On the sequence convergence of the Cantor and Aleksandrov cube on an arbitrary complete Boolean algebra

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Four of the Questions

are written on the wall

One Sequence to answer

them all

Convergence

Definition

For $X \neq \emptyset$, a mapping $\lambda : X^\omega \to P(X)$ is a **convergence** on X.
A convergence λ is called a **topological convergence** iff there exists a topology $\mathcal{O}$ on X such that $\lambda = \lim_{\mathcal{O}}$.

Theorem

Each topological convergence λ satisfies conditions:

(L1) $\forall a \in X \; a \in \lambda(\langle a \rangle)$.

(L2) $\forall x \in X^\omega \; \forall y \prec x \; \lambda(x) \subset \lambda(y)$.

(L3) $\forall x \in X^\omega \; (\forall y \prec x \; \exists z \prec y \; a \in \lambda(z)) \Rightarrow a \in \lambda(x)$.

If $|\lambda(x)| \leq 1$, then those are also sufficient conditions. (Kisyński, 1960)

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Generating topology

Theorem

For each convergence $\lambda : X^\omega \to P(X)$ there exists a maximal topology O_λ such that $\forall x \in X^\omega \ \lambda(x) \subset \lim_{\mathcal{O}_\lambda} x$.

(L1) $\lambda'(x) = \begin{cases} \lambda(x) \cup \{a\} & \text{if } x = \langle a \rangle \text{ for some } a \in X \\ \lambda(x) & \text{otherwise.} \end{cases}$

(L2) $\lambda'^{-}(x) = \bigcup_{x \prec y} \lambda(y)$

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A convergence λ satisfying (L1) and (L2) such that λ^* is a topological convergence will be called a **weakly topological convergence.**

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If a convergence λ satisfy (L1) and (L2) and we have $|\lambda(x)| \leq 1$ for each sequence x, then λ is a weakly topological convergence.

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The Cantor cube

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The Cantor cube of weight κ, denoted by $\langle 2^\kappa, \tau_C \rangle$ is the Tychonov product of κ-many copies of two point discrete space $2 = \{0, 1\}$

Let $\xi : 2^\kappa \to P(\kappa)$ be a bijection defined by $f(x) = x^{-1}[\{1\}]$.

ξ is a homeomorphism between 2^κ and $P(\kappa)$ (as Boolean algebra)

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For the sequence of sets $\langle X_n : n \in \omega \rangle \in (P(\kappa))^\omega$ let

$$\liminf_{n\in\omega} X_n = \bigcup_{k\in\omega} \bigcap_{n\geq k} X_n \quad \limsup_{n\in\omega} X_n = \bigcap_{k\in\omega} \bigcup_{n\geq k} X_n$$

Fact

A sequence $\langle x_n : n \in \omega \rangle$ converges to the point $x \in 2^\kappa$ iff

$$\liminf_{n\in\omega} X_n = \limsup_{n\in\omega} X_n = X,$$

where $X_n = \xi(x_n)$, and $X = \xi(x)$.

$\langle 2^\kappa, \tau_C \rangle$ is sequential iff $\kappa = \omega$.

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Generalization of the Cantor cube

Let $\mathbb{B}$ be a complete Boolean algebra, and $x = \langle x_n : n \in \omega \rangle$.

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$$\liminf x = \bigvee_{k\in\omega} \bigwedge_{n\geq k} x_n \qquad \limsup x = \bigwedge_{k\in\omega} \bigvee_{n\geq k} x_n$$

$$\lambda_s(x) = \begin{cases} \{\limsup x\} & \text{if } \liminf x = \limsup x \\ \emptyset & \text{if } \liminf x < \limsup x \end{cases}$$

Definition

Topology $\mathcal{O}_{\lambda_s}$ is the well known **sequential topology**, and usually denoted by τ_s.

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Theorem

λ_s ia a topological convergence iff $\mathbb{B}$ is $(\omega, 2)$-distributive.

$$a \in \lim_{\tau_s}(x) \Leftrightarrow \forall y \prec x \; \exists z \prec y \; a \in \lambda_s(z).$$

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A sequence x is $\limsup$-stable iff for each $y \prec x$ $\limsup y = \limsup x$.

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Complete Boolean algebra $\mathbb{B}$ satisfies condition $(\hbar)$ iff each sequence has a $\limsup$-stable subsequence.

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$\mathfrak{t}$-cc $\Rightarrow (\hbar) \Rightarrow \mathfrak{s}$-cc.

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The Alexandrov cube

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The Alexandrov cube of weight κ, denoted by $\langle 2^\kappa, \tau_A \rangle$ is the Tychonov product of κ many copies of two point space $2 = \{0, 1\}$ with topology $\mathcal{O}_A = \{\emptyset, \{0\}, \{0, 1\}\}$.

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Generalization of the Alexandrov cube and its dual

Definition

Let
$$\lambda_{ls}(x) = (\limsup x)\uparrow, \quad \lambda_{li}(x) = (\liminf x)\downarrow .$$

Theorem

Set $F \in \mathcal{F}_{ls}$ iff it is upward closed and $\bigwedge_{n\in\omega} x_n \in F$, for each decreasing $x \in F^\omega$.
Set $F \in \mathcal{F}_{li}$ iff it is downward closed and $\bigvee_{n\in\omega} x_n \in F$, for each increasing $x \in F^\omega$.

Open set in $\mathcal{O}_{\lambda_{ls}}$ is downward closed and contains $\mathbf{0}$.
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$\mathcal{O}_{\lambda_{ls}}, \mathcal{O}_{\lambda_{li}} \subset \tau_S$.

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Theorem

λ_{ls} and λ_{li} satisfy (L1) and (L2).

λ_{ls} and λ_{li} are topological convergences iff $\mathbb{B}$ is $(\omega, 2)$-distributive.

(L3)

$$\lambda^*_{ls}(x) = \bigcap_{y \prec x} \bigcup_{z \prec y} \lambda_{ls}(z), \quad \lambda^*_{li}(x) = \bigcap_{y \prec x} \bigcup_{z \prec y} \lambda_{li}(z)$$

Question 1.

Are λ^*_{ls} and λ^*_{li} topological convergences?

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Topology $\mathcal{O}^*$

Definition

Let the family $\mathcal{P}^* = \mathcal{O}_{\lambda_{ls}} \cup \mathcal{O}_{\lambda_{li}}$ be a subbase for a topology $\mathcal{O}^*$.

Theorem

$$\mathcal{O}^* \subset \tau_s$$

$$\lim_{\mathcal{O}^*} = \lim_{\lambda_{ls}} \cap \lim_{\lambda_{li}}$$

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If $\mathbb{B}$ satisfies $(\hbar)$ or it is $(\omega, 2)$-distributive, then $\lim_{\mathcal{O}^*} = \lim_{\tau_s}$.

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Is it always $\lim_{\mathcal{O}^*} = \lim_{\tau_s}$?

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Theorem

For Boolean algebra $P(\omega)$ we have $\mathcal{O}^* = \tau_s$.

Proof: Both spaces, $\langle P(\omega), \tau_s \rangle$ and $\langle P(\omega), \mathcal{O}^* \rangle$ are Hausdorff, $O^* \subset \tau_s$ and $\langle P(\omega), \tau_s \rangle$ is homeomorphic to the Cantor cube, so it is compact, and as a compact space, its minimality in the class of Hausdorff spaces implies $\mathcal{O}^* = \tau_s$.

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Theorem

If a Boolean algebra carries strictly positive Maharam submeasure μ, we have $\mathcal{O}^* = \tau_s$.

Proof: For $O \in \tau_s$ and $a \in O$ let $B(a,r) = \{x \in \mathbb{B} : \mu(x \triangle a) < r\} \subset O$ and

$$O_1 = \{x \in \mathbb{B} : \mu(x \setminus a) < r/2\}, \quad O_2 = \{x \in \mathbb{B} : \mu(a \setminus x) < r/2\}$$

So, $a \in O_1 \cap O_2 \subset B(a,r) \subset O$.
Also we have $O_1 \in \mathcal{O}_{\lambda_{ls}}$ and $O_2 \in \mathcal{O}_{\lambda_{li}}$.

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Does there exist a complete Boolean algebra such that $\mathcal{O}^* \neq \tau_s$?

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Definition

$\mathcal{N}_0$ is the family of all neighborhoods of the point $\mathbf{0}$ in topology τ_s.
$\mathcal{N}_0^d = \{U \in \mathcal{N}_0 : U = U\downarrow\}$.

Theorem (Balcar, Glówczyński, Jech)

If $\langle \mathbb{B}, \tau_s \rangle$ is a Frechét space, then for each $V \in \mathcal{N}_0$ exists $U \in \mathcal{N}_0^d$ such that $U \subset V$. So, then $\mathcal{N}_0^d$ is a neighborhood base at the point $\mathbf{0}$.

$\mathcal{N}_0^d = \mathcal{O}_{\lambda_{ls}}$ is a neighborhood base at the point $\mathbf{0}$ for the topology $\mathcal{O}^*$.

If in a topological space $\langle \mathbb{B}, \tau_s \rangle$ the family $\mathcal{N}_0^d$ is not a neighborhood base at $\mathbf{0}$, then $\tau_s \neq \mathcal{O}^*$.

Question 4.

Does there exist a c.B.a. such that $\mathcal{N}_0^d$ is not a neighborhood base of $\mathbf{0}$?

Base matrix tree

A base matrix tree is a tree $\langle \mathcal{T}, {}^*\supset \rangle$ of height $\mathfrak{h}$ such that $\mathcal{T}$ is dense in a pre-order $\langle [\omega]^\omega, \subset^* \rangle$. Levels are MAD families, and maximal chains are towers.

By Balcar, Pelant and Simon, such tree always exists.

Let us denote by $Br(\mathcal{T})$ a set of all maximal branches of $\mathcal{T}$ and let $\kappa = |Br(\mathcal{T})|$.

$2^{\omega_1} \leq \kappa \leq \mathfrak{c}^{\mathfrak{h}} = 2^{\mathfrak{h}}$.

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A construction of the Sequence

Let $\mathbb{B}$ is a c.B.a. such that $\mathrm{cc}(\mathbb{B}) > 2^{\mathfrak{h}}$ and $1 \Vdash_{\mathbb{B}} |((\mathfrak{h})^V)\check{}| < \mathfrak{t}$.

So, $1 \Vdash |\check{T}_\alpha| < \mathfrak{t}$

$1 \Vdash \exists X \in [\check{\omega}]^{\check{\omega}} \ \forall B \in \check{T}_\alpha \ X \subset^* B$

By the Maximum principle there exists a name σ_α such that

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Let $\langle b_\alpha : \alpha < \kappa \rangle$ be a maximal antichain in $\mathbb{B}$. Then, by Mixing lemma there exists name τ such that

$$\forall \alpha < \kappa \ b_\alpha \Vdash \tau = \sigma_\alpha$$

$x_n = \|\check{n} \in \tau\|$ and for $\tau_x = \{\langle \check{n}, x_n \rangle : n \in \omega\}$ we have

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Properties of the Sequence

$0 \notin \lambda^*_{ls}(x)$

$0 \in \lim_{\mathcal{O}_{\lambda_{ls}}}(x)$

Answer 1.

λ^*_{ls} is not a topological convergence.

$0 \in \lim_{\mathcal{O}_{\lambda_{ls}}}(x) \cap \lim_{\mathcal{O}_{\lambda_{li}}}(x) = \lim_{\mathcal{O}^*}(x)$ and $0 \notin \lim_{\tau_s}(x)$

Answer 2.

$\lim_{\tau_s} \neq \lim_{\mathcal{O}^*}$

Answer 3.

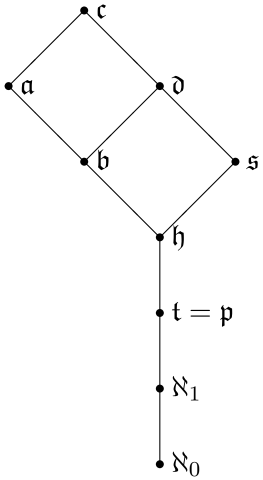
$\tau_s \neq \mathcal{O}^*$

If $X = \{x_n : n \in \omega\}$, then $\mathbb{B} \setminus X \in \tau_s$, but it is not downward closed and each downward closed neighborhood of $\mathbf{0}$ intersects X.

Answer 4.

There exists a Boolean algebra in which $\mathcal{N}_0^d$ is not a neighborhood base of $\mathbf{0}$ for topology τ_s.

Small cardinals



$\mathfrak{s}$

- Set $S \subset \omega$ splits a set $A \subset \omega$ iff $|A \cap S| = \omega$ and $|A \setminus S| = \omega$.
- $\mathcal{S} \subset [\omega]^\omega$ is a splitting family iff each $A \in [\omega]^\omega$ is splitted by some element of $\mathcal{S}$.
- Splitting number, $\mathfrak{s}$, is the minimal cardinality of a splitting family.

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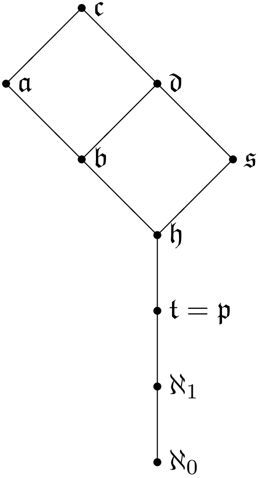

$\mathfrak{t}$

- $\mathcal{T} \subset [\omega]^\omega$ is a tower iff $\langle \mathcal{T}, {}^* \supsetneq \rangle$ well-ordered and the family $\mathcal{T}$ has no pseudointersection.
- Tower number, $\mathfrak{t}$, is the minimal cardinality of a tower.

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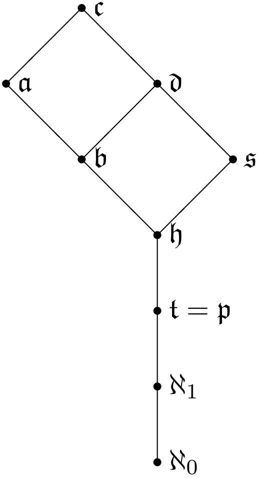

$\mathfrak{b}$

- For functions $f, g \in \omega^\omega$, $f \leq^* g$ denotes $\exists n_0 \in \omega \; \forall n \geq n_0 \; f(n) \leq g(n)$.
- $\mathcal{B} \subset \omega^\omega$ is unbounded family iff there does not exist $g \in \omega^\omega$ such that $f \leq^* g$ for each $f \in \mathcal{B}$.
- Bounding number, $\mathfrak{b}$, is the minimal cardinality of unbounded family.

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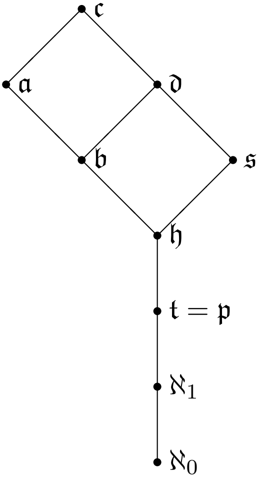

$\mathfrak{h}$

$$\mathfrak{h} = \min\{|\mathcal{H}| : \mathcal{H} \text{ is a family of open dense subsets of the order } \langle [\omega]^\omega, \subset^* \rangle \text{ and } \bigcap \mathcal{H} \text{ is not dense}\}.$$