Generic Extensions of Models of **ZFC**

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Notations and Terminology

M is a model if $\langle M, \in\rangle$ is a transitive model of **ZFC** , M is either countable set, or a class.

M_2 is an extension of M_1 if $M_1 \subseteq M_2$ are models with same ordinals $\mathrm{On}^{M_1} = \mathrm{On}^{M_2}$.

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M_2 is an extension of M_1, $a \in M_2$, $a \subseteq M_1$ (or $a \subseteq On$). Then $M_1[a]$ is the smallest model of **ZFC** such that $M_1 \subseteq M_1[a]$ and $a \in M_1[a]$. Note that for $a, b \subseteq M_1$, $a, b \in M_2$ we have $M_1[a][b] = M_1[b][a]$.

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Let κ be an uncountable regular cardinal of M_1.

M_2 is a κ-generic extension of M_1 if there exists a poset $P \in M_1$, $|P| \leq \kappa$ and an M_1-generic ultrafilter G on P such that $M_2 = M_1[G]$.

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M_2 is a κ-C.C. generic extension of M_1 if there exists a κ-C.C. (every antichain has cardinality smaller than κ) Boolean algebra $B \in M_1$, complete in M_1 and an M_1-generic ultrafilter G on B such that $M_2 = M_1[G]$.

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$Bd_{M_1,M_2}(\kappa)$ says that

$$(\forall u \subseteq On, u \in M_2)(\exists y \in M_2)(\exists a \in M_1)$$
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$Apr_{M_1,M_2}(\kappa)$ says that

$$(\forall f \in M_2, f \text{ a function}, \mathrm{dom}(f) \in M_1, \mathrm{rng}(f) \subseteq M_1)(\exists g \in M_1,$$
$$\mathrm{dom}(g) = \mathrm{dom}(f))(\forall x \in \mathrm{dom}(f))\,(f(x) \in g(x) \wedge |g(x)|^{M_1} < \kappa).$$

Main Results

Theorem 1 (essentially P. Vopěnka)

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In [B2], we have proved the following

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Recently, another proof of Theorem 2 was given by S.D. Friedman, S. Fuchino and H. Sakai [FFS]. We present the idea of a proof of Theorem 2 that is different from those of [B2] and [FFS].

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$$a = \{\{\xi : \langle t, \xi \rangle \in r\} : t \in P\}, \quad y = \{\{\xi : \langle x, \xi \rangle \in r\} : x \in G\}.$$

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If $M_2 = M_1[G]$, where G is a generic ultrafilter on a M_1-complete κ-C.C. Boolean algebra $B \in M_1$, then for every $f : \alpha \longrightarrow \beta$, there exists a function $h : \alpha \times \beta \longrightarrow B$, $h \in M_1$ such that $f = h^{-1}(G)$.

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If M_2 is a generic extension of M_1 and $Apr_{M_1,M_2}(\kappa)$ holds true, then M_2 is a κ-C.C. generic extension of M_1

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Lemma 4

If $Apr_{M_1,M_2}(\kappa)$ holds true and $\mathcal{P}(\kappa) \cap M_2 \subseteq M_1$, then $M_1 = M_2$.

The assertion of the lemma is same as that of Theorem 4.1 of [B2].

Main Lemma

The basic result is contained in

Lemma 5 (Main Lemma)

If $Apr_{M_1,M_2}(\kappa)$ holds true then for any set $a \in M_2$, $a \subseteq M_1$, the model $M_1[a]$ is a generic extension of M_1.

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The proofs of this lemma in [B2] and [FFS] are different. We present still another proof of this lemma.
Independently J.L. Krivine found similar proof of a weaker result.

Support

A set $\sigma \subseteq M_1$, $\sigma \in M_2$ is a **support** if for any relations $r_1, r_2 \in M_1$ there exists a relation $r \in M_1$ such that $r''\sigma = r_1''\sigma \setminus r_2''\sigma$, where $r''\sigma = \{u : (\exists v \in \sigma)\,[v,u] \in r\}$.

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If $M_2 = M_1[G]$, where G is a generic ultrafilter on a poset, then G is a support. Actually, for every $y \subseteq M_1$, $y \in M_1[G]$, there exists a relation $r \in M_1$ such that $y = r''G$. If G is a generic ultrafilter on a Boolean algebra, then for any $x \in M_2 \cap \mathcal{P}(M_1)$ even $x = f^{-1}(G)$, $f \in M_1$, where f is a function.

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If $\sigma \subseteq M_1$, $\sigma \in M_2$ is a support, then $M_1[\sigma]$ is a generic extension of M_1.

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Nice simple proof was given by B. Balcar [Ba].

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If G is an M_1 generic ultrafilter on a complete Boolean algebra B, we let

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(i) r is a symmetric antireflexive relation.

(ii) $r''\{x\} \subseteq B \setminus G$ for any $x \in G$.

(iii) For any $u \subseteq B \setminus G$, $u \in M_1$, there exists an $x \in G$ such that $u \subseteq r''\{x\}$.

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Assume that $\sigma \subseteq a \in M_1$ is a support. Set

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Then

$$\mathcal{P}(a) \cap M_1 = r_1''\sigma \text{ and } (\mathcal{P}(a) \setminus \mathcal{P}(a \setminus \sigma)) \cap M_1 = r_2''\sigma.$$

Support

Since σ is a support, there exists a relation $r_3 \in M_1$ such that

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It is easy to show that σ is an M_1-generic ultrafilter on $\langle a, \leq\rangle$. Thus $M_1[\sigma]$ is a generic extension of M_1.

Main Lemma

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Proof. Let $\mathcal{B}$ denote the family of Borel subsets of ${}^{\omega_0}2$. There exist a mapping $\# : \mathcal{B}^{M_1} \longrightarrow \mathcal{B}^{M_2}$ preserving complement and unions of countable families belonging to M_1 – see R. Solovay [So].

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If $a \in {}^{\omega_0}2 \cap M_2$ we set

$$j(a) = \{A \in \mathcal{B}^{M_1} : a \in \#(A)\}.$$

Evidently $M_1[a] = M_1[j(a)]$. We show that $j(a)$ is a support.

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Then there exists a set $\{a_\eta : \eta \in \lambda\} \in M_1$ of pairwise disjoint sets such that

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We let $f(\eta)$ to be the smallest ξ such that $u_\xi \in j(a) \cap v_\eta$ if $j(a) \cap v_\eta \neq \emptyset$ and $f(\eta) = 0$ otherwise. By the assumptions, there exists a function $g \in M_1$, $\mathrm{dom}(g) = \lambda$ and such that for each $\xi \in \lambda$ we have $f(\xi) \in g(\xi)$ and $|g(\xi)|^{M_1} \leq \aleph_0$.

For $t \in a_\eta$ we set $h(t) = \bigcup\{u_\xi : u_\xi \in v_\eta \wedge \xi \in g(\eta)\}$. Evidently $h \in M_1$ and $\mathrm{rng}(h) \subseteq \mathcal{B}^{M_1}$.

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$$h(t) = \begin{cases} h_1(t) \setminus h_2(t) & \text{if } t \in \mathrm{dom}(h_1) \cap \mathrm{dom}(h_2), \\ h_1(t) & \text{if } t \in \mathrm{dom}(h_1) \setminus \mathrm{dom}(h_2). \end{cases}$$

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The theorem follows by Theorem 6. □

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Since $Apr_{M_1,M_1[a]}(\lambda)$, every set from $([\beta]^{\leq\lambda})^{M_1[a]}$ is a subset of a set from $([\beta]^{\leq\lambda})^{M_1}$.
So, we may assume that all values of g are in $([\beta]^{\leq\lambda})^{M_1}$. Now, by $Apr_{M_1,M_1[a]}(\lambda)$ there exists a function $h : \alpha \longrightarrow [([\beta]^{\leq\lambda})^{M_1}]^{<\lambda}$ such that $g(\xi) \in h(\xi)$ for each $\xi \in \alpha$.

A Note

Set $d(\xi) = \bigcup h(\xi)$. Then $d \in M_1$ and $f(\xi) \in d(\xi)$ for each $\xi \in \alpha$. Since $|d(\xi)|^{M_1} \leq \lambda$ we have $|d(\xi)|^{M[G]} \leq \aleph_0$.

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A proof of Lemma 4 is based on the following

Lemma 8

If B is a complete atomless κ-C.C. Boolean algebra, then the first cardinal λ such that B is not $(\lambda, \kappa))$-distributive is $\lambda \leq \kappa$.

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A complete ω_1-C.C. $(\aleph_0, \aleph_0)$-distributive $(\aleph_1, \aleph_0)$-non-distributive Boolean algebra produces a Souslin tree. Therefore

Corollary 9

If $\mathcal{P}(\omega_0) \cap M_2 \subseteq M_1$, $\mathcal{P}(\omega_1) \cap M_2 \not\subseteq M_1$ and $Apr_{M_1,M_2}(\omega_1)$ holds true, then there exists a Souslin continuum in M_1.

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The proof of Lemma 5 in [B2] is based on an embedding of the free κ-complete Boolean algebra with λ generators constructed in M_1 preserving $<\kappa$ unions of sets from M_1 into the similar Boolean algebra constructed in M_2.

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The proof of Lemma 5 in [B2] is based on an embedding of the free κ-complete Boolean algebra with λ generators constructed in M_1 preserving $<\kappa$ unions of sets from M_1 into the similar Boolean algebra constructed in M_2.
The presented proof reduced this problem to the $\aleph_1$-free Boolean algebra with $\aleph_0$ generators $\mathcal{B}$.

Thanks for attention

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