Van der Waerden spaces and their relatives

Jana Flašková

Department of Mathematics
University of West Bohemia in Pilsen

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Ψ-spaces

For a given maximal almost disjoint (MAD) family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ we define the space $\Psi(\mathcal{A})$ as follows:

- The underlying set is $\mathbb{N} \cup \{p_A : A \in \mathcal{A}\}$.
- Every point in $\mathbb{N}$ is isolated.
- Every point p_A has neighborhood base of all sets $\{p_A\} \cup A \setminus K$ where K is a finite subset of A.

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Note: $\Psi(\mathcal{A})$ is regular, first countable and separable.

Sequentially compact spaces

All topological spaces are Hausdorff.

Definition.

A topological space X is called sequentially compact if for every sequence $\langle x_n \rangle_{n\in\omega}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k\in\omega}$.

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Is it possible to choose the subsequence in such a way that the set of indices is "large"?

AP-sets and van der Waerden spaces

$A \subseteq \mathbb{N}$ is an AP-set if A contains arithmetic progressions of arbitrary length.

- (van der Waerden theorem)
  Sets that are not AP-sets form an ideal
- van der Waerden ideal is an F_σ-ideal

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  Sets that are not AP-sets form an ideal
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Definition A. (Kojman)

A topological space X is called van der Waerden if for every sequence $\langle x_n \rangle_{n\in\omega}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k\in\omega}$ so that $\{n_k : k \in \omega\}$ is an AP-set.

Sequentially compact $\neq$ van der Waerden

Every van der Waerden space is sequentially compact.

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Theorem (Kojman)

There exists a compact, sequentially compact, separable space which is first-countable at all points but one, which is not van der Waerden.

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Theorem (Kojman)

There exists a compact, sequentially compact, separable space which is first-countable at all points but one, which is not van der Waerden.

Proof. Consider the one-point compactification of $\Psi(\mathcal{A})$ for a suitable MAD family $\mathcal{A}$.

Van der Waerden spaces – a sufficient condition

Theorem (Kojman)

If a Hausdorff space X satisfies the following condition

$(*)$ The closure of every countable set in X is compact and first-countable.

Then X is van der Waerden.

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For example, compact metric spaces or every succesor ordinal with the order topology satisfy $(*)$.

$\mathcal{I}_{1/n}$-spaces

$$\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{a \in A} \frac{1}{a} < \infty\}$$

The summable ideal $\mathcal{I}_{1/n}$ is an F_σ-ideal and P-ideal.

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Definition B.

A topological space X is called $\mathcal{I}_{1/n}$-space if for every sequence $\langle x_n \rangle_{n \in \omega}$ in X there exists a converging subsequence $\langle x_{n_k} \rangle_{k \in \omega}$ so that $\{n_k : k \in \omega\}$ does not belong to $\mathcal{I}_{1/n}$.

Sequentially compact $\neq \mathcal{I}_{1/n}$-space

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Theorem 2.

If a Hausdorff space X satisfies the following condition

$(*)$ The closure of every countable set in X is compact and first-countable.

Then X is a $\mathcal{I}_{1/n}$-space.

$\mathcal{I}_{1/n}$-spaces – a sufficient condition

Theorem 2.

If a Hausdorff space X satisfies the following condition

(∗) The closure of every countable set in X is compact and first-countable.

Then X is a $\mathcal{I}_{1/n}$-space.

Theorems 1. and 2. remain true if the summable ideal is replaced by an arbitrary tall F_σ-ideal on ω which contains all finite sets.

$\mathcal{I}_{1/n}$ vs van der Waerden spaces

Erdős-Turán Conjecture.

Every set $A \notin \mathcal{I}_{1/n}$ is an AP-set.

If Erdős-Turán Conjecture is true then every $\mathcal{I}_{1/n}$-space is van der Waerden.

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Theorem 3.

($\mathrm{MA}_{\sigma-cent.}$) There exists a van der Waerden space which is not an $\mathcal{I}_{1/n}$-space.

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($\mathrm{MA}_{\sigma-cent.}$) There exists a van der Waerden space which is not an $\mathcal{I}_{1/n}$-space.

Theorem 3. is true for an arbitrary F_σ P-ideal on ω.

Outline of the proof

Lemma

Assume $A \subseteq \mathbb{N}$ is an AP-set and $f : \mathbb{N} \to \mathbb{N}$.
There is an AP-set $C \subseteq A$ such that

(1) either f is constant on C

(2) or f is finite-to-one on C and $f[C] \in \mathcal{I}_{1/n}$.

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Proposition

($\mathrm{MA}_{\sigma-cent.}$) There exists a MAD family $\mathcal{A} \subseteq \mathcal{I}_{1/n}$ so that for every AP-set $B \subseteq \mathbb{N}$ and every finite-to-one function $f : B \to \mathbb{N}$ there exists an AP-set $C \subseteq B$ and $A \in \mathcal{A}$ so that $f[C] \subseteq A$.

Strongly van der Waerden spaces

Definition C. (Kojman)

A topological space X is called strongly van der Waerden if for every AP-set $A \subseteq \mathbb{N}$ and every sequence $\langle x_n \rangle_{n \in A}$ in X there exists a converging subsequence $\langle x_n \rangle_{n \in B}$ where $B \subseteq A$ is an AP-set.

Proposition (Kojman)

A topological space X is van der Waerden if and only if it is strongly van der Waerden.

Product of van der Waerden spaces

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The product of two van der Waerden spaces is van der Waerden.

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Question: What is the minimal number of van der Waerden spaces such that their product is not van der Waerden?

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The product of two van der Waerden spaces is van der Waerden.

Any finite or countable product of van der Waerden spaces is again van der Waerden.

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The upper bound is certainly less or equal to $\mathfrak{h}$.

Strongly $\mathcal{I}_{1/n}$-spaces

Definition D.

A topological space X is called strongly $\mathcal{I}_{1/n}$-space if for every $\mathcal{I}_{1/n}$-positive set $A \subseteq \mathbb{N}$ and every sequence $\langle x_n \rangle_{n \in A}$ in X there exists a converging subsequence $\langle x_n \rangle_{n \in B}$ where $B \subseteq A$ is does not belong to $\mathcal{I}_{1/n}$.

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A topological space X is called strongly $\mathcal{I}_{1/n}$-space if for every $\mathcal{I}_{1/n}$-positive set $A \subseteq \mathbb{N}$ and every sequence $\langle x_n \rangle_{n \in A}$ in X there exists a converging subsequence $\langle x_n \rangle_{n \in B}$ where $B \subseteq A$ is does not belong to $\mathcal{I}_{1/n}$.

Question: Is every $\mathcal{I}_{1/n}$-space a strongly $\mathcal{I}_{1/n}$-space?

Some questions

Question: Is the product of two $\mathcal{I}_{1/n}$-spaces an $\mathcal{I}_{1/n}$-space?

If $\mathcal{I}_{1/n}$-spaces and strongly $\mathcal{I}_{1/n}$-spaces coincide then the answer is obviously positive.

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If $\mathcal{I}_{1/n}$-spaces and strongly $\mathcal{I}_{1/n}$-spaces coincide then the answer is obviously positive.

Question: What is the minimal number of $\mathcal{I}_{1/n}$-spaces such that their product is not $\mathcal{I}_{1/n}$-space?

References

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