Rosenthal compact spaces

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1. K *is metrizable.*
2. K *has a countable basis of open sets.*
3. *There is a countable family of open sets that disjointly separates the points of* K.
4. $K \subset C(X)$ *for some Polish* X.

$$C(X) = \{f : X \longrightarrow \mathbb{R} \text{ continuous } \}$$

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If you construct a compact space without using any strange set-theoretic device (like AC), then most probably you got a Rosenthal compactum.

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$+$ every Rosenthal K is Fréchet-Urysohn space: every point in the closure of a set A is the limit of sequence from A.

Three Rosenthal compacta

Theorem (Todorcevic)

If K is a nonmetrizable separable Rosenthal compactum, then

1. *either* $K \supset A(\mathfrak{c})$
2. *or* $K \supset S$
3. *or* $K \supset D$

- $A(\mathfrak{c})$ is the one-point compactification of the discrete set of size $\mathfrak{c}$.
- S is the split Cantor set: $2^{\omega+1}$ ordered lexicographically, with the order topology.
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Our aim: Multidimensional versions of this result.

$\varphi : K \longrightarrow L$ continuous.

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- S_n and D_n are not separable (except S_2). So this is not a *basis result*.

The open degree

Definition

A compact space K has open degree $\leq n$ iff there exists a countable family $\mathscr{F}$ of open sets such that for every different $x_0, \ldots, x_n \in K$ there exist repective neighborhoods $V_0, \ldots, V_n \in \mathscr{F}$ such that $V_0 \cap \cdots \cap V_n = \emptyset$.

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- $odeg(K) \leq 1$ if and only if K is metrizable.
- If the open sets in $\mathscr{F}$ are G_δ, then this means that K is an at most n-to-1 preimage of a metric space.

Finite basis theorem

Theorem (A.-Todorcevic)

Given $n < \omega$, there is a finite list

$$K_1^{(n)}, \dots, K_{p_n}^{(n)}$$

of separable Rosenthal compacta of open degree n, such that every separable Rosenthal K with $odeg(K) \geq n$ contains one from the list.

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$p_1 = 1,\ p_2 = 3,\ p_3 = 4,\ p_4 = 8,\ \ldots$

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3. Only in some cases, an infinity point ∞.

The game (simplified version)

Given n, K and a fix countable dense set D, two players play

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If K is Rosenthal, we can use determinacy[1].

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The winning strategy of Player II means that $odeg(K) < n$.

The winning strategy of Player I produces a tree structure in K, that we must reduce to a canonical form.

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