Luzin and Sierpiński sets

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Let $\mathcal{I}$ be a σ-ideal of subsets of $\mathbb{R}$ ($\mathbb{R}^2$) and $\mathcal{B}$ a family of Borel sets. We say that $\mathcal{I}$:

- is translation invariant, if for each $x \in \mathbb{R}$ and $I \in \mathcal{I}$ we have $x + I \in \mathcal{I}$,
- is scale invariant, if for each $x \in \mathbb{R}$ and $I \in \mathcal{I}$ we have $xI \in \mathcal{I}$,
- has Borel base if $(\forall I \in \mathcal{I})(\exists B \in \mathcal{B} \cap \mathcal{I})(I \subseteq B)$,
- has Steinhaus property if $Int(A - B) \neq \emptyset$ for each $A, B \in \mathcal{B} \setminus \mathcal{I}$

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Example

Meager sets $\mathcal{M}$ and null sets $\mathcal{N}$ have these properties.

Definition

A is

- $\mathcal{I}$-nonmeasurable if $A \notin \sigma(\mathcal{B} \cup \mathcal{I})$,
- completely $\mathcal{I}$-nonmeasurable if $A \cap B$ is $\mathcal{I}$-nonmeasurable for every $B \in \mathcal{B} \setminus \mathcal{I}$,
- $\mathcal{I}$-Luzin set if $|A| = \mathfrak{c}$ and for every $I \in \mathcal{I}$ a set $A \cap I$ is countable,
- strong $\mathcal{I}$-Luzin set if A is an $\mathcal{I}$-Luzin and its intersection with every Borel $\mathcal{I}$-positive set is uncountable.

Definition

A is:

- a Luzin set if $|L| = \mathfrak{c}$ and every intersection of L and a meager set is countable,
- a strong Luzin set if A is a Luzin set and every intersection of A and a $\mathcal{M}$-positive Borel set is uncountable,
- a Sierpiński set if $|S| = \mathfrak{c}$ and every intersection of S and a null set is countable,
- a strong Sierpiński set if A is a Sierpiński set and every intersection of A and a $\mathcal{N}$-positive Borel set is uncountable,
- a Bernstein set if for each perfect set P we have $A \cap P \neq \emptyset$ and $A^c \cap P \neq \emptyset$.

Fact

Let B be a Borel $\mathcal{I}$-positive set and let D be a countable dense set. Then $B + D$ is an $\mathcal{I}$-residual set.

Corolary

Let L be a $\mathcal{I}$-Luzin set. Then $L + \mathbb{Q}$ is a strong $\mathcal{I}$-Luzin set.

Fact (CH)

There exists a partition of $\mathbb{R}$ into $\mathfrak{c}$ many strong $\mathcal{I}$-Luzin sets.

Theorem (CH)

There exists a set $A \subseteq \mathbb{R}^2$ such that each horizontal slice A^y is a strong $\mathcal{I}$-Luzin set and each vertical slice A_x is a cocountable set. Such a set is $\mathcal{M}$ and $\mathcal{N}$-nonmeasurable. Moreover, in the case $\mathcal{I} = \mathcal{M}$, A is completely $\mathcal{M}$-nonmeasurable, and in the case $\mathcal{I} = \mathcal{N}$, A is completely $\mathcal{N}$-nonmeasurable.

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Theorem (CH)

There exists a set $A \subseteq \mathbb{R}^2$ such that each vertical slice A_x is cocountable and A is completely $\mathcal{M}$, $\mathcal{N}$-nonmeasurable.

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There exists a set $A \subseteq \mathbb{R}^2$ such that each horizontal slice A^y is a strong Luzin set and each vertical slice A_x is strong Sierpiński set. Moreover, A is completely $\mathcal{M}$- and $\mathcal{N}$-nonmeasurable.

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There exists a set $A \subseteq \mathbb{R}^2$ such that each horizontal slice A^y is a strong Luzin set and each vertical slice A_x is strong Sierpiński set. Moreover, A is completely $\mathcal{M}$- and $\mathcal{N}$-nonmeasurable.

Proof

Let $\{L_\alpha : \alpha < \mathfrak{c}\}$ and $\{S_\alpha : \alpha < \mathfrak{c}\}$ be a partition of $\mathbb{R}$ into strong Luzin sets and strong Sierpiński sets respectively. Let us set:

$$A = \bigcup_{\alpha < \mathfrak{c}} (L_\alpha \times S_\alpha).$$

□

Theorem

- Assume that a Luzin set exists. Then there exists a set $A \subseteq \mathbb{R}^2$ such that for each straight line l a set $A \cap l$ is a strong Luzin set.

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- (CH) There exists a set $A \subseteq \mathbb{R}^2$ such that for each homeomorphism $h : \mathbb{R} \to \mathbb{R}^2$ on its image a set $h(\mathbb{R}) \cap A$ is a strong Luzin set and A is a Hamel basis.

Theorem (CH)

There exist a set $A \subseteq \mathbb{R}^2$ such that for every increasing continuous function f $A \cap f$ is a strong Luzin set and for each decreasing locally absolutely continuous function g $A \cap g$ is a strong Sierpiski set and A is a Hamel basis.

Theorem

- Assume that a Sierpiński set exists. Then there exists a set $A \subseteq \mathbb{R}^2$ such that for each straight line l a set $A \cap l$ is a strong Sierpiński set.
- (CH) There exists a set $A \subseteq \mathbb{R}^2$ such that for each straight line l on the plane a set $l \cap A$ is a strong Sierpiński set and A is a Hamel basis.

Fact

- Let L be an $\mathcal{I}$-Luzin set. Then there exists a linearly independent $\mathcal{I}$-Luzin set.
- Let L be an $\mathcal{I}$-Luzin set. Then there exists a linearly independent strong $\mathcal{I}$-Luzin set.

Problem

Does the existence of an $\mathcal{I}$-Luzin set imply the existence of an $\mathcal{I}$-Luzin set which is a Hamel base?

Fact (CH)

There is an $\mathcal{I}$-Luzin set L such that L is a linear subspace of $\mathbb{R}$.

Theorem

It is consistent that $2^\omega = \omega_2$ and there is a Luzin set which is a linear subspace of $\mathbb{R}$.

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Proof.

Let us work in a model V' obtained from a model V of CH by adding ω_2 Cohen reals $\{c_\alpha : \alpha < \omega_2\}$. Set

$$L = span_{\mathbb{Q}}(\{c_\alpha : \alpha < \omega_2\}).$$

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$\square$

Problem

Does the existance of a Luzin set imply the existance of a Luzin set which is a linear subspace of $\mathbb{R}$?

Theorem (CH)

For each $\mathcal{I}$-Luzin set L there exists an $\mathcal{I}$-Luzin set X such that $\{x + L : x \in X\}$ is a partition of $\mathbb{R}$.

Theorem (CH)

There exists an $\mathcal{I}$-Luzin set L such that $L + L$ is an $\mathcal{I}$-Luzin set.

Theorem (CH)

There exists an $\mathcal{I}$-Luzin set L such that $L + L = \mathbb{R}$.

Theorem (CH)

For each $n \in \mathbb{N} \setminus \{0\}$ There exists an $\mathcal{I}$-Luzin set L such that $\bigoplus^n L$ is an $\mathcal{I}$-Luzin set and $\bigoplus^{n+1} L = \mathbb{R}$.

Theorem (CH)

There exists an $\mathcal{I}$-Luzin set L such that $span(L)$ is an $\mathcal{I}$-Luzin set.

Corolary (CH)

1. There exists an $\mathcal{I}$-Luzin set L such that $\bigoplus^{n+1} L$ is an $\mathcal{I}$-Luzin for each $n \in \mathbb{N}$,
2. There exists an $\mathcal{I}$-Luzin set L such that $L + L = L$,
3. There exists an $\mathcal{I}$-Luzin set L such that $\langle \bigoplus^{n+1} L : n \in \mathbb{N} \rangle$ is a ascending sequence of $\mathcal{I}$-Luzin sets.

Theorem (CH)

- There exists a Luzin set L such that $L + L$ is a Bernstein set.
- There exists a Sierpiński set S such that $S + S$ is a Bernstein set.

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Proof.

$Perf = \{P_\alpha : \alpha < \mathfrak{c}\}$, $\mathcal{M} \cap \mathcal{B} = \{M_\alpha : \alpha < \mathfrak{c}\}$.
We choose sequences $\{l_\alpha : \alpha < \mathfrak{c}\}$, $\{l'_\alpha : \alpha < \mathfrak{c}\}$ and $\{p_\alpha : \alpha < \mathfrak{c}\}$ such that for each $\xi < \mathfrak{c}$:

1. $l_\xi, l'_\xi \notin \bigcup_{\alpha<\xi} M_\alpha$,
2. $(\bigcup_{\alpha\leq\xi}\{l_\alpha, l'_\alpha\} + \bigcup_{\alpha\leq\xi}\{l_\alpha, l'_\alpha\}) \cap \{p_\alpha : \alpha < \xi\} = \emptyset$,
3. $l_\xi + l'_\xi \in P_\xi$,
4. $p_\xi \in P_\xi$.

Proof...

Let us denote:

$$
\begin{aligned}
M_1 &= \bigcup_{\alpha<\xi} M_\alpha, \\
M_2 &= \bigcup_{\alpha<\xi} M_\alpha \cup (\{p_\alpha\}_{\alpha<\xi} - \{l_\alpha, l'_\alpha\}_{\alpha<\xi}) \cup \frac{1}{2}\{p_\alpha\}_{\alpha<\xi}, \\
P &= P_\xi,
\end{aligned}
$$

Does there exist $l' \in M_2^c$ such that a set $M_1^c \cap (P - l')$ has cardinality $\mathfrak{c}$?

Proof...

We extend our universe V (via generic extension) to V' such that $V' \models cov(\mathcal{M}) \geq \omega_2$.

We will work in V'. Let us now fix a set $A \subseteq P$ of cardinality ω_1. Notice that for every $a \in A$ a set $\{l : \ a - l \in M_1^c\} = -M_1^c + a$ is comeager. Since $cov(\mathcal{M}) > \omega_1$

$$\bigcap_{a \in A} \{l : \ a - l \in M_1^c\} \cap M_2^c \neq \emptyset.$$

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$$\bigcap_{a \in A} \{l : \ a - l \in M_1^c\} \cap M_2^c \neq \emptyset.$$

It shows that $V' \models \exists l' \in M_2^c \ |M_1^c \cap (P - l')| \geq \omega_1$.
So, V' models the following sentence:

$$(\exists l')_{\mathbb{R}}(\exists T)_{Perf}(\forall x)_{\mathbb{R}}(l' \in M_2^c \wedge (x \in T \to x \in M_1^c \ \wedge \ x + l' \in P))$$

By Shoenfield absoluteness theorem it is also true in V. □

Theorem

There are no Luzin set L and Sierpiński set S such that $L + S$ is a Bernstein set.

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Follows from

Babinkostova L., Sheepers M. Products and selection principles, Topology Proceedings, Vol. 31 (2007), 431-443.

Lemma

Let A be a null set. We can find a perfect set P such that for every n

$$A + \underbrace{P + P + \cdots + P}_{n} \in \mathcal{N}.$$

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Proof of lemma

We can assume that A is Borel. Let V be our universe. We enlarge it (via forcing) to V' satisfying $V' \models add(\mathcal{N}) = \omega_3$.

Proof of lemma...

Let us work now in V'. Take $X \subseteq \mathbb{R}$ of cardinality ω_2. Then $A + X \in \mathcal{N}$, so we can find a null Borel set B, such that $A + X \subseteq B$. Notice that $\{x : \ x + A \subseteq B\}$ is a coanalytic set of cardinality ω_2, hence, it contains a perfect set P_0.

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Now, set $A_1 = A_0 + P_0$. We want to find a perfect set $P_1 \subseteq P_0$ such that $A_1 + P_1 \in \mathcal{N}$. Moreover, we require that the first splitting node in P_0 is still a splitting node in P_1.

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Now, set $A_1 = A_0 + P_0$. We want to find a perfect set $P_1 \subseteq P_0$ such that $A_1 + P_1 \in \mathcal{N}$. Moreover, we require that the first splitting node in P_0 is still a splitting node in P_1.

We procced by a simple induction on n-th step finding for a null set A_n and a perfect set P_n a perfect set $P_{n+1} \subseteq P_n$ such that $A_{n+1} = A_n + P_{n+1}$ is null and all splitting nodes from first $n+1$ levels in P_n remains splitting nodes in P_{n+1}.

Proof of lemma...

We get a sequence of perfect sets $(P_n, n \in \omega)$ such that $P = \bigcap_{n\in\omega} P_n$ is a perfect set. Moreover, we can find a null G_δ B such that $B \supseteq \bigcup_{n\in\omega} A_n$. Notice that

$$V' \models (\exists P \in Perf)(\exists B)(\forall n)(\forall x)(\forall a)(\forall b)(B \text{ is null } G_\delta \wedge$$

$$(a \in A \wedge b \notin B \wedge x_0, x_1, \ldots, x_n \in P \to a + x_0 + \cdots x_n \neq b)),$$

where $x_0, x_1, \ldots, x_n$ are naturally coded by x e.g. by the formula $x_i(k) = x(kn + i)$.
Above formula is Σ^1_2. □

Marcin Michalski, Szymon Żeberski, “Luzin and Sierpiński sets, some nonmeasurable subsets of the plane“, arXiv.org/abs/1406.3062