Reconstructing the topology of polymorphism clones of homogeneous structures

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Clones

Given a set A.

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$O_A^{(n)} := A^{(A^n)}$, $\qquad O_A := \bigcup_{n \in \mathbb{N} \setminus \{0\}} O_A^{(n)}$,

Projections

$e_i^n \in O_A^{(n)} : (x_1, \ldots, x_n) \mapsto x_i$ $\qquad$ (where $n \in \mathbb{N} \setminus \{0\}$, $1 \leq i \leq n$).

J_A denotes the set of all projections on A.

Clones

$C \subseteq O_A$ is called clone if

1. $J_A \subseteq C$,
2. it is closed with respect to composition.

Clone isomorphisms

A clone isomorphism between clones C and D is a bijection that preserves projections and composition.

Relational structures

Relational signatures

A relational signature is a pair $\underline{\Sigma} = (\Sigma, \mathrm{ar})$, where

- Σ is a set of relational symbols,
- $\mathrm{ar} : \Sigma \to \mathbb{N} \setminus \{0\}$.

Relational structures

A $\underline{\Sigma}$-structure is a pair $\mathbf{A} = (A, (\varrho^{\mathbf{A}})_{\varrho \in \Sigma})$, where

- A is a set,
- $\varrho^{\mathbf{A}} \subseteq A^{\mathrm{ar}(\varrho)}$, for each $\varrho \in \Sigma$.

Polymorphism clones

Given a relational signature $\underline{\Sigma}$, and a $\underline{\Sigma}$-structure $\mathbf{A}$.

Polymorphisms

$f \in O_A^{(n)}$ is called n-ary polymorphism of $\mathbf{A}$ if

$$f : \mathbf{A}^n \to \mathbf{A}.$$

The set of n-ary polymorphisms of $\mathbf{A}$ is denoted by $\mathrm{Pol}^{(n)}(\mathbf{A})$.

Polymorphism clones

$\mathrm{Pol}(\mathbf{A}) := \bigcup_{n \in \mathbb{N} \setminus \{0\}} \mathrm{Pol}^{(n)}(\mathbf{A})$ is a clone.

It is called the polymorphism clone of $\mathbf{A}$.

Topology on clones

Given a set A, equipped with the discrete topology.

Topology on $O_A^{(n)}$

- for every finite $M \subseteq A^n$ and for every $h : M \to A$:

$$\Phi_{M,h} := \{f \in O_A^{(n)} \mid f{\restriction}_M = h\}.$$

- together all $\Phi_{M,h}$ form the basis of a topology — the topology of pointwise convergence on $O_A^{(n)}$,

Topology on O_A

- O_A can be considered as the topological sum of the $O_A^{(n)}$.
- Composition of functions is continuous.

Topology on clones (cont.)

Topology on clones

- Every clone $C \leq O_A$ can be considered as topological subspace of O_A.
- Thus, every clone is canonically equipped with a topology, with respect to which the composition is continuous.

Metrization of the canonical topology on $O_A^{(n)}$ when $|A| = \omega$

- Let $\overline{w} = (\bar{a}_i)_{i<\omega}$ be an enumeration of A^n.
- Define $D_{\overline{w}} : O_A^{(n)} \times O_A^{(n)} \to \omega + 1$:

$$D_{\overline{w}}(f,g) := \begin{cases} \min\{i \in \omega \mid f(\bar{a}_i) \neq g(\bar{a}_i)\} & f \neq g \\ \omega & f = g. \end{cases}$$

- Then the following defines an ultrametric on $O_A^{(n)}$:

$$d_{\overline{w}}(f,g) := \begin{cases} 2^{-D_n(f,g)} & f \neq g \\ 0 & f = g. \end{cases}$$

Reconstruction and automatic homeomorphicity

Let $C \leq O_A$ be a closed clone.

Clones with reconstruction

C has reconstruction if whenever C is isomorphic to another closed subclone $D \leq O_A$, then C and D are isomorphic as topological clones.

Definition (Bodirsky, Pinsker, Pongrácz)

C has automatic homeomorphicity if every clone isomorphism from C to another closed clone on A is a homeomorphism.

Some clones with automatic homeomorphicity

Theorem (Bodirsky, Pinsker, Pongrácz)

The following clones have automatic homeomorphicity:

1. *every closed clone on A that contains $O_A^{(1)}$,*
2. *the polymorphism clone of the Rado graph,*
3. *the Horn-clone (= the smallest clone on a countable set A that contains all injective functions from O_A).*

More examples

Theorem (CP+MP)

Let $\mathbf{U}$ *be a countable homogeneous relational structure. If*

1. $\mathrm{Pol}(\mathbf{U})$ *contains all constant functions,*
2. $\mathrm{Age}(\mathbf{U})$ *has the free amalgamation property,*
3. $\mathrm{Age}(\mathbf{U})$ *is closed with respect to finite products,*
4. $\mathrm{Age}(\mathbf{U})$ *has the* HAP,

then $\mathrm{Pol}(\mathbf{U})$ *has automatic homeomorphicity.*

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Homogeneity

$\mathbf{U}$ is homogeneous if every local isomorphism of $\mathbf{U}$ extends to an automorphism of $\mathbf{U}$.

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Age($\mathbf{U}$)

The age of a relational structure $\mathbf{U}$ is the class of all finite relational structures of the same type as $\mathbf{U}$, that embedd into $\mathbf{U}$.

More examples

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Free amalgamation

$\mathrm{Age}(\mathbf{U})$ has the free amalgamation property if it is closed with respect to amalgamated free sums.

More examples

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HAP

A class $\mathcal{C}$ of structures has the HAP if for all $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{C}$,...

$$\begin{array}{ccc} \mathbf{B} & & \\ {\scriptstyle \iota}\big\uparrow & & \\ \mathbf{A} & \xrightarrow{g} & \mathbf{C}. \end{array}$$

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then $\mathrm{Pol}(\mathbf{U})$ *has automatic homeomorphicity.*

Example

The following structures have automatic homeomorphicity:

- the Rado graph with all loops added,
- the universal homogeneous digraph with all loops added.

Slightly changing the argument, it can be shown that also the countable generic poset $(\mathbb{P}, \leq)$ has automatic homeomorphicity.

Strong gate coverings (simplified definition)

Definition

Let $\mathbf{U}$ be a countable homogeneous structure.
A strong gate covering of $\mathrm{Pol}(\mathbf{U})$ consists of a family $(f_n)_{n \in \mathbb{N}_+}$ such that for all $n \in \mathbb{N}_+$:

- $f_n \in \mathrm{Pol}^{(n)}(\mathbf{U})$,
- for each convergent sequence $(g_j)_{j \in \mathbb{N}}$ in $\mathrm{Pol}^{(n)}(\mathbf{U})$ there exists a convergent sequence $(\iota_j)_{j \in \mathbb{N}}$ in $\mathrm{Emb}(\mathbf{U})$, such that for all $(x_1, \ldots, x_n) \in U^n$ we have

$$g_j(x_1, \ldots, x_n) = f_n(\iota_j(x_1), \ldots, \iota_j(x_n)).$$

Which structures have strong gate coverings?

Existence of strong gate coverings

Proposition (CP+MP)

Let $\mathbf{U}$ *be a countable homogeneous structure. If*

1. $\mathrm{Age}(\mathbf{U})$ *has the free amalgamation property,*
2. $\mathrm{Age}(\mathbf{U})$ *is closed with respect to finite products,*
3. $\mathrm{Age}(\mathbf{U})$ *has the* HAP*,*

then $\mathrm{Pol}(\mathbf{U})$ *has a strong gate covering*

Remark

The proof uses axiomatic Fraïssé-theory to show the existence of universal homogeneous polymorphisms of every arity. From this the existence of a strong gate covering follows at once.

Thus, all mentioned polymorphism clones have a strong gate covering.

Automatic homeomorphicity from strong gate coverings

Proposition (CP+MP)

Let $\mathbf{U}$ *be a countable homogeneous structure such that* $\mathrm{Pol}(\mathbf{U})$ *has a strong gate covering. Then* $\mathrm{Pol}(\mathbf{U})$ *has automatic homeomorphicity iff every clone isomorphism from* $\mathrm{Pol}(\mathbf{U})$ *to another closed clone on* U *is open.*

Proposition (Bodirsky, Pinsker, Pongrácz)

Let $\mathbf{U}$ *be a relational structure such that* $\mathrm{Pol}(\mathbf{U})$ *contains all constant functions. Then every isomorphism from* $\mathrm{Pol}(\mathbf{U})$ *to another closed clone on* U *is open.*

Thus the theorem is proved.

Even more examples

Theorem (CP+MP)

Let $\mathbf{U}$ *be a countable homogeneous relational structure. If*

1. $\mathrm{Aut}(\mathbf{U})$ *acts oligomorphically and transitively on* U,
2. $\mathrm{Emb}(\mathbf{U})$ *has automatic homeomorphicity,*
3. $\mathrm{Age}(\mathbf{U})$ *has the free amalgamation property,*
4. $\mathrm{Age}(\mathbf{U})$ *is closed with respect to finite products,*
5. $\mathrm{Age}(\mathbf{U})$ *has the* HAP,

then $\mathrm{Pol}(\mathbf{U})$ *has automatic homeomorphicity.*

Example

The following structures have automatic homeomorphicity:

- the Rado graph (already known from BPP),
- the universal homogeneous digraph,
- the universal homogeneous k-uniform hypergraph (for all $k \geq 2$).

Sketch of the proof

Let

- $C := \mathrm{Pol}(\mathbf{U})$,
- $D \leq O_U$ a closed clone,
- $h : C \to D$ a clone isomorphism.

Structure of the proof

h is continuous:

- $\mathrm{Emb}(\mathbf{U})$ has automatic homeomorphicity.
- Thus, $h\restriction_{\mathrm{Emb}(\mathbf{U})}$ is continuous.
- We need to "lift" continuity from $h\restriction_{\mathrm{Emb}(\mathbf{U})}$ to h.
- This is achieved using strong gate coverings.

h is open:

- This uses the topological Birkhoff Theorem by Bodirsky and Pinsker.

Lifting the continuity

Lemma (Bodirsky, Pinsker, Pongrácz)

Given

- *a countable homogeneous structure* $\mathbf{U}$,
- *a countable structure* $\mathbf{V}$,
- $\xi : \mathrm{Pol}(\mathbf{U}) \to \mathrm{Pol}(\mathbf{V})$, *such that* $\xi\restriction_{\mathrm{Emb}(\mathbf{U})}$ *is continuous.*

If $\mathrm{Pol}(\mathbf{U})$ *has a strong gate covering, then* ξ *is continuous.*

Thus, our particular h is continuous.

We still need to show that h is open.

How to obtain openness?

Proposition (CP+MP)

Let $\mathbf{U}$ *be a countable homogeneous relational structure. If*

1. $\mathrm{Aut}(\mathbf{U})$ *acts oligomorphically and transitively on* U,
2. $\mathbf{U}$ *has quantifier elimination for primitive positive formulae (QEPPF),*
3. $\mathrm{Age}(\mathbf{U})$ *has the free amalgamation property,*
4. $\mathrm{Age}(\mathbf{U})$ *is closed with respect to finite products,*

then every continuous isomorphism from $\mathrm{Pol}(\mathbf{U})$ *to another closed clone* $D \leq O_U$ *is open.*

Remark

- *The proof generalizes a neat idea from the proof of automatic homeomorphicity for the polymorphism clone of the Rado-graph in BPP.*
- *It uses a topological Birkhoff Theorem due to Bodirsky and Pinsker.*

Showing QEPPF

It only remains, to show QEPPF.

First observation:

Theorem (Romov)

A countable ω-categorical relational structure $\mathbf{U}$ has quantifier elimination for primitive positive formulae if and only if it is polymorphism homogeneous.

Remark

$\mathbf{U}$ is polymorphism homogeneous if every partial polymorphism of $\mathbf{U}$ with finite domain extends to a global polymorphism.

Showing QEPPF (2)

Second observation:

Lemma (folklore)

$\mathbf{U}$ *is polymorphism homogeneous if and only if* $\mathbf{U}^n$ *is homomorphism homogeneous, for every* $n \geq 1$.

Third observation:

Theorem (Dolinka)

A countable homogeneous structure $\mathbf{U}$ *is homomorphism homogeneous if and only if* $\mathrm{Age}(\mathbf{U})$ *has the* HAP.

Fourth obervation

Lemma (folklore)

Retracts of homomorphism homogeneous structures are homomorphism homogeneous, too.

Showing QEPPF (3)

Proposition (CP+MP)

Let $\mathbf{U}$ *be a countable homogeneous structure. If*

1. $\mathrm{Age}(\mathbf{U})$ *has the free amalgamation property,*
2. $\mathrm{Age}(\mathbf{U})$ *is closed with respect to finite products,*
3. $\mathrm{Age}(\mathbf{U})$ *has the* HAP,

then $\mathbf{U}^n$ *is isomorphic to a retract of* $\mathbf{U}$, *for every* $n > 1$.

Remark

The proof of this uses axiomatic Fraïssé-theory in order to show the existence of universal homogeneous retractions from $\mathbf{U}$ *to* $\mathbf{U}^n$.

Thus our particular structure $\mathbf{U}$ has QEPPF.
It follows that our particular h is open.
This finishes proof of the second theorem.

Open problems

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   - $(\mathbb{Q}, \leq)$ has automatic homeomorphicity - proved by Behrisch, Truss and Vargas

Open problems

1. Does the polymorphism clone of the rational Urysohn space have a strong gate covering (and hence automatic homeomorphicity)?
2. What about the polymorphism clone of rationals?
   - $(\mathbb{Q}, \leq)$ has automatic homeomorphicity - proved by Behrisch, Truss and Vargas
   - For $(\mathbb{Q}, <)$ it is still not known.