Additivity of the ideal of microscopic sets

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Definition (J. Appell)

A set $M \subseteq \mathbb{R}$ is called microscopic ($M \in \mathcal{M}$) if for all $\varepsilon \in (0,1)$ there is a sequence of intervals $(I_k)_k$ such that $M \subseteq \bigcup_k I_k$ and $|I_k| \leqslant \varepsilon^k$ for all $k \in \mathbb{N}$.

Fact

$\mathcal{M}$ is a σ-ideal.

Question (G. Horbaczewska)

Is $add(\mathcal{M}) = 2^\omega$ under Martin's axiom?

$$\text{add}(\mathcal{I}) = \min\left\{\text{card}(\mathcal{A}) : \ \mathcal{A} \subseteq \mathcal{I} \ \wedge \ \bigcup \mathcal{A} \notin \mathcal{I}\right\}$$

Fact

$2^\omega = non(\mathcal{M}) = cov(\mathcal{M}) = cof(\mathcal{M})$ under Martin's axiom.

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$2^\omega = \mathit{non}(\mathcal{M}) = \mathit{cov}(\mathcal{M}) = \mathit{cof}(\mathcal{M})$ under Martin's axiom.

Let $(f_n)_n$ be a sequence of increasing functions $f_n\colon (0,1) \to (0,1)$ such that the following definition makes sense
(i.e., $\lim_{x\to 0^+} f_n(x) = 0$ for all n and there is $x_0 \in (0,1)$ such that for all $0 < x < x_0$ the sequence $(f_n(x))_n$ is non-increasing and $\sum_n f_n(x) < +\infty$).

Definition (G. Horbaczewska)

A set $M \subseteq \mathbb{R}$ is called (f_n)-microscopic ($M \in \mathcal{M}(f_n)$) if for all $\varepsilon \in (0,1)$ there is a sequence of intervals $(I_k)_k$ such that $M \subseteq \bigcup_k I_k$ and $|I_k| \leqslant f_k(\varepsilon)$ for all $k \in \mathbb{N}$.

$\mathcal{F}$ is the family of all $(f_n)_n$ satisfying the above conditions.

Proposition (G. Horbaczewska)

$\bigcap_{(f_n)_n\in\mathcal{F}} \mathcal{M}(f_n)$ is the family of all sets of strong measure zero.

Proposition (Czudek, K., Mrożek, Wołoszyn)

$\bigcup_{(f_n)_n\in\mathcal{F}} \mathcal{M}(f_n)$ is the family of all Lebesgue null sets.

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Definition

A set $M \subseteq \mathbb{R}$ is called nanoscopic if it is $\left(x^{2^n}\right)$-microscopic.

Question (G. Horbaczewska)

Is the family of all nanoscopic sets an ideal?

Theorem (Czudek, K., Mrożek, Wołoszyn)

No!

Question

How does the ideal/σ-ideal generated by nanoscopic sets look like? Is it equal to $\mathcal{M}(g_n)$ for some $(g_n)_n \in \mathcal{F}$?

Question

Let $f_n(\varepsilon) = \frac{\varepsilon}{2^n}$ for all $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$. Is $\mathcal{M}(f_n)$ an ideal?

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Theorem (Czudek, K., Mrożek, Wołoszyn)

If A is nanoscopic and $B \in \mathcal{SMZ}$, then $A \cup B$ is nanoscopic.

$M \subseteq \mathbb{R}$ is of strong measure zero ($M \in \mathcal{SMZ}$) if for every sequence $(\varepsilon_n)_n$ of positive reals there exists a sequence of intervals $(I_k)_k$ such that $M \subseteq \bigcup_k I_k$ and $|I_k| \leqslant \varepsilon_k$ for all $k \in \mathbb{N}$.

Theorem (Czudek, K., Mrożek, Wołoszyn)

There are an $\left(x^{n!}\right)$-microscopic (picoscopic) set X and a point $x \in \mathbb{R}$ such that $X \cup \{x\}$ is not picoscopic anymore!

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Theorem (Czudek, K., Mrożek, Wołoszyn)

Let $(f_n)_n \in \mathcal{F}$. Assume that $X \in \mathcal{M}(f_n)$ satisfies at least one of the following conditions:

- *X can be covered by an (f_n)-microscopic $\mathbf{F}_\sigma$ set;*
- *$\overline{X}$ is an unbounded interval;*
- *X is bounded.*

Then $X \cup Y \in \mathcal{M}(f_n)$ for any $Y \in \mathcal{SMZ}$.

Question

Let $(f_n)_n \in \mathcal{F}$ and $X \in \mathcal{M}(f_n)$ be such that $I \subseteq \overline{X}$ for some interval I. Is $X \cup Y \in \mathcal{M}(f_n)$ for any $Y \in \mathcal{SMZ}$?

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Definition

A set is in $\mathcal{M}^\star(f_n)$ if it can be covered by an (f_n)-microscopic $\mathbf{F}_\sigma$ set.

$\mathcal{M}(f_n) \setminus \mathcal{M}^\star(f_n) \neq \emptyset$ for any $(f_n)_n \in \mathcal{F}$.

Theorem (Czudek, K., Mrożek, Wołoszyn)

$\mathcal{M}^\star(f_n)$ is a σ-ideal for any $(f_n)_n \in \mathcal{F}$.

Proposition (K.)

Assume Martin's axiom. Then $\mathrm{add}\left(\mathcal{M}\left(x^{\ln(n+1)}\right)\right) = 2^\omega$.

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Definition

A set $M \subseteq \mathbb{R}$ is in $\mathcal{M}'$ if for every $\varepsilon \in (0,1)$ there are a set $D \subseteq \mathbb{N}$ of asymptotic density zero and a sequence of intervals $(I_k)_{k\in D}$ such that $M \subseteq \bigcup_{k\in D} I_k$ and $|I_k| \leqslant \varepsilon^k$ for all $k \in D$.

$D \subseteq \mathbb{N}$ is of asymptotic density zero if $\lim_n \frac{|D\cap\{1,\ldots,n\}|}{n} = 0$.

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A. Kwela,
Additivity of the ideal of microscopic sets.
Top. App. 204 (2016), 51-62.

K. Czudek, A. Kwela, N. Mrożek, W. Wołoszyn
Ideal-like properties of generalized microscopic sets.
Submitted.

Thank you for your attention!