $M$, denoted $\mathrm{Spec}(M)$, is the set of all prime submodules of $M$. Also for a prime ideal $p$ of $R$, $\mathrm{Spec}_p(M)$ will denote the set of all $p$-prime submodules of $M$. The intersection of all prime submodules of $M$ containing $N$, denoted $\mathrm{rad}\, N$, is called the *radical* of $N$. If there is no prime submodule containing $N$, $\mathrm{rad}\, N$ is defined to be $M$. In the ideal case, the radical of $I$ is denoted by $\sqrt{I}$. As a generalization of a primary ideal one hand and a generalization of the prime submodule on the other hand, a proper submodule $Q$ of $M$ is called a *primary-like submodule*, if for $r \in R$ and $m \in M$, $rm \in Q$ implies either $r \in (Q : M)$ or $m \in \mathrm{rad}\, Q$ [9]. We say that a submodule $N$ of a non-zero $R$-module $M$ satisfies the *primeful property* if for each prime ideal $p$ of $R$ with $(N : M) \subseteq p$, there exists a prime submodule $P$ containing $N$ such

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that $(P : M) = p$. If the zero submodule of $M$ satisfies the primeful property, then $M$ is called a *primeful module* [12]. For example, finitely generated modules and projective modules over domains are two classes of primeful modules [12, Propositin 3.8, Corollary 4.3]. The *primary-like spectrum* of $M$, denoted $\mathcal{PS}(M)$, is defined to be the set of all primary-like submodules of $M$ satisfying the primeful property. If $N$ is a submodule of $M$ satisfying the primeful property, then $(\operatorname{rad} N : M) = \sqrt{(N : M)}$ [12, Proposition 5.3]. It is easily seen that if $Q$ is a primary-like submodule satisfying the primeful property, then $p = \sqrt{(Q : M)}$ is a prime ideal of $R$. Therefore by a *p-primary-like submodule* $Q$ of $M$, we mean that $Q$ is a primary-like submodule satisfying the primeful property with $p = \sqrt{(Q : M)}$. The set of such submodules is denoted by $\mathcal{PS}_p(M)$. It should be noted that if $Q \in \mathcal{PS}_p(M)$ and $m$ be a maximal ideal of $R$ containing $p$, then there is a prime submodule $P$ containing $Q$ such that $(P : M) = m$. It follows that $\operatorname{rad} Q \neq M$ for all $Q \in \mathcal{PS}(M)$.

In recent years, several generalizations of the Zariski topology from rings to modules have been introduced and studied from various points of views (see, for example, [2, 5, 7, 11, 13, 9]).

One of them is the *Zariski topology* on Spec$(M)$ which is described by taking the set $\{V(N) \mid N \text{ is a submodule of } M\}$ as the set of closed sets of Spec$(M)$, where $V(N) = \{P \in \operatorname{Spec}(M) \mid (P : M) \supseteq (N : M)\}$ [11, 7].

Now, we set $\nu(N) = \{Q \in \mathcal{PS}(M) \mid \sqrt{(Q : M)} \supseteq \sqrt{(N : M)}\}$ for every submodule $N$ of $M$. As in the case of the Zariski topology on Spec$(M)$, the class of varieties $\Omega(M) = \{\nu(N) \mid N \text{ is a submodule of } M\}$ satisfies all axioms of closed sets in a topological space [9, Lemma 1]. Throughout this paper, it is assumed that $\mathcal{PS}(M)$ is equipped with this topology which enjoys analogs of many of the properties of the Zariski topology on Spec$(M)$. We have already obtained some of the topological properties of this space in [9]. For instance, in [9, Lemma 5], it has been shown that the set $\mathcal{B} = \{\mathcal{PS}(M) \setminus \nu(rM) \mid r \in R\}$ forms a basis for this topology on $\mathcal{PS}(M)$. Furthermore, every finite intersection of the elements of $\mathcal{B}$ a quasi-compact subspaces of $\mathcal{PS}(M)$ [9, Theorem 3].

In this paper, we examine the properties of certain mappings between the primary-like spectrum $\mathcal{PS}(M)$ of $M$ and the spectrums Spec$(R/Ann(M))$ and Spec$(M)$, in particular considering when these mappings are continuous or homeomorphisms (Proposition 2.8, Theorem 2.9 and Corollary 2.10). It is shown that $\mathcal{PS}(M)$ is connected if and only if Spec$(R/Ann(M)$ is a connected space (Proposition 2.12). Hochster's characterization of a spectral space involves an irreducibility discussion in $\mathcal{PS}(M)$. It is shown that for any finitely generated module $M$, every irreducible subspace of $\mathcal{PS}(M)$ is the closure of a singleton set (Theorem 3.8). In particular, if $M$ is a finitely generated $R$-module, then $\mathcal{PS}(M)$ is a spectral space, i.e., $\mathcal{PS}(M)$ is homeomorphic with Spec$(S)$ for some commutative ring $S$ (Theorem 4.4).

## 2. Continuous mappings between spectrums

As shown in [11, Proposition 3.1], $\psi : \operatorname{Spec}(M) \to \operatorname{Spec}(R/\operatorname{Ann}(M))$ defined by $\psi(P) = (P : M)/Ann(M)$ is a continuous mapping. In [9, Proposition1], we have introduced the mappings $\phi : \mathcal{PS}(M) \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ by $\phi(Q) = \sqrt{(Q : M)}/Ann(M)$ which is continuous, and plays a role analogous to that of $\psi$. Here, we introduce $\rho : \mathcal{PS}(M) \to \mathrm{Spec}(M)$ defined by $\rho(Q) = S_p(Q + pM)$, in which $p = \sqrt{(Q : M)}$ and

$$S_p(Q + pM) = \{m \in M \mid \exists c \in R \setminus p,\ cm \in Q + pM\}.$$

By [12, Proposition 4.4], $\rho$ is well defined. Note that $\phi = \psi \circ \rho$. It is shown that $\rho$ is a continuous mapping (Proposition 2.8), and the conditions under which $\rho$ is injective, surjective, closed and open are examined.

An $R$-module $M$ is called a multiplication module, if every submodule of $M$ has the form $IM$. In this case, we can take $I = (N : M)$ (see, for example, [8]). It is easy to see that if $M$ is a multiplication $R$-module, then $\psi$ is injective.

**Proposition 2.1.** *Let $M$ be an $R$-module. Consider the following statements.*

*(1) If $\nu(Q) = \nu(Q')$ for $Q, Q' \in \mathcal{PS}(M)$, then $Q = Q'$.*

*(2) $|\mathcal{PS}_p(M)| \leq 1$ for every $p \in Spec(R)$.*

*(3) $\phi$ is injective.*

*(4) $\rho$ is injective.*

*Then* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4). *Moreover, if $M$ is a multiplication $R$-module, then* (4) $\Rightarrow$ (3).

*Proof.* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) Follows from [9, Proposition 2].
(3) $\Rightarrow$ (4) Clear.
(4) $\Rightarrow$ (3) Since $M$ is a multiplication module , $\psi$ is injective. Now since $\phi = \psi \circ \rho$ and $\rho$ is injective, we conclude that $\phi$ is injective. □

The following example shows that (4) $\Rightarrow$ (3) in Proposition 2.1 is not true in general.

**Example 2.2.** Let $V$ be a vector space over a field $F$ with $\dim_F V > 1$. It is evident that $\mathcal{PS}(V)$ and $\mathrm{Spec}(V)$ are the set of all proper vector subspaces of $V$. Now, since $\phi(Q) = \phi(Q') = 0$ for all distinct subspaces $Q, Q' \in \mathcal{PS}(V)$, $\phi$ is not injective. On the other hand $\rho$ is injective, because if $\rho(Q) = \rho(Q')$ for $Q, Q' \in \mathcal{PS}(V)$, then $S_{(0)}(Q) = S_{(0)}(Q')$ which follows that $Q = Q'$.

**Proposition 2.3.** *Let $M$ be an $R$-module. Consider the following statements:*

*(1) $\mathcal{PS}_p(M) \neq \emptyset$ for every $p \in V(Ann(M))$.*

*(2) $\phi$ is surjective.*

*(3) $\psi$ is surjective.*

*(4) $pMp \neq Mp$ for every $p \in V(Ann(M))$.*

*(5) $S_p(pM)$ is a $p$-prime submodule of $M$ for every $p \in V(Ann(M))$.*

*(6)* $\mathrm{Spec}_p(M) \neq \emptyset$*, for every* $p \in V(Ann(M))$*.*

*Then* $(1) \Leftrightarrow (2) \Rightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6)$*. Moreover, all of these conditions are equivalent in the following cases:*

*(a)* $M$ *is a multiplication* $R$*-module.*

*(b)* $M$ *is a projective* $R$*-module.*

*(c)* $M$ *is a faithfully flat* $R$*-module.*

*Proof.* $(1) \Leftrightarrow (2) \Rightarrow (3)$ Clear.
$(3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6)$ Follows from [12, Theorem 2.1].
$(6) \Rightarrow (1)$ (a) Let $M$ be a multiplication $R$-module. Let $p \in V(Ann(M))$ and $P \in \mathrm{Spec}_p(M)$. Since $M$ is a multiplication module, we have $P = pM$. Suppose $p' \in \mathrm{Spec}(R)$ and $p \subseteq p'$. By [12, Theorem 2.1], there exists a prime submodule $P'$ of $M$ such that $(P' : M) = p'$. It follows that $P = pM \subseteq p'M = P'$. Hence $P$ satisfies the primeful property and so $\mathcal{PS}_p(M) \neq \emptyset$.
(b) Let $M$ be a projective $R$-module. Let $p \in V(Ann(M))$ and $P \in \mathrm{Spec}_p(M)$. By [1, Corollary 2.3], $pM$ is a prime submodule of $M$. Also, $pM$ satisfies the primeful property and $(pM : M) = p$ by [12, Corollary 4.3 and Proposition 4.5]. Therefore $pM \in \mathcal{PS}_p(M)$.
(c) Let $M$ be a faithfully flat $R$-module. Let $p \in V(Ann(M))$ and $P \in \mathrm{Spec}_p(M)$. By [4, Corollary 2.6 (ii)], $pM$ is a prime submodule of $M$. Also, $pM$ satisfies the primeful property and $(pM : M) = p$ by [12, Corollary 4.3 and Proposition 4.5]. Therefore $pM \in \mathcal{PS}_p(M)$. □

The following example shows that $(3) \Rightarrow (2)$ in Proposition 2.3 is not true in general.

**Example 2.4.** Let $\Omega$ be the set of all prime integers $p$ and $M = \prod_{p\in\Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$. By [12, Example1] the submodule $(0)$ of $M$ satisfies the primeful property (i.e., $\psi$ is surjective) and $\mathrm{rad}(0) = (0)$ . It follows that $(0)$ is not a primary-like submodule of $M$, since it is not a prime submodule of $M$. Thus $(0) \notin \mathcal{PS}(M)$. Now, we show that $\phi$ is not surjective. If on the contrary $\phi(Q) = \underline{(0)}$ for some $Q \in \mathcal{PS}(M)$, then $(Q : M) = 0$. Let $(Q : M) \subseteq p$ for $(0) \neq p \in \mathrm{Spec}(\mathbb{Z})$. Since $pM$ is the only prime submodule with $(pM : M) = p$ and $Q$ satisfies the primeful property, we have $Q \subseteq pM$. Thus $Q \subseteq \cap_{p\in\Omega} pM = (0)$ and so $Q = (0)$. This is a contradiction, since $(0) \notin \mathcal{PS}(M)$.

**Corollary 2.5.** *If* $\mathcal{PS}_p(M)$ *is a singleton set for every* $p \in Spec(R)$*, then* $\phi$ *is a bijective map and therefore* $\mathrm{Spec}(R/\mathrm{Ann}(M))$ *is a singleton set.*

*Proof.* By Proposition 2.1 and Proposition 2.3. □

**Corollary 2.6.** *Let* $M$ *be a multiplication* $R$*-module. If* $\rho$ *is injective and* $\psi$ *is surjective, then* $\phi$ *is bijective. In this case* $\mathcal{PS}(M)$ *and* $\mathrm{Spec}(R/\mathrm{Ann}(M))$ *are homeomorphic.*

*Proof.* By Proposition 2.1, Proposition 2.3 and [9, Theorem 1]. □

**Proposition 2.7.** *Let $M$ be a finitely generated multiplication $R$-module. Then $\phi$ and $\rho$ are surjective, and $\psi$ is bijective.*

*Proof.* Since $M$ is a multiplication $R$-module, it is evident that $\psi$ is injective. Let $\bar{p} \in \mathrm{Spec}(R/\mathrm{Ann}(M))$. Since $M$ is finitely generated, by [8, Theorem 2.5], $pM \neq M$. Thus by [8, Corollary 2.11], $pM \in \mathrm{Spec}(M)$ and $\psi(pM) = \bar{p}$. Hence $\psi$ is surjective and so by Proposition 2.3, $\phi$ is surjective. Now, let $P \in \mathrm{Spec}(M)$. Then there exists a $Q \in \mathcal{PS}(M)$ such that $\phi(Q) = \psi(P)$, i.e., $\sqrt{(Q : M)} = (P : M)$. By [12, Proposition 4.4], $(S_p(Q + pM) : M) = (P : M)$ where $p = (P : M)$. Since $M$ is a multiplication $R$-module, we have $\rho(Q) = S_p(Q + pM) = P$. Hence $\rho$ is surjective. □

Let $M$ be an $R$-module. From now on, we will denote $R/\mathrm{Ann}(M)$ by $\bar{R}$ and any ideal $I/\mathrm{Ann}(M)$ of $\bar{R}$ by $\bar{I}$. By [11, Proposition 3.1], $\psi^{-1}(V(\bar{I})) = V(IM)$, for every ideal $I \in V(Ann(M))$. Also, by [9, Proposition 1], we have $\phi^{-1}(V(\bar{I})) = \nu(IM)$, for every ideal $I \in V(Ann(M))$. Therefore both $\psi$ and $\phi$ are continuous. Now we give a similar result for $\rho$.

**Proposition 2.8.** *Let $M$ be a $R$-module. Then $\rho^{-1}(V(N)) = \nu(N)$, for every submodule $N$ of $M$. Therefore $\rho$ is a continuous mapping.*

*Proof.* Let $Q \in \rho^{-1}(V(N))$. Then $\rho(Q) \in V(N)$, and so $(S_p(Q + pM) : M) \supseteq (N : M)$ in which $p = \sqrt{(Q : M)}$. Hence we have

$$\sqrt{(Q : M)} \supseteq \sqrt{(S_p(Q + pM) : M)} \supseteq \sqrt{(N : M)}.$$

Thus $Q \in \nu(N)$, so that $\rho^{-1}(V(N)) \subseteq \nu(N)$. For the reverse inclusion, let $Q \in \nu(N)$. It follows that, $p = \sqrt{(Q : M)} \supseteq \sqrt{(N : M)} \supseteq (N : M)$. Thus

$$(S_p(Q + pM) : M) \supseteq (pM : M) \supseteq ((N : M)M : M) = (N : M),$$

which shows that $S_p(Q + pM) \in V(N)$, i.e., $\rho(Q) \in V(N)$. Thus $Q \in \rho^{-1}(V(N))$ so that $\nu(N) \subseteq \rho^{-1}(V(N))$. □

In [11, Theorem 3.6], it has been shown that that if $\psi$ is a surjective map, then $\psi(V(N)) = V(\overline{\sqrt{(N : M)}})$ and $\psi(\mathrm{Spec}(M) - V(N)) = \mathrm{Spec}(\bar{R}) - V(\overline{\sqrt{(N : M)}})$, for every submodule $N$ of $M$. Also, by [9, Theorem 1], $\phi(\nu(N)) = V(\overline{\sqrt{(N : M)}})$ and $\phi(\mathcal{PS}(M) - \nu(N)) = \mathrm{Spec}(\bar{R}) - V(\overline{\sqrt{(N : M)}})$, for every submodule $N$ of $M$. Now we give a similar result for $\rho$.

**Theorem 2.9.** *Let $M$ be an $R$-module. Then if $\rho$ is surjective, then for every submodule $N$ of $M$, $\rho(\nu(N)) = V(N)$ and $\rho(\mathcal{PS}(M) - \nu(N)) = \mathrm{Spec}(M) - V(N)$. Therefore $\rho$ is closed and open.*

*Proof.* Let $N$ be a submodule of $M$. Using Proposition 2.8, $\rho^{-1}(V(N)) = \nu(N)$. Then $\rho(\nu(N)) = \rho(\rho^{-1}(V(N))) = V(N)$. Also, we have

$$\begin{aligned}\rho(\mathcal{PS}(M) - \nu(N)) &= \rho(\rho^{-1}(\mathrm{Spec}(M)) - \rho^{-1}(V(N))) \\ &= \rho(\rho^{-1}(\mathrm{Spec}(M) - (V(N)))) \\ &= \mathrm{Spec}(M) - V(N).\end{aligned}$$

□

**Corollary 2.10.** *Let $\rho$ be as before. Then $\rho$ is a bijection if and only if $\rho$ is a homeomorphism.*

*Proof.* By Theorem 2.9. □

**Corollary 2.11.** *Let $M$ be a finitely generated multiplication $R$-module. Then $\rho$ is a homeomorphism if and only if $\phi$ is a homeomorphism.*

*Proof.* $\Rightarrow$) By Proposition 2.1, Proposition 2.7 and [9, Proposition 1].
$\Leftarrow$) By Proposition 2.1, Proposition 2.7 and Theorem 2.9. □

**Proposition 2.12.** *Let $\phi$ be a surjective map. Then the following statements are equivalent.*

*(1) $\mathcal{PS}(M)$ is connected;*

*(2) $\mathrm{Spec}(\bar{R})$ is connected;*

*(3) The ring $\bar{R}$ contains no idempotent other than $\bar{0}$ and $\bar{1}$;*

*(4) $\mathrm{Spec}(M)$ is connected.*

*Proof.* (1)$\Rightarrow$(2) Since $\phi$ is a continuous map, $\phi$ preserves connectedness. Hence $\mathrm{Spec}(\bar{R})$ is connected.
(2)$\Rightarrow$(1) If $\mathcal{PS}(M)$ is disconnected, then $\mathcal{PS}(M)$ must contain a non-empty proper subset $Y$ that is both open and closed. Accordingly, $\phi(Y)$ is a non-empty subset of $\mathrm{Spec}(\bar{R})$ that is both open and closed by Theorem 2.9. Since $Y$ is open, $Y = \mathcal{PS}(M) - \nu(N)$ for some submodule $N$ of $M$ whence by Theorem 2.9, $\phi(Y) = \mathrm{Spec}(\bar{R}) - \phi^{-1}(V(\overline{\sqrt{(N:M)}}))$. Therefore, if $\phi(Y) = \mathrm{Spec}(\bar{R})$, then $V(\overline{\sqrt{(N:M)}}) = \emptyset$. Thus $\overline{\sqrt{(N:M)}} = \bar{R}$, and so $N = M$. It follows that $Y = \mathcal{PS}(M) - \nu(N) = \mathcal{PS}(M) - \nu(M) = \mathcal{PS}(M)$ which is impossible, since $Y$ is a proper subset of $\mathcal{PS}(M)$. Thus $\phi(Y)$ is a proper subset of $\mathrm{Spec}(\bar{R})$ so that $\mathrm{Spec}(\bar{R})$ is disconnected, a contradiction.
(2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follows from [11, Corollary 3.8]. □

**Lemma 2.13.** *Let $M$ and $M'$ be $R$-modules and $N'$ a submodule of $M'$. Let $f : M \to M'$ be an epimorphism. Then the following hold.*

*(1) If $N'$ is a primary-like submodule of $M'$, then $f^{-1}(N')$ is a primary-like submodule of $M$.*

*(2) If $N'$ satisfies the primeful property, then so does $f^{-1}(N')$.*

*Proof.* (1) Suppose $rm' \in f^{-1}(N')$ and $r \notin (f^{-1}(N') : M)$. Thus $rf(m') \in N'$ and $r \notin (N' : M')$. Now, since $f^{-1}(\text{rad}\, N') \subseteq \text{rad}(f^{-1}(N'))$, we have $m' \in \text{rad}(f^{-1}(N'))$ as required.
(2) Let $p$ be a prime ideal such that $(f^{-1}(N') : M) \subseteq p$. Suppose $r \in (N' : M')$. Hence $f(rM) = rf(M) = rM' \subseteq N'$. Thus $rM \subseteq f^{-1}(N')$ and so $r \in p$. Therefore $(N' : M') \subseteq p$. Then there exists a prime submodule $P'$ of $M'$ containing $N'$ such that $(P' : M') = p$. It is easily seen that $(f^{-1}(P') : M) = p$. Thus $f^{-1}(N')$ satisfies the primeful property. □

**Theorem 2.14.** *Let $M$ and $M'$ be $R$-modules and $f : M \to M'$ be a epimorphism. Then the following hold:*

*(1) The mapping $\sigma_f : \text{Spec}(M') \to \text{Spec}(M)$ defined by $P' \mapsto f^{-1}(P')$ is an injective continuous map.*

*(2) The mapping $\mu_f : \mathcal{PS}(M') \to \mathcal{PS}(M)$ defined by $Q' \mapsto f^{-1}(Q')$ is an injective continuous map.*

*(3) If $g : M' \to M''$ is an epimorphism, then $\mu_{g \circ f} = \mu_f \circ \mu_g$ and $\sigma_{g \circ f} = \sigma_f \circ \sigma_g$.*

*(4) $\mu(1_{\text{Spec}(M)}) = 1_{\mathcal{PS}(M)}$ and $\sigma(1_M) = 1_{\text{Spec}(M)}$ in which $1_{\mathcal{PS}(M)}$ and $1_{\text{Spec}(M)}$ are identity maps over $\text{Spec}(M)$ and $\mathcal{PS}(M)$ respectively.*

*(5) $\rho_M \circ \mu_f = \sigma_f \circ \rho_{M'}$ in which $\rho_M$ and $\rho_{M'}$ are the same $\rho$ related to $M$ and $M'$ respectively.*

*Proof.* (1) Follows from [11, Proposition 3.9].
(2) It is clear that $\mu_f$ is well-defined by Lemma 2.13. It is also injective, since $f$ is surjective. Let $Q' \in \mathcal{PS}(M')$ and $\nu(N)$ be a closed subset of $\mathcal{PS}(M)$. Now, by using the fact that $\nu(N) = \nu(\sqrt{(N : M)}M)$, we have

$$\begin{aligned} Q' \in {\mu_f}^{-1}(\nu(N)) &\Leftrightarrow Q' \in {\mu_f}^{-1}(\nu(\sqrt{(N : M)}M)) \\ &\Leftrightarrow f^{-1}(Q') \supseteq \sqrt{(N : M)}M \\ &\Leftrightarrow Q' \supseteq f(\sqrt{(N : M)}M) \\ &\Leftrightarrow Q' \supseteq \sqrt{(N : M)}M' \\ &\Leftrightarrow Q' \in \nu(\sqrt{(N : M)}M'). \end{aligned}$$

Therefore ${\mu_f}^{-1}(\nu(N)) = \nu(\sqrt{(N : M)}M')$ and so $\mu_f$ is continuous.
(3), (4) Clear.
(5) Let $Q' \in \mathcal{PS}(M')$. Then we have $(\rho_M \circ \mu_f)(Q') = S_p(f^{-1}(Q') + pM)$ and $(\sigma_f \circ \rho_{M'})(Q') = f^{-1}(S_{p'}(Q' + p'M'))$, where $p = \sqrt{(f^{-1}(Q') : M)}$ and

$p' = \sqrt{(Q' : M')}$. It is easily seen that that $p = p'$, and then we have

$$\begin{aligned} x \in S_p(f^{-1}(Q') + pM) &\Leftrightarrow cx \in f^{-1}(Q') + pM \text{ for some } c \in R - p \\ &\Leftrightarrow cf(x) \in Q' + pM \text{ for some } c \in R - p \\ &\Leftrightarrow f(x) \in S_p(Q' + pM') \\ &\Leftrightarrow x \in f^{-1}(S_p(Q' + pM')) \\ &\Leftrightarrow x \in f^{-1}(S_{p'}(Q' + p'M')). \end{aligned}$$

Thus $\rho_M \circ \mu_f = \sigma_f \circ \rho_{M'}$. □

**Lemma 2.15.** *Let $M$ and $M'$ be $R$-modules. Let $f : M \to M'$ be a epimorphism and $N$ a submodule of $M$ containing $Kerf$. Then the following hold.*

*(1) If $N$ is a primary-like submodule of $M$, then $f(N)$ is a primary-like submodule of $M'$.*

*(2) If $N$ satisfies the primeful property, then $f(N)$ satisfies the primeful property.*

*Proof.* (1) First note that $f(N)$ is a proper submodule of $M'$, since $N$ is a proper submodule containing $Kerf$. Assume that $rf(m) \in f(N)$ for $r \in R$ and $m \in M$. Thus there exists $n \in N$ such that $rm - n \in Kerf$. Hence $rm \in N$, and thus $r \in (N : M)$ or $m \in \operatorname{rad} N$. Since $(N : M) = (f(N) : M')$ and $f(\operatorname{rad}N) = \operatorname{rad}(f(N))$, then $f(N)$ is a primary-like submodule of $M'$.
(2) Let $p$ be a prime ideal containing $(f(N) : M')$. Then $p$ is a prime ideal containing $(N : M)$ and so there is a prime submodule $P$ containing $N$ such that $(P : M) = p$. Since $P$ contains $Kerf$, the submodule $f(P)$ of $M'$ is a prime submodule containing $f(N)$ such that $(f(P) : M') = p$. □

**Theorem 2.16.** *Let $M$ and $M'$ be $R$-modules. Let $f : M \to M'$ be an epimorphism. Then $\mathcal{PS}(M')$ is homeomorphic to the topological subspace $\mathcal{W}$ of $\mathcal{PS}(M)$ consists of all primary-like submodules of $M$ containing $Kerf$.*

*Proof.* Consider $\delta_f : \mathcal{W} \to \mathcal{PS}(M')$ defined by $\delta_f(Q) = f(Q)$. By Lemma 2.15, $\delta_f$ is well-defined. Also $\delta_f$ is continuous. Indeed, since $f$ is surjective and $Q$ is a primary-like submodule containing $Kerf$, we have

$$\begin{aligned} Q \in {\delta_f}^{-1}(\nu(f(N))) &\Leftrightarrow f(Q) \in \nu(f(N)) \\ &\Leftrightarrow \sqrt{(f(Q) : M')} \supseteq \sqrt{(f(N) : M')} \\ &\Leftrightarrow \sqrt{(Q : M)} \supseteq \sqrt{(N : M)} \\ &\Leftrightarrow Q \in \nu(N). \end{aligned}$$

It shows that ${\delta_f}^{-1}(\nu(f(N))) = \nu(N) \cap \mathcal{W}$. Moreover, by letting $\mu_f$ as in Theorem 2.14, $(\delta_f \circ \mu_f)(Q') = f(f^{-1}(Q')) = Q'$ for all $Q' \in \mathcal{PS}(M')$ . Thus $\delta_f \circ \mu_f = 1_{\mathcal{PS}(M')}$. Also if $Q \in \mathcal{W}$, then $(\mu_f \circ \delta_f)(Q) = f^{-1}(f(Q)) \supseteq Q$. For the reverse inclusion, let $x \in f^{-1}(f(Q))$. Then $f(x) = f(q)$, for some $q \in Q$ so that $x - q \in Kerf$. It follows that $x \in Q$, since $Kerf \subseteq Q$. Hence $(\mu_f \circ \delta_f)(Q) = Q$ for every $Q \in \mathcal{W}$, and so $\mu_f \circ \delta_f = 1_{\mathcal{W}}$. Thus $\mathcal{PS}(M')$ is homeomorphic to $\mathcal{W}$. □

## 3. Irreducibliy in $\mathcal{PS}(M)$

A topological space $X$ is called *irreducible* if $X \neq \emptyset$ and every finite intersection of non-empty open sets of $X$ is non-empty. A subset $Y$ of a topological space $X$ is called *irreducible* if the subspace $Y$ of $X$ is irreducible. Equivalently, a subspace $Y$ of $X$ is irreducible if for every pair of closed subsets $C_1, C_2$ of $X$ with $Y \subseteq C_1 \cup C_2$, we have $Y \subseteq C_1$ or $Y \subseteq C_2$ (see, for example, [6, P. 94]).

Let $M$ be an $R$-module and $Y$ be a subset of $\mathcal{PS}(M)$. We will denote the closure of $Y$ in $\mathcal{PS}(M)$ by $cl(Y)$, and also the intersection of all elements of $Y$ by $\gamma(Y)$ (note that if $Y = \emptyset$, then $\gamma(Y) = M$).

**Proposition 3.1.** *Let $M$ be an $R$ module. Then for every $Q \in \mathcal{PS}(M)$, $\nu(Q)$ is irreducible.*

*Proof.* Since $\{Q\}$ is an irreducible subset of $\mathcal{PS}(M)$, $\overline{\{Q\}}$ is an irreducible subset of $\mathcal{PS}(M)$ by [3, page 13, Exercise 20]. Now, by [9, Corollary 2], $cl(\{Q\}) = \nu(Q)$. Therefore $\nu(Q)$ is an irreducible subset of $\mathcal{PS}(M)$. □

Let $C$ be a closed subset of a topological space $X$. An element $x \in C$ is called a *generic point* of $X$ if $C = cl(\{x\})$.

**Corollary 3.2.** *Let $M$ be an $R$-module and $Q, Q' \in \mathcal{PS}(M)$. If $\sqrt{(Q : M)} = \sqrt{(Q' : M)}$. Then $Q$ is a generic point for the irreducible closed subset $\nu(Q')$.*

*Proof.* First note that, by proposition 3.1, $\nu(Q')$ is an irreducible closed subset of $\mathcal{PS}(M)$. Also, by [9, Corollary 2], $\{\overline{Q}\} = \nu(Q) = \nu(Q')$. Thus $Q$ is a generic point of $\nu(Q')$. □

**Proposition 3.3.** *Let $M$ be an $R$-module and $Y \subseteq \mathcal{PS}(M)$. If $\gamma(Y)$ is a primary-like submodule of $M$, then $Y$ is irreducible in $\mathcal{PS}(M)$.*

*Proof.* Suppose that $\gamma(Y)$ is a primary-like submodule of $M$. Then by [9, Proposition3], $\nu(\gamma(Y)) = cl(Y)$. Hence $cl(Y)$ is irreducible by Proposition 3.1. Thus $Y$ is irreducible by [3, page 13, Exersise 20]. □

**Corollary 3.4.** *Let $M$ be an $R$-module and $Y \subseteq \mathcal{PS}(M)$. If $Y$ is linearly ordered by inclusion, then $Y$ and $\mathcal{PS}_p(M)$ are irreducible in $\mathcal{PS}(M)$ for every prime ideal $p$ of $R$.*

*Proof.* Since $\gamma(Y)$ and $\gamma(\mathcal{PS}_p(M))$ are primary-like submodule of $M$, we conclude that $Y$ and $\mathcal{PS}_p(M)$ are irreducible by Proposition 3.3. □

**Corollary 3.5.** *Let $m$ be a maximal ideal of $R$ and $M$ an $R$-module. Then $\mathcal{PS}_m(M)$ is an irreducible closed subset of $\mathcal{PS}(M)$.*

*Proof.* Let $m$ is a maximal ideal of $R$. Then $\mathcal{PS}_m(M)$ is irreducible by Corollary 3.4. Also, since

$$\begin{aligned}\nu(mM) &= \{Q \in \mathcal{PS}(M) \mid \sqrt{(Q : M)} \supseteq \sqrt{(mM : M)}\} \\ &= \{Q \in \mathcal{PS}(M) \mid \sqrt{(Q : M)} \supseteq m\} \\ &= \{Q \in \mathcal{PS}(M) \mid \sqrt{(Q : M)} = m\} \\ &= \mathcal{PS}_m(M),\end{aligned}$$

$\mathcal{PS}_m(M)$ is a closed subset of $\mathcal{PS}(M)$. □

**Proposition 3.6.** *Let $M$ be an $R$-module, $Y \subseteq \mathcal{PS}(M)$ and $p = \sqrt{(\gamma(Y):M)}$. If $p$ is a prime ideal of $R$ and $\mathcal{PS}_p(M) \neq \emptyset$, then $Y$ is irreducible in $\mathcal{PS}(M)$.*

*Proof.* Let $Q \in \mathcal{PS}_p(M)$. Since $p = \sqrt{(Q:M)} = \sqrt{(\gamma(Y):M)}$, we have $\nu(Q) = \nu(\gamma(Y)) = cl(Y)$ by [9, Proposition3]. Hence, $cl(Y)$ is irreducible and so $Y$ is irreducible by [3, page 13, Exersise 20]. □

**Proposition 3.7.** *Let $M$ be an $R$-module, $Y$ be an irreducible subset of $\mathcal{PS}(M)$ and $A = \{\sqrt{(Q:M)} : Q \in Y\}$. Then $A$ is an irreducible subset of Spec$(R)$, and thus $\gamma(A) = \sqrt{(\gamma(Y):M)}$ is a prime ideal of $R$.*

*Proof.* Suppose that $Y$ is irreducible. Since $\phi$ is continuous by Proposition 2.8, $\phi(Y) = Y'$ is an irreducible subset of Spec$(\bar{R})$, . Therefore, we have $\gamma(Y') = \overline{\sqrt{(\gamma(Y):M)}}$, and so $\gamma(Y')$ is a prime ideal of $\bar{R}$ by [6, page 129, Proposition 14]. Thus $\gamma(A) = \sqrt{(\gamma(Y):M)}$ for some subset $A$ of Spec$(R)$, and hence $\gamma(A)$ is a prime ideal of $R$. Thus by [6, page 129, Proposition 14], $A$ is an irreducible subset of Spec$(R)$. □

**Theorem 3.8.** *Let $M$ be an $R$-module and $\phi$ be surjective. If $Y \subseteq \mathcal{PS}(M)$, then $Y$ is an irreducible closed subset of $\mathcal{PS}(M)$ if and only if $Y = \nu(Q)$ for some $Q \in \mathcal{PS}(M)$. In particular, every irreducible closed subset of $\mathcal{PS}(M)$ has a generic point.*

*Proof.* By Proposition 3.1, for any $Q \in \mathcal{PS}(M)$, $\nu(Q)$ is an irreducible closed subset of $\mathcal{PS}(M)$. Conversely, if $Y$ is an irreducible closed subset of $\mathcal{PS}(M)$, then $Y = \nu(N)$ for some submodule $N$ of $M$ in which $\sqrt{(\gamma(\nu(N)):M)} = \sqrt{(\gamma(Y):M)} = p$, and this ideal is a prime ideal of $R$ by Proposition 3.7. Since $\phi$ is surjective, there exists a $p$-primary-like submodule $Q \in \mathcal{PS}(M)$ such that $\sqrt{(Q:M)} = p$. It follows that $p = \sqrt{(\gamma(\nu(N)):M)} = \sqrt{(Q:M)}$. Hence $\nu(\gamma(\nu(N))) = \nu(Q)$. Thus $Y = \nu(Q)$, by [9, Proposition3]. □

**Proposition 3.9.** *Let $M$ be an $R$ module and $\phi$ be surjective. Then*

*(1) The assignment $Q \mapsto \nu(Q)$ is a surjection from $\mathcal{PS}(M)$ to the set of all irreducible closed subsets of $\mathcal{PS}(M)$.*

*(2) The assignment $\nu(Q) \mapsto \overline{\sqrt{(Q:M)}}$ is a bijection from the set of all irreducible closed subsets of $\mathcal{PS}(M)$ to* Spec$(\bar{R})$.

*(3) The assignment $\nu(Q) \mapsto V(\psi^{-1}(\overline{\sqrt{(Q:M)}}))$ is a bijection from the set of all irreducible closed subsets of $\mathcal{PS}(M)$ to the set of all irreducible closed subsets of* Spec$(M)$.

*Proof.* (1) By Theorem 3.8.
(2) It is easy to see that the given assignment is well-defined and an injection. Suppose $\bar{p} \in$ Spec$(\bar{R})$. Since $\phi$ is surjective, $\bar{p} = \overline{\sqrt{(Q:M)}}$ for some $Q \in \mathcal{PS}(M)$. Thus the assighnment $\nu(Q) \mapsto \bar{p}$, and so the given assignment is a

surjection.
(3) First note that, for any $Q \in \mathcal{PS}(M)$, the closed subset $V(\psi^{-1}(\overline{\sqrt{(Q:M)}}))$ of $\mathrm{Spec}(M)$ is irreducible by [11, Theorem 5.7]. Thus the given assignment is well defined, since for $Q, Q' \in \mathcal{PS}(M)$, $\nu(Q) = \nu(Q')$ implies that $\sqrt{(Q:M)} = \sqrt{(Q':M)}$. Now, let $\psi^{-1}(\overline{\sqrt{(Q:M)}}) = P$ and $\psi^{-1}(\overline{\sqrt{(Q':M)}}) = P'$ for $Q, Q' \in \mathcal{PS}(M)$. If $V(P) = V(P')$, then $(P:M) = (P':M)$. Thus, since $\psi$ is surjective, $\sqrt{(Q:M)} = \sqrt{(Q':M)}$. It follows that $\nu(Q) = \nu(Q')$, and then the given assignment is injective. Next, for the surjectivity, let $Q \in \mathcal{PS}(M)$ and $V(\psi^{-1}(\overline{\sqrt{(Q:M)}})) = V(P)$. Thus $\nu(P)$ is mapped to $V(\psi^{-1}(\overline{\overline{\sqrt{(P:M)}}}))$ which is $V(P)$. Thus the given assignment is a bijection. □

**Theorem 3.10.** *Let $M$ be a finitely generated $R$-module. Then the following statements are equivalent.*

*(1)* $\mathrm{Spec}(M)$ *is an irreducible space;*

*(2)* $\mathcal{PS}(M)$ *is an irreducible space;*

*(3)* $\mathrm{Supp}(M)$ *is an irreducible space;*

*(4)* $\sqrt{Ann(M)}$ *is a prime ideal of $R$;*

*(5)* $\mathcal{PS}(M) = \nu(pM)$ *for some* $p \in \mathrm{Supp}(M)$*;*

*(6)* $\mathrm{Spec}(M) = V(pM)$ *for some* $p \in \mathrm{Supp}(M)$.

*Proof.* (1) ⇒ (2) Since $\mathrm{Spec}(M)$ is an irreducible space, by [11, Theorem 5.7], $\mathrm{Spec}(M) = V(P)$ for some $P \in \mathrm{Spec}(M)$. Let $\bar{p} = \psi(P)$. Since $\phi$ is surjective, there is an element $Q \in \mathcal{PS}(M)$ such that $\phi(Q) = \overline{\sqrt{(Q:M)}} = \bar{p}$. We show that $\mathcal{PS}(M) = \nu(Q)$. Suppose that $Q' \in \mathcal{PS}(M)$ and $p' = \sqrt{(Q':M)}$. Now, since $\rho(Q') \in V(P)$, we have

$$\overline{\sqrt{(Q':M)}} = \overline{p'} = \overline{(S_{p'}(Q'+p'M):M)} = \overline{(\rho(Q'):M)}$$
$$\supseteq \overline{(P:M)} = \bar{p} = \overline{\sqrt{(Q:M)}},$$

which follows that $\sqrt{(Q':M)} \supseteq \sqrt{(Q:M)}$. Thus $Q' \in \nu(Q)$, and therefore $\mathcal{PS}(M) = \nu(Q')$, i.e., $\mathcal{PS}(M)$ is an irreducible space by Theorem 3.8.
(2) ⇒ (3) Since $\phi$ is a surjective continuous map, $\mathrm{Spec}(\bar{R})$ is irreducible by the assumption. Now since by [3, page 13, Ex. 21], $\mathrm{Spec}(\bar{R})$ and $V(Ann(M))$ are homeomorphic and by [12, Proposition 3.4] $\mathrm{Supp}(M) = V(Ann(M))$, we conclude that $\mathrm{Supp}(M)$ is an irreducible space.
(3) ⇒ (4) By [6, page 102, Proposition 14], $\gamma(\mathrm{Supp}(M))$ is a prime ideal of $R$. Now since $\gamma(\mathrm{Supp}(M)) = \gamma(V(Ann(M))) = \sqrt{Ann(M)}$, we are done.
(4) ⇒ (5) First note that $\sqrt{Ann(M)} \subseteq \sqrt{(Q:M)}$, for each $Q \in \mathcal{PS}(M)$. Since $M$ is finitely generated, by [12, Proposition 3.8], there exists $P \in \mathrm{Spec}(M)$ such

that $(P:M)=\sqrt{(P:M)}=\sqrt{Ann(M)}$. Therefore

$$\begin{aligned}\mathcal{PS}(M)&=\{Q\in\mathcal{PS}(M)\mid\sqrt{(Q:M)}\supseteq\sqrt{(P:M)}\}\\&=\nu(P)=\nu(\sqrt{(P:M)}M)=\nu(\sqrt{Ann(M)}M).\end{aligned}$$

Thus $\mathcal{PS}(M)=\nu(pM)$ in which $p=\sqrt{Ann(M)}$ and $p\in\mathrm{Supp}(M)$.
(5) $\Rightarrow$ (6) Let $P\in\mathrm{Spec}(M)$. Since $\phi$ is surjective, there is an element $Q\in\mathcal{PS}(M)$ such that $\phi(Q)=\overline{(P:M)}$, and so $\overline{\sqrt{(Q:M)}}=\overline{(P:M)}$. Now, since $\mathcal{PS}(M)=\nu(pM)$ for some $p\in\mathrm{Supp}(M)$, we have

$$(pM:M)\subseteq\sqrt{(pM:M)}\subseteq\sqrt{(Q:M)}=(P:M).$$

Thus $P\in V(pM)$ so that $\mathrm{Spec}(M)=V(pM)$ for some $p\in\mathrm{Supp}(M)$.
(6) $\Rightarrow$ (1) Let $\mathrm{Spec}(M)=V(pM)$ for some $p\in\mathrm{Supp}(M)$. Since $\psi$ is surjective, there exists $P\in X$ such that $(P:M)=p$. Hence by [11, Result 3], we have $\mathrm{Spec}(M)=V(pM)=V((P:M)M)=V(P)$. Thus, by [11, Theorem 5.7], $\mathrm{Spec}(M)$ is irreducible. □

## 4. $\mathcal{PS}(M)$ as a spectral space

A topological space $X$ is a $T_0$*-space* if and only if for any two distinct points in $X$ there exists an open subset of $X$ which contains one of the points but not the other. It is well-known that, for any ring $R$, $Spec(R)$ is a $T_0$-space for the Zariski topology. In [11, page 429], it has been shown that if $M$ is a vector space, then $\mathrm{Spec}(M)$ is not a $T_0$-space. This example can be used again to show that $\mathcal{PS}(M)$ is not also a $T_0$ space. In fact, if $M$ is a vector space, then $\nu(N)=\mathcal{PS}(M)$ for every proper subspace $N$ of $M$ so that $\mathcal{PS}(M)$ has the trivial topology.

**Proposition 4.1.** *Let $M$ be a multiplication $R$-module. If for every $Q\in\mathcal{PS}(M)$ the ideal $(Q:M)$ is a radical ideal, then $\mathcal{PS}(M)$ is a $T_0$-space.*

*Proof.* Let $Q\in\mathcal{PS}(M)$ and $(Q:M)$ is a radical ideal of $R$. Then $Q$ is a prime submodule of $M$, and so $\mathcal{PS}(M)$ is a topological subspace of $\mathrm{Spec}(M)$. Thus $\mathcal{PS}(M)$ is a $T_0$-space by [11, Corollary 6.2]. □

**Theorem 4.2.** *Let $M$ be an $R$-module. Then the following statements are equivalent.*

*(1) $\mathcal{PS}(M)$ is a $T_0$-space.*

*(2) If $\nu(Q)=\nu(Q')$ for $Q,Q'\in\mathcal{PS}(M)$, then $Q=Q'$.*

*Proof.* (1) Let $Q\neq Q'$ for some $Q,Q'\in\mathcal{PS}(M)$. Since $\mathcal{PS}(M)$ is a $T_0$-space, $cl(Q)\neq clQ'$. Thus by [9, Corollary 2], we have $\nu(Q)\neq\nu(Q')$.
(2) Let $Q\neq Q'$ for some $Q,Q'\in\mathcal{PS}(M)$. Then by the assumption $\nu(Q)\neq\nu(Q')$. Therefore, by [9, Corollary 2], $cl(Q)\neq cl(Q')$. Hence $\mathcal{PS}(M)$ is a $T_0$-space. □

Recall that a *spectral space* is a topological space homeomorphic to the prime spectrum of a ring equipped with the Zariski topology. By Hochster's characterization [10], the topological space $X$ is spectral if and only if the following statements hold:

(1) $X$ is a $T_0$-space.

(2) $X$ is quasi-compact.

(3) the quasi-compact open subsets of $X$ are closed under finite intersection and form an open base.

(4) each irreducible closed subset of $X$ has a generic point.

For any ring $R$, $Spec(R)$ is well-known to satisfy these condition (see [6, P. 401-403]).

**Theorem 4.3.** *Let $M$ be a finitely generated multiplication $R$-module. Then $\mathcal{PS}(M)$ is a spectral space.*

*Proof.* By [9, Theorem3], Theorem 3.8, [9, Corollary3] and [11, Corollary 6.2]. □

**Theorem 4.4.** *Let $M$ be an $R$-module. Consider the following statements:*

*(1) $\mathcal{PS}(M)$ is a spectral space;*

*(2) $\mathcal{PS}(M)$ is a $T_0$-space;*

*(3) If $\nu(Q) = \nu(Q')$ for $Q, Q' \in \mathcal{PS}(M)$, then $Q = Q'$;*

*(4) $|\mathcal{PS}_p(M)| \leq 1$ for every $p \in Spec(R)$;*

*(5) $\phi$ is injective.*

*Then* $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5)$.
*Moreover if $\phi$ is surjective , then* $(5) \Rightarrow (1)$.

*Proof.* (1)⇒(2) Clear.
(2)⇒(3) By Theorem 4.2.
(3) ⇔ (4) ⇔ (5) By Proposition 2.1.
(5) ⇒ (1) By[9, Theorem 1], $\mathcal{PS}(M)$ is homeomorphic to the spectral space $\mathrm{Spec}(\bar{R})$. Thus $\mathcal{PS}(M)$ is a spectral space. □

**Proposition 4.5.** *Let $M$ be an $R$-module and $|\mathcal{PS}_p(M)| = 1$ for every $p \in Spec(R)$. Then $\mathcal{PS}(M)$ is a spectral space.*

*Proof.* Since $\mathrm{Spec}(\bar{R})$ is a spectral space, $\mathcal{PS}(M)$ is also a spectral space by Corollary 2.5 and Corollary 2.10. □

**Theorem 4.6.** *Let $M$ be a multiplication $R$-module such that $\mathcal{PS}(M)$ is a non-empty finite set. Then $\mathcal{PS}(M)$ is a spectral space if and only if $|\mathcal{PS}_p(M)| \leq 1$ for every $p \in Spec(R)$.*

*Proof.* Since $\mathcal{PS}(M)$ is a non-empty finite set, then the conditions (2) and (3) in Hochster's characterization are clearly true. Suppose $Y = \{y_1, y_2, ..., y_n\}$ is an irreducible closed subset of $\mathcal{PS}(M)$. Thus $Y = \overline{\{y_i\}}$ for some $i$ where $1 \leq i \leq n$, i.e., $Y$ has a generic point. Hence, by [9, Theorem 4.3], $\mathcal{PS}(M)$ is a spectral space if and only if $\mathcal{PS}(M)$ is a $T_0$-space if and only if $|\mathcal{PS}_p(M)| \leq 1$ for every $p \in Spec(R)$. □

**Theorem 4.7.** *Let $M$ be an $R$-module and $Im(\phi)$ be a closed subset of* $\mathrm{Spec}(\bar{R})$. *Then $\mathcal{PS}(M)$ is a spectral space if and only if $\phi$ is injective.*

*Proof.* Let $Y = Im(\phi)$ be a closed subset of $\mathrm{Spec}(\bar{R})$. Then $Y$ is a spectral subspace of $\mathrm{Spec}(\bar{R})$. Assume that $\phi$ is injective. Then the bijection $\phi : \mathcal{PS}(M) \to Y$ is continuous by Proposition 2.8. We show that $\phi$ is a closed map. Let $N$ be a submodule of $M$, and $Y' = Y \cap V(\overline{\sqrt{(N : M)}})$. Then $Y'$ is a closed subset of $Y$, and so by Proposition 2.8 we have

$$\phi^{-1}(Y') = \phi^{-1}(Y) \cap \phi^{-1}(V\overline{\sqrt{(N : M)}}) = \nu(\sqrt{(N : M)}M) = \nu(N).$$

Hence $\phi(\nu(N)) = \phi(\phi^{-1}(Y')) = Y'$ is a closed subset of $Y$. Thus $\phi : \mathcal{PS}(M) \to Y$ is a homeomorphism and so $\mathcal{PS}(M)$ is a spectral space. Conversely, assume $\mathcal{PS}(M)$ is a spectral space. Hence by Theorem 4.4, $\phi$ is injective. □

*Comment* 4.8.

*Example* 4.9. (1) Every proper submodule of $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is primary-like. However $Spec_L(\mathbb{Z}(p^\infty)) = Spec(\mathbb{Z}(p^\infty)) = \emptyset$.

(2) For $\mathbb{Z}$-module $\mathbb{Q}$, $Spec(\mathbb{Q}) = \{0\}$ and $Spec_L(\mathbb{Q}) = \emptyset$, because $\mathbb{Q}$ have no submodules satisfying the primeful property.

(3) For a vector space $V$ over a field $F$, $Spec_L(V) = Spec(V)$= the set of all proper vector subspaces of $V$.

(4) Let $M = \mathbb{Q} \oplus \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the cyclic group of order $p$. Then $\mathrm{Spec}(M) = \{\mathbb{Q} \oplus 0, 0 \oplus \mathbb{Z}_p\}$ [13, Example 2.6]. Although $\{0 \oplus 0, \mathbb{Q} \oplus 0, 0 \oplus \mathbb{Z}_p\} \cup \{N : N \nsubseteq \mathbb{Q} \oplus 0,\ N \nsubseteq 0 \oplus \mathbb{Z}_p\}$ is the set of all primary like submodules of $M$. However $\mathcal{PS}(M) = \emptyset$.

(5) Let $M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p$. Then $M$ is not multiplication $\mathbb{Z}$-module [13, Example 3.7]. $\mathrm{Spec}(M) = pM = \mathbb{Z}(p^\infty) \oplus 0$. By an easy verification $\{< 1/p^i + \mathbb{Z} > \oplus 0 :\ i \in \mathbb{Z}\} \cup \{\mathbb{Z}(p^\infty) \oplus 0, 0 \oplus \mathbb{Z}_p\} \cup \{N : N \nsubseteq \mathbb{Z}(p^\infty) \oplus 0\}$ is the set of all primary-like submodules of $M$. But $N = 0 \oplus \mathbb{Z}_p \notin \mathcal{PS}(M)$. Also $\mathrm{rad}\, N = M$ and $(N : M) = 0$. Thus for the primary-like submodule $N$, $\sqrt{(N : M)} \subsetneq (\mathrm{rad}\, N : M)$.

(6) Let $M = \prod_{p \in \Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$ and $M' = \bigoplus_{p \in \Omega} \frac{\mathbb{Z}}{p\mathbb{Z}}$ be $\mathbb{Z}$-modules where $\Omega$ is the set of prime integers. Then $M'$ is a 0-prime submodule of $M$ which dose not satisfy the primeful property. In fact $\mathrm{Spec}(M) = \{M'\} \cup \{pM : p \in \Omega\}$. However $(\mathrm{rad}\, M' : M) = \sqrt{(M' : M)} = 0$ [9, Example 2.12].

It is easy to see that the zero submodule 0 satisfies the primeful property and $\operatorname{rad} 0 = 0$. But 0 is not a primary-like submodule of $M$, because 0 is not a prime submodule of $M$.

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