Square sequences and simultaneous stationary reflection

Chris Lambie-Hanson

Einstein Institute of Mathematics
Hebrew University of Jerusalem

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Fruška Gora
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joint work with Yair Hayut

Reflection/compactness principles

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Canonical inner models, such as L, typically exhibit large degrees of incompactness, while the existence of large cardinals tends to imply compactness and reflection principles.

Stationary reflection

Definition

Let β be an ordinal of uncountable cofinality.

1. $S \subseteq \beta$ is *stationary* (in β) if $S \cap C \neq \emptyset$ for all closed, unbounded $C \subseteq \beta$.

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2. Suppose $S \subseteq \beta$ is stationary and $\alpha < \beta$ has uncountable cofinality. S *reflects at* α if $S \cap \alpha$ is stationary in α.

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3. Suppose $\mathcal{T}$ is a collection of stationary subsets of β and $\alpha < \beta$ has uncountable cofinality. $\mathcal{T}$ *reflects simultaneously at* α if S reflects at α for all $S \in \mathcal{T}$.

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Suppose $\kappa \leq \lambda$ are cardinals, with λ regular, and $S \subseteq \lambda$ is stationary. $\mathrm{Refl}(< \kappa, S)$ is the statement that, whenever $\mathcal{T}$ is a collection of stationary subsets of S and $|\mathcal{T}| < \kappa$, then $\mathcal{T}$ reflects simultaneously at some $\alpha < \lambda$.

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Square principles

Definition (Jensen, Schimmerling)

Suppose κ, μ are cardinals, with μ infinite. $\square_{\mu,<\kappa}$ is the assertion that there is a sequence $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \mu^+ \rangle$ such that:

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Note that, if $\vec{C}$ is a $\square_{\mu,<\kappa}$-sequence, then there cannot be a *thread* through $\vec{C}$, i.e. a club $D \subseteq \mu^+$ such that, for all $\alpha \in \lim(D)$, $D \cap \alpha \in \mathcal{C}_\alpha$.

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Theorem (Schimmerling, Foreman-Magidor)

Suppose $\square_{\aleph_\omega,<\omega}$ *holds. Then* $\mathrm{Refl}(1, S)$ *fails for every stationary* $S \subseteq \aleph_{\omega+1}$.

Square and stationary reflection

Theorem (Cummings-Foreman-Magidor)

Assuming the consistency of infinitely many supercompact cardinals, it is consistent that $\square_{\aleph_\omega,\omega}$ *and* $\mathrm{Refl}(<\omega, \aleph_{\omega+1})$ *both hold.*

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Theorem (CFM)

Suppose $n < \omega$ *and* $\square_{\aleph_\omega,\aleph_n}$ *holds. Then* $\mathrm{Refl}(\omega, S)$ *fails for every stationary* $S \subseteq \aleph_{\omega+1}$*.*

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Assuming the consistency of infinitely many supercompact cardinals, it is consistent that $\square^*_{\aleph_\omega}$ holds and $\mathrm{Refl}(<\aleph_\omega, S^{\aleph_{\omega+1}}_{<\aleph_n})$ holds for all $n < \omega$.

($S^\lambda_\kappa = \{\alpha < \lambda \mid \mathrm{cf}(\alpha) = \kappa\}$.)

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Principles of the form $\Box_{\mu,<\kappa}$ can be weakened by replacing the order-type restrictions with the requirement that the sequence have no thread.

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$\Box(\lambda, <\kappa^+) \equiv \Box(\lambda, \kappa)$.

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$\square(\lambda, <\kappa^+) \equiv \square(\lambda, \kappa)$. $\square(\lambda, 1) \equiv \square(\lambda)$.

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Suppose $\kappa < \lambda$ *are uncountable cardinals, with* λ *regular, and* $\square(\lambda, <\kappa)$ *holds. Then* $\mathrm{Refl}(<\kappa, S)$ *fails for every stationary* $S \subseteq S^{\lambda}_{\geq\kappa}$.

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Suppose $\kappa < \lambda$ are uncountable cardinals, with λ regular, and $\square(\lambda, < \kappa)$ holds. Then $\mathrm{Refl}(< \kappa, S)$ fails for every stationary $S \subseteq S^{\lambda}_{\geq\kappa}$.

Proof sketch: Suppose $\vec{\mathcal{C}} = \langle \mathcal{C}_\alpha \mid \alpha < \lambda \rangle$ is a $\square(\lambda, < \kappa)$-sequence and $S \subseteq S^{\lambda}_{\geq\kappa}$ is stationary such that $\mathrm{Refl}(< \kappa, S)$ holds.

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For all $\alpha < \lambda$, let $S_\alpha = \{\beta \in S \mid \alpha \in D_\beta\}$, and let $A = \{\alpha < \lambda \mid S_\alpha \text{ is stationary}\}$. A is unbounded in λ.

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Proof of claim: Find $\delta < \lambda$ such that $\mathcal{S} = \{S_\alpha \mid \alpha \in X \cup \{\gamma\}\}$ reflects simultaneously at δ, and fix $E \in \mathcal{C}_\delta$. For every $\alpha \in X \cup \{\gamma\}$, there is $\beta_\alpha \in \lim(E) \cap S_\alpha$. Then, since $\alpha \in D_{\beta_\alpha}$ and $\beta_\alpha \in \lim(E)$, we have $\alpha \in \lim(E)$. In particular, $E \cap \gamma \in \mathcal{C}_\gamma$ and $X \subseteq E \cap \gamma$. $\square$

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Claim: Suppose $\gamma \in A$. Then there is $C \in \mathcal{C}_\gamma$ such that $A \cap \gamma \subseteq C$.
Proof of claim: Suppose not. For each $C \in \mathcal{C}_\gamma$, find $\alpha_C \in (A \cap \gamma) \setminus C$. Let $X = \{\alpha_C \mid C \in \mathcal{C}_\gamma\}$. Now $X \in [A \cap \gamma]^{<\kappa}$, but there is no $C \in \mathcal{C}_\gamma$ such that $X \subseteq C$. □

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But now $\bigcup_{\gamma \in \lambda \cap \lim(A)} \{C \in \mathcal{C}_\gamma \mid A \cap \gamma \subseteq C\}$, ordered by the initial segment relation, is a tree of height λ, with levels of size $< \kappa$. It therefore has a cofinal branch, which corresponds to a thread through $\vec{\mathcal{C}}$. $\square$

Full square sequences

Conjecture

Suppose $\kappa < \lambda$ *are uncountable cardinals, with* λ *regular, and* $\square(\lambda, <\kappa)$ *holds. Then* $\mathrm{Refl}(<\kappa, S)$ *fails for every stationary* $S \subseteq \lambda$*.*

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Full square sequences and reflection

Theorem (Hayut-LH)

Suppose $\kappa < \lambda$ are uncountable cardinals, with λ regular, and there is a full $\square(\lambda, < \kappa)$-sequence. Then $\mathrm{Refl}(< \kappa, S)$ fails for every stationary $S \subseteq \lambda$.

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Theorem (Hayut-LH)

Suppose $\kappa < \lambda$ are uncountable cardinals, with λ regular, and there is a non-full $\square(\lambda, <\kappa)$-sequence. Then $\mathrm{Refl}(2, \lambda)$ fails.

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1. $\square(\aleph_{\omega+1}) + \mathrm{Refl}(1, \aleph_{\omega+1})$.
2. $\square(\aleph_{\omega+1}, 2)$+*whenever S is a stationary, co-stationary subset of $\aleph_{\omega+1}$, $\{S, \aleph_{\omega+1} \setminus S\}$ reflects simultaneously.*

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Assume the consistency of infinitely many supercompact cardinals. Then each of the following is consistent.

1. $\square(\aleph_{\omega+1}) + \mathrm{Refl}(1, \aleph_{\omega+1})$.
2. $\square(\aleph_{\omega+1}, 2)$+*whenever S is a stationary, co-stationary subset of $\aleph_{\omega+1}$, $\{S, \aleph_{\omega+1} \setminus S\}$ reflects simultaneously.*
3. $\square(\aleph_{\omega+1}, \aleph_m) + \forall(n < \omega)\mathrm{Refl}(< \aleph_m, S^{\aleph_{\omega+1}}_{<\aleph_n})$, *where $m < \omega$.*

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Analogous results can be obtained at other successors of singular cardinals, at successors of regular cardinals, and at inaccessible cardinals.

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As before, analogous results can be obtained for other successors of singulars, successors of regulars, and inaccessible cardinals.

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Thank you!