Group actions on Polish spaces

Robert Rałowski and Szymon Żeberski

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Notation and Terminology

$(G, \cdot)$ be any uncountable Polish group, X is a Polish space and $I \subseteq \mathscr{P}(X)$ s.t

- I is σ-ideal with a Borel base and
- I contains all singletons and
- I translation invariant.

(X, I) is Polish ideal space
Let $\mathcal{B}_+(I) = Borel(X) \setminus I$ be set of all I-positive Borel sets.
$Perf(X)$ stands for set of all perfect subsets of X

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Definition (Cardinal coefficients)

Let X - Polish space and $I \subseteq \mathscr{P}(X)$ be σ-ideal.

$$non(I) = min\{|A| : A \subset X \wedge A \notin I\}$$

$$cov(I) = min\{|\mathscr{A}| : \mathscr{A} \subset I \wedge \bigcup \mathscr{A} = X\}$$

$$cov_h(I) = min\{|\mathscr{A}| : \mathscr{A} \subset I \wedge (\exists B \in \mathcal{B}_+(I)) \bigcup \mathscr{A} = B\}$$

$$cof(I) = min\{|\mathscr{B}| : \mathscr{B} \subseteq I \wedge (\forall A \in I)(\exists B \in \mathscr{B})\ A \subseteq B\}$$

$\mathscr{N}$ σ-ideal of null sets and
$\mathscr{M}$ σ-ideal of all meager subsets of X.
$cov(\mathscr{M}) = cov_h(\mathscr{M})$, $cov(\mathscr{N}) = cov_h(\mathscr{N})$.

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Definition

Let (X, I) be Polish ideal space. Assume that $C \subseteq D \subseteq X$. We say that C is completely I-nonmeasurable in D iff

$$(\forall B \in \mathscr{B}_I^+(X))\,(B \cap D \notin I \to (B \cap C \notin I \wedge B \cap (D \setminus C) \notin I)).$$

Definition (Group action)

We say that $(G, \cdot)$ acts on Polish space X if

1. $(\forall x \in X) ex = x$,
2. $(\forall x \in X)(\forall g, h \in G)\ \ (gh)x = g(hx)$

Definition (Orbit)

Let G acts on X and $A \subset X$ then

$$GA = \{gx \in X : \ \ (g, x) \in G \times A\}$$

is called an orbit of the set A by the group G and whenever $A = \{x\}$ is singleton then we will write Gx instead of $G\{x\}$ for convenience.

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Many orbits case

Theorem

Let (X, I) be Polish ideal space and $(G, \cdot)$ be any Polish group acting on X.

Let

$$(\forall B \in \mathscr{B}_I^+(X))\ cof(I) \leq |\{Gb :\ b \in B\}|.$$

Then there exists $H \leq G$ and $A \subset X$
such that A and HA are completely I-nonmeasurable subsets of X.
Moreover if (G, J) is a Polish ideal space and
there exists Borel bases $\mathscr{B}_G \subset \mathscr{B}_J^+(G)$ and $\mathscr{B}_X \subset \mathscr{B}_I^+(X)$ with

$$|\mathscr{B}_G| = |\mathscr{B}_X| \leq |\{Gb :\ b \in B\}|,$$

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Proof

Let us enumerate bases $\mathscr{B}_G = \{C_\alpha : \alpha < \lambda\}$ and $\mathscr{B}_X = \{B_\alpha : \alpha < \lambda\}$ where $\lambda \leq |\{Gb : \ b \in B\}|$.

Build transfinite sequence:

$$\langle (a_\xi, d_\xi, h_\xi, c_\xi) \in B_\xi \times B_\xi \times C_\xi \times C_\xi : \ \ \xi < \lambda \rangle$$

with the following conditions:

1. the collection of orbits $\{G a_\xi : \xi < \lambda\} \cup \{G d_\xi : \xi < \lambda\}$ is pairwise disjoint,
2. $\langle h_\xi : \ \xi < \lambda \rangle_G \cap \{c_\xi : \xi < \lambda\} = \emptyset$.

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$\alpha < \lambda$ step

By assumption

$$(\forall B \in \mathscr{B}_I^+(X))\ cof(I) \leq |\{Gb : \ b \in B\}|.$$

we can find

$$a_\alpha, d_\alpha \in B_\alpha \setminus \bigcup(\{Ga_\xi : \xi < \alpha\} \cup \{Gd_\xi : \xi < \alpha\})$$

and $h_\alpha \in C_\alpha \setminus \langle h_\xi : \ \xi < \alpha \rangle_G$ because $|\langle Z \rangle_G| \leq \aleph_0 \cdot |Z|$ for any set $Z \subset G$.

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Now let us take the following sets:
$H = \langle h_\alpha \in G : \alpha < \lambda \rangle_G$, $A = \{a_\alpha \in X : \alpha < \lambda\}$ and
$D = \{d_\alpha \in X : \ \alpha < \lambda\}$.
Then we have

- H is completely J-nonmeasurable subgroup of G,
- A and D are completely I-nonmeasurable subsets of the Polish space X.

Moreover by the inclusion

$$A \subset HA \subset D^c$$

HA is completely I-nonmeasurable subset of X.

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Theorem

Let (X, I) be Polish ideal space and $(G, \cdot)$ acting on X. If we have

$$(\forall B \in \mathscr{B}^+_I(X))\, cof(I) \leq |\{Gb : \ b \in B\}|.$$

Then there exists subgroup $H \leq B$ and the pairwise disjoint family $\{A_\alpha : \alpha < cof(I)\} \subset \mathscr{P}(X)$ such that:

1. $(\forall \alpha < cof(I))\, A_\alpha, HA_\alpha$ *are completely I-nonmeasurable in X,*
2. $(\forall \alpha, \beta)\alpha < \beta < cof(I) \to HA_\alpha \cap HA_\beta = \emptyset.$

Moreover if (G, J) is a Polish space and there exists Borel bases $\mathscr{B}_G \subset \mathscr{B}^+_J(G)$ and $\mathscr{B}_X \subset \mathscr{B}^+_I(X)$ with

$$|\mathscr{B}_G| = |\mathscr{B}_X| \leq |\{Gb : \ b \in B\}|,$$

then H is completely J-nonmeasurable in the group G.

Few orbits case

Theorem

Let $(G, \cdot)$ be any group and let (X, I) be a Polish ideal space.
If for some (every) $x \in X$ $Gx = X$ and

$$(\exists \lambda < 2^\omega)(\forall x, y \in X)\ x \neq y \rightarrow |G_{x,y}| \leq \lambda$$

where $G_{x,y} = \{g \in G :\ y = gx\}$.
Then there exists a subgroup $H \leq G$ and a subset $A \subset X$ such that A and HA are complete I-nonmeasurable sets in X.
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Theorem

Let (X, I) is a Polish ideal space and $(G, \cdot)$ group acting on X. Assume that $X = \bigcup\{Gx_n : n \in \omega\}$ is a union of the countable many I-positive and I-measurable orbits.
Suppose that

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$$A^n_\alpha \text{ and } HA^n_\alpha \text{ is completely } I - \text{nonmeasurable in } Gx_n.$$

Moreover, if (G, J) forms Polish ideal space then our H is completely J-nonmeasurable in G.

Theorem

Let $(G, \cdot, J)$ be a Polish ideal group which acts on the Polish ideal space (X, I). Let us assume that

1. $cov_h(J) = cov_h(I) = cof(I) = cof(J)$,
2. *for any* $n \in \omega$, $s \in \mathbb{Z}^n$ *there exists* $G' \subseteq G$ *such that* $G \setminus G' \in J$ *and for every* $g \in G'$, $a \in G'^n$ *the following condition holds*

$$\{h \in G : \prod_{i \in n} a_i \cdot h^{s_i} = g\} \in J.$$

Then there is a completely J-nonmeasurable subgroup H in G and completely I-nonmeasurable subset $A \subseteq X$ such that HA is completely I-nonmeasurable in the space X.

$Non(I) < cov(I)$

Theorem

Let (X, I) be a Polish ideal space and $non(I) < cov_h(I)$.
Assume that $(G, \cdot)$ is a group which acts on X.
If $H \leq G$ and $A \in I$ are such that HA contains a Borel set $B \notin I$ then there is a subgroup $H' \leq H$ such that $H'A$ is completely I-nonmeasurable in some I-positive Borel set.

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Proof.

Let $B \in \mathscr{B}_I^+(X)$ such that $B \subseteq HA$.
Let $T \subset B$ witness of $non(I)$.
Define $F : T \to H$ such that

$$t \in F(t)A \text{ holds for any } t \in T.$$

If $H' = \langle F[T] \rangle_H$ then

$$|H'| = |F[T]| \leq |T| = non(I) < cov_h(I).$$

Then $T \subseteq F[T]A \subseteq H'A$. So, $H'A \notin I$.
Notice that

- $H'A = \bigcup \mathcal{F}$, where $\mathcal{F} = \{hA : \; h \in H'\} \subseteq I$
- $|\mathcal{F}| < cov_h(I)$

Then any I-positive Borel set can not be covered by the family $\mathcal{F}$. □

Applications

X topological space
$\mathscr{H}(X)$ space of all homeomorphisms on X with compact-open topology:

$$\{V(K,U):\ K \subseteq X \text{ is compact and } U \subseteq X \text{ is open in } X\},$$

where

$$V(K,U) = \{f \in \mathscr{H}(X):\ f[K] \subseteq U\}).$$

When X is compact Polish space then $\mathscr{H}(X)$ is also Polish one.
A metric on $\mathscr{H}(X)$:

$$d(f,g) = \sup_{x \in X}\{d(f(x),g(x))\} + \sup_{y \in X} d(f^{-1}(y),g^{-1}(y)).$$

Proposition

Let $(G,\cdot)$ *be a Polish space. Fix* $n \in \omega$, $s \in \mathbb{Z}^n$*. Then there exists comeager* $G' \subseteq G$ *such that for every* $g \in G'$, $a \in G'^n$ *the following set*

$$\{h \in G : \prod_{i \in n} a_i^{s_i} \cdot h = g\}$$

is meager.

Corollary

Assume that $\mathrm{cov}(\mathscr{M}) = \mathrm{cof}(\mathscr{M})$. *Let* X *be a compact Polish space without isolated points.*

Then there exist a completely $\mathscr{M}$*-nonmeasurable subgroup* $H < \mathscr{H}(X)$

and a completely $\mathscr{M}$*-nonmeasurable subset* $A \subseteq X$ *such that* HA *is completely* $\mathscr{M}$*-nonmeasurable.*

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From the other side the following example is a simple corollary from Theorem 6 when we can find many different orbits.

Corollary

If G is a subgroup of the group of all isometries on the Cantor space 2^ω defined as follows

$$G = \{T_X : X \in \mathscr{P}(\{n \in \omega : n \equiv 0 \mod 2\})\}$$

where for any $x \in 2^\omega$ and $n \in \omega$

$$T_X(x)(n) = \begin{cases} x(n) & \text{when } n \notin X \\ 1 - x(n) & \text{when } n \in X. \end{cases}$$

Then there is a subgroup H of G and uncountable many pairwise disjoint subsets $\{A_\alpha \subset 2^\omega : \alpha < cof(\mathscr{M})\}$ such that HA_α are completely $\mathscr{M}$-nonmeasurable in the Cantor space 2^ω for any $\alpha < cof(\mathscr{M})$. Moreover, $\{HA_\alpha : \alpha < cof(\mathscr{M})\}$ forms a pairwise disjoint family of subsets of the Cantor space.

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Thank You

R. Rałowski and Sz. Żeberski, Group action on Polish spaces, arXiv:1406.3063