Measures on Suslinean spaces

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Suslin's Problem

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A Boolean algebra is called *Suslinean* if it is ccc and not σ-centered.

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There is no Suslinean linearly ordered space. (There is no Suslinean interval algebra)

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- $\diamondsuit \implies \neg$SH [Jensen, 1972]
- MA(ω_1) $\implies$ SH [Solovay, Tennenbaum, 1971]

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Theorem [Juhasz, Sapirovsky, Tall 70's]

- $MA(\omega_1) \implies$ no Suslinean space of π-weight ω_1;
- $MA(\omega_1) \implies$ no Suslinean first countable space;
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Small spaces

Definition

A compact space is *small* if it cannot be mapped continuously onto $[0,1]^{\omega_1}$. A Boolean algebra is *small* if it does not contain an uncountable independent family.

Examples

linearly ordered spaces, first-countable spaces, spaces of countable tightness, metrizable spaces, scattered spaces, Corson compacta, monotonically normal spaces, HS, HL, ...

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Small spaces in Suslinean context

Fact

Let $\kappa > \mathfrak{c}$. The space 2^κ is Suslinean.

The ultimate version of Suslin's hypothesis

USH = "there is no small Suslinean space".

Theorem [Todorčević 2000]

$\neg$USH.

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Is there a small Suslinean space supporting a measure?

Definition

A compact space supports a measure if there is a (Radon) measure μ such that each nonempty open subset is μ-positive. A Boolean algebra supports a measure if there is a finitely additive measure μ such that each nonzero element is μ-positive.

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Definition

A Boolean algebra $\mathfrak{A}$ is *minimally generated* if $\mathfrak{A} = \bigcup \mathfrak{A}_\alpha$, where

- $\mathfrak{A}_0$ is trivial,
- $\mathfrak{A}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$ for limit γ,
- there is no $\mathfrak{B}$ such that $\mathfrak{A}_\alpha \subsetneq \mathfrak{B} \subsetneq \mathfrak{A}_{\alpha+1}$.

Examples of Stone spaces of m.g. algebras (colored)

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Uniformly regular measures

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A measure μ on a Boolean algebra $\mathfrak{A}$ is *uniformly regular* if there is a countable family $\mathcal{B} \subseteq \mathfrak{A}$ such that

$$\mu(A) = \sup\{\mu(B) \colon B \in \mathcal{B}, B \subseteq A\}$$

for each $A \in \mathfrak{A}$.

Remark

If $\mathfrak{A}$ supports a uniformly regular measure, then it has a countable π-base (and, consequently, is σ-centered).

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Theorem [PBN]

Suslin Hypothesis is equivalent to "every ccc minimally generated Boolean algebra supports a uniformly regular measure".

Proof of " $\Longrightarrow$ " (sketch):

- Assume $\mathfrak{A}$ is minimally generated;
- $\mathfrak{A}$ contains a dense tree algebra $\mathfrak{T}$ such that $\mathfrak{A}$ is minimally generated over $\mathfrak{T}$;
- If $\mathfrak{T}$ is ccc and SH holds, then $\mathfrak{T}$ is countable;
- One can define a measure ν on $\mathfrak{T}$;
- Since $\mathfrak{A}$ is m.g. over $\mathfrak{T}$, ν can be extended to μ on $\mathfrak{A}$ such that $\mathfrak{T}$ approximates μ from below.

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Fact

If $\mathfrak{A}$ is minimally generated and supports a measure, then $\mathfrak{A}$ supports a uniformly regular measure (and, consequently, is σ-centered).

Proof:

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- $\mathfrak{A}$ contains a dense tree algebra $\mathfrak{T}$ such that $\mathfrak{A}$ is minimally generated over $\mathfrak{T}$;
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Kunen's space

Theorem [Kunen 1981]

There is a Corson compact space supporting a non-separable measure

Definition

A measure μ is separable if there is a countable family $\mathcal{A}$ of measurable sets such that $\inf\{\mu(A \triangle E) \colon A \in \mathcal{A}\} = 0$ for every measurable E.

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The standard measure on 2^{ω_1} is non-separable.

Fact

If K is big, then it carries a non-separable measure.

Theorem [Fremlin 1997]

$\mathrm{MA}(\omega_1)$ implies that small spaces carry only separable measures.

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The characterization

Theorem [Kunen, van Mill 1995]

The following are equivalent

- $\neg \mathrm{MA}_{\text{measure algebras}}(\omega_1)$,
- 2^{ω_1} can be covered by ω_1 nullsets,
- ω_1 is not a precaliber of measure algebras,

Definition

ω_1 is a precaliber for $\mathfrak{A}$ if every uncountable family of elements of $\mathfrak{A}$ has an uncountable centered subfamily.

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Definition

K is Corson compact if it is homeomorphic to a compact subset of $\Sigma([0,1]^\kappa)$ for some κ, where

$$\Sigma([0,1]^\kappa) = \{x \in [0,1]^\kappa \colon |\mathrm{supp}(x)| = \aleph_0\}.$$

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- there is a Suslinean Corson compactum supporting a measure.

Fact

If a Corson compactum is separable, then it is metrizable. There is no non-separable measure on a metrizable space.

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Suslinean Corson compactum

Theorem [Džamonja, Plebanek 2004]

Assume ω_1 is not a precaliber of measure algebras. Then there is a Corson compact non-separable space supporting a measure.

Proof.

- Let $(A_\alpha)_{\alpha<\omega_1}$ be a witness for $\neg$ precaliber$_{ma}(\omega_1)$;
- Let $X = \{x \in 2^{\omega_1} : \{A_\alpha : x(\alpha) = 1\}$ is centered$\}$;
- X is Corson compact;
- X is not separable;
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- X is not separable;
- X supports a measure.

General framework

$\mathrm{cov}(\mathcal{N}_{\omega_1}) = \omega_1$ implies that there is a Corson compact space supporting a non-separable measure.

- Assume $(N_\alpha)_{\alpha<\omega_1} \subseteq \mathcal{N}_{\omega_1}$ is increasing and covers 2^{ω_1};
- Let $F_\alpha \subseteq 2^{\omega_1} \setminus \bigcup_{\xi<\alpha} N_\xi$ be closed such that $\inf\{\lambda(F_\alpha \triangle A)\colon A \in \mathfrak{A}_\alpha\} = 0$, where $\mathfrak{A}_\alpha$ is the algebra generated by $\{F_\xi \colon \xi < \alpha\}$;
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Non-separable measure without $MA_{ma}(\omega_1)$

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Theorem [Plebanek 1997]

The assumption on covering can be relaxed to "there is a family $(N_\alpha)_{\alpha<\omega_1} \subseteq \mathcal{N}_{\omega_1}$ whose union intersects each perfect subset of 2^{ω_1}". Consequently, it is consistent that $MA_{ma}(\omega_1)$ holds and there is a small space with a non-separable measure.

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Theorem [Kamburelis 1989]

Assume $\mathrm{cov}(\mathcal{N}_{\omega_1}) > \omega_1$. Let $\mathfrak{A}$ be a Boolean algebra supporting a measure and let $|\mathfrak{A}| = \omega_1$. Then $\mathfrak{A}$ is σ-centered.

Proof.

- WLOG $\mathfrak{A} \subseteq \mathrm{Borel}(2^{\omega_1})/\mathcal{N}_{\omega_1}$;
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Problem

Is it consistent that there is no small Suslinean space supporting a measure?

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A sample construction of small Suslinean space

Theorem [essentially Bell 1996]

In a model obtained by adding a single Cohen real there is a small Suslinean space.

Total ideal spaces in algebraic language

For $A, B \subseteq \omega$ such that $A \cap B = \emptyset$ let

$$\rho(A, B) = \{x \in 2^\omega : n \in A \implies x(n) = 0, n \in B \implies x(n) = 1\}.$$

Having $(A_\alpha)_{\alpha<\kappa}$, $(B_\alpha)_{\alpha<\kappa}$ we can define

$$\mathfrak{A} = \mathrm{alg}\{\rho(A_\alpha, B_\alpha) : \alpha < \kappa\}$$

and then consider the Stone space. . .

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Gap

Definition

A sequence $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is a *gap* if for each $\alpha < \beta < \omega_1$

- L_α, $R_\alpha \subseteq \omega$;
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Destructible gap from a Cohen real

Notation

A gap $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ has property $(\mathfrak{s})$ if for each uncountable $X \subseteq \omega_1$ there are $\alpha < \beta \in X$ such that

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In a model obtained by adding a single Cohen real there is a gap with property $(\mathfrak{s})$.

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Proof

- Let $(L'_\alpha, R'_\alpha)_{\alpha<\omega_1}$ be a gap from the ground-model.
- if $c \subseteq \omega$ is a Cohen real, $(L'_\alpha \cap c, R'_\alpha \cap c)$ will have the desired property.

Destructible gap

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- if $c \subseteq \omega$ is a Cohen real, $(L'_\alpha \cap c, R'_\alpha \cap c)$ will have the desired property.

Destructible gap

Theorem [Todorčević]

In a model obtained by adding a single Cohen real there is a gap with property $(\mathfrak{s})$.

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Small Suslinean space

Theorem [ess. Bell 1996]

Let $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ be a gap with property $(\mathfrak{s})$. Let $\mathfrak{A}$ be the Boolean algebra generated by $\mathcal{G} = \{\rho(L, R) \colon L =^* L_\alpha, R =^* R_\alpha, \alpha < \omega_1\}$. Then $\mathfrak{A}$ is ccc, not σ-centered and all measures on $\mathfrak{A}$ are separable.

Proof of ccc (sketch):

- $\mathcal{G}$ forms a π-base for $\mathfrak{A}$.
- Consider $\mathcal{B} = \{\rho(L_\alpha, R_\alpha) \colon \alpha \in X\}$ for an uncountable X.
- Because of property $(\mathfrak{s})$ there are $\alpha < \beta \in X$ such that $\rho(L_\beta, R_\beta) \subseteq \rho(L_\alpha, R_\alpha)$. So $\mathcal{B}$ is not an antichain.

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- Assume $\mathfrak{A}$ carries a non-separable measure.
- WLOG there is $\varepsilon > 0$ and an uncountable X such that $\mu(\rho(L_\alpha, R_\alpha) \triangle \rho(L_\beta, R_\beta)) > \varepsilon$ for each $\alpha \neq \beta \in X$.
- WLOG there is r such that for all $\alpha \in X$ we have $|\mu(\rho(L_\alpha, R_\alpha)) - r| < \varepsilon/2$.
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Reformulation of the main problem

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Is it consistent that the following dichotomy holds: each subalgebra of $\mathrm{Borel}(2^\omega)/\mathcal{N}$ is either σ-centered or big?

Remark

There is a small subalgebra of $\mathrm{Borel}(2^\omega)/\mathcal{N}$ without a countable π-base.

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Thank you for your attention