Countably compact + countably tight and proper forcing

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Outline

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$$\text{C-closed} \quad \text{vs} \quad \mathfrak{t} = \omega \quad \text{vs} \quad h\pi\chi = \omega$$

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3. discuss new results including that PFA implies that

   $\mathrm{h}\pi\chi = \omega$ spaces are C-closed.

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5. [MoMr statement:] "compact + $t = \omega$ $\Rightarrow$ sequential"

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Remark: C-closed is only interesting in countably compact spaces, but "not C-closed" is always interesting

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   PFA no help: meet ω_1-dense sets is just an old ρ

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(H) ctbly cpt + separable + $h\pi\chi=\omega$ $\overset{PFA}{\Longrightarrow}$ $t=\omega$ indestructible by forcing with ${}^{<\omega_1}\omega_2$

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but, if we are hoping for $\vec{Y}$ to have an ω_1 limit ...

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proof of proper with $t = \omega$ is like “no S-space”

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and then, using $h\pi\chi = \omega$ and countable $\mathcal{M}_p$ poset was totally proper etc. and satisfied the $\aleph_2$-p.i.c.
and the generic free sequence converges to z by weight $\leq \aleph_1$

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and the generic free sequence converges to z by weight $\leq \aleph_1$

what to do under PFA?

We continue to study $X = Y \cup \{z\}$ with Y countably compact and now $t = \omega$ for X. Also, WLOG ω is dense in Y hence X has weight $\mathfrak{c}$

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allowing us to produce a model of

$$\text{CH} + h\pi\chi = \omega \;\Rightarrow\; \text{C-closed} \quad \text{and MoMr}$$

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set up for a Picture proof using our new idea to get z a limit

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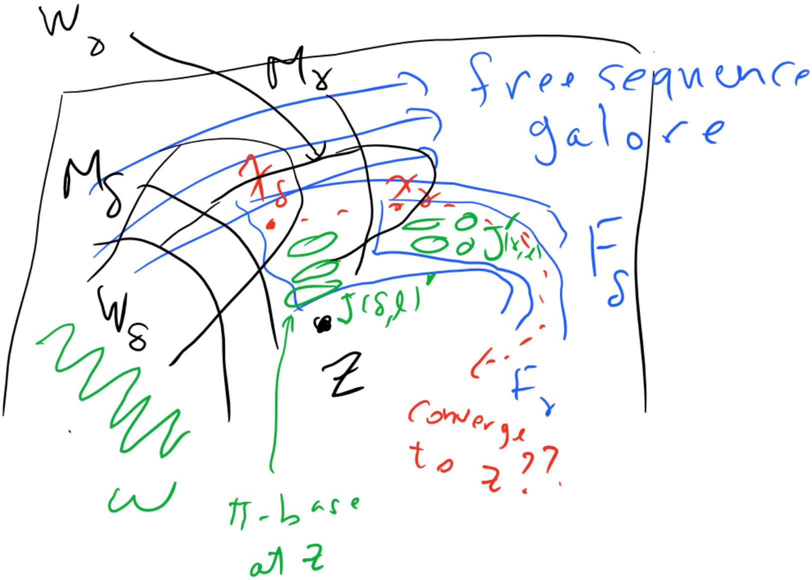
5. $\{M_\alpha : \alpha \in \omega_1\}$ is a continuous $\in$-chain of countable $M \prec (H(\omega_2), \in, \lhd, \mathcal{F}, \mathcal{I}, ...)$ – hence $[\omega_1]^{\aleph_0} \subset \bigcup_\alpha M_\alpha$.

Well, I did say there'd be a picture



Hausdorff-Luzin gaps to the rescue

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which means z is a CAP of the free sequence $\{Y_\delta : \delta \in C\}$

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4. [with Hart] it is consistent (Mahlo) to have a model of MoMr

   i.e. all compact countably tight are sequential

   with a compact sequential space with no G_δ-points.