Regular families of small subsets of Polish spaces

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Notation and Terminology

Let X is a Polish space and $I \subseteq \mathscr{P}(X)$ s.t

- I is σ-ideal with a Borel base and
- I contains all singletons,

then (X, I) is Polish ideal space
Let $\mathcal{B}_+(I) = Borel(X) \setminus I$ be set of all I-positive Borel sets.
$Perf(X)$ stands for set of all perfect subsets of X

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Definition (Cardinal coefficients)

Let X - Polish space and $I \subseteq \mathscr{P}(X)$ be σ-ideal and $\mathscr{F} \subset I$ let

$$cov(\mathcal{F}) = min\{|\mathscr{A}| : \mathscr{A} \subset \mathcal{F} \wedge \bigcup \mathscr{A} = X\}$$

$$cov_h(\mathcal{F}) = min\{|\mathscr{A}| : \mathscr{A} \subset \mathcal{F} \wedge (\exists B \in \mathcal{B}_+(I)) \bigcup \mathscr{A} = B\}$$

$$cof(I) = min\{|\mathscr{B}| : \mathscr{B} \subseteq I \wedge (\forall A \in I)(\exists B \in \mathscr{B})\ A \subseteq B\}$$

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$\mathscr{N}$ σ-ideal of null sets and $\mathscr{M}$ σ-ideal of all meager subsets of X.
$cov(\mathscr{M}) = cov_h(\mathscr{M})$, $cov(\mathscr{N}) = cov_h(\mathscr{N})$.

Theorem (Cichoń-Pawlikowski)

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Complete I-nonmeasurability

Definition

Let (X, I) be Polish ideal space. We say that $A \subseteq X$ is completely I-nonmeasurable in X iff

$$(\forall B \in \mathcal{B}_+(I))\ A \cap B \neq \emptyset \wedge A^c \cap B \neq \emptyset.$$

- $A \subseteq X$ is complete $[X]^{\leq\omega}$-nonmeasurable iff A is Bernstein subset of X,
- $A \subseteq [0, 1]$ is complete $\mathcal{N}$-nonmeasurable iff $\lambda_*(A) = 0$ and $\lambda^*(B) = 1$,
- $A \subseteq X$ is complete $\mathcal{M}$-nonmeasurable if $\emptyset \neq U \subseteq X$ then $A \cap U$ does not have Baire property.

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Cichoń, Morayne, RR, Ryll-Nardzewski, Żeberski

Theorem (1)

Let X be an uncountable Polish space, $I \subseteq P(X)$ be σ-ideal with Borel base. Let $\mathcal{A} \subseteq I$ be a family of subsets of the space X such that

1. $X \setminus \bigcup \mathcal{A} \in I$,
2. $(\forall x \in X)\ (\bigcup\{A \in \mathcal{A} : x \in A\} \in I)$,
3. $cov_h(\{\bigcup\{A \in \mathcal{A} : x \in A\} : x \in X\}) \geq Cof(I)$.

Then there exists $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is completely I-nonmeasurable in space X.

J. Cichoń, M. Morayne, R. Rałowski, Cz. Ryll-Nardzewski, Sz. Żeberski, *On nonmeasurable unions*, Topology and its Applications, 154 (2007), pp.884-893,

Regular families

Theorem (RR and Żeberski)

Let X and Y be a Polish space and I be an c.c.c. σ-ideal with a Borel base. Let $F \subseteq X \times Y$ be an analytic relation such that

- $X \setminus \{x \in X : (\exists y \in Y)((x,y) \in F)\} \in I$,
- $(\forall y \in Y)(\{x \in X : (x,y) \in F\} \in I)$,
- $(\forall x \in X)(|\{y \in Y : (x,y) \in F\}| < \aleph_0)$.

Then there exists $Z \subseteq Y$ such that
$\{x \in X : (\exists y \in Z) : (x,y) \in F\}$ is completely I-nonmeasurable in X.

Rałowski, R., Żeberski, Sz., *Complete nonmeasurability in regular families*. Houston Journal of Mathematics, Vol. 34 No. 3 (2008), 773–780.

Proof

Choose $\mathcal{F} = \{F^y : \ y \in Y\}$ - is point finite.
F is analytic then

$$(\forall B \in Bor(X) \setminus I)\ B \subseteq [\bigcup \mathcal{F}]_I \to |\{A \in \mathcal{F} : \ B \cap A \neq \emptyset\}| = \mathfrak{c}.$$

If $D \subseteq X$ then $]D[_I$ is maximal (mod I) Borel set contained in D.
Define $\{\mathcal{A}_\xi \subseteq \mathcal{F} : \xi < \gamma\}$ such that

- $|\mathcal{A}_\xi| < \mathfrak{c}$,
- $\mathcal{A}_\xi \subseteq \mathcal{F} \setminus \bigcup_{\eta<\xi} \mathcal{A}_\eta$,
- $]\bigcup \mathcal{A}_\xi[_I$ is maximal element of $\{]\bigcup \mathcal{A}[_I\!: \ \mathcal{A} \subset \mathcal{F} \setminus (\bigcup_{\eta<\xi} \mathcal{A}_\eta) \wedge |\mathcal{A}| < \mathfrak{c}\}$

I is $c.c.c.$ then $]\bigcup \mathcal{A}_\xi[_I$ exists.

Because $\{\mathcal{A}_\xi \subseteq \mathcal{F} : \xi < \gamma\}$ is such that

- $|\mathcal{A}_\xi| < \mathfrak{c}$,
- $\mathcal{A}_\xi \subseteq \mathcal{F} \setminus \bigcup_{\eta<\xi} \mathcal{A}_\eta$,
- $]\bigcup \mathcal{A}_\xi[_I$ is maximal element of $\{]\bigcup \mathcal{A}[_I\colon\ \mathcal{A} \subset \mathcal{F} \setminus (\bigcup_{\eta<\xi} \mathcal{A}_\eta) \wedge |\mathcal{A}| < \mathfrak{c}\}$

Then

- if $\eta < \xi$ then $]\bigcup \mathcal{A}_\xi[\subseteq]\mathcal{A}_\eta[_I$,
- $\mathcal{F}$ is point finite then $]\mathcal{A}_\omega[_I = \emptyset$.

Set $\mathcal{F}_0 = \mathcal{F} \setminus \bigcup_{n\in\omega} \mathcal{A}_n$. Because

$$(\forall B \in Bor(X) \setminus I)\ B \subseteq [\bigcup \mathcal{F}]_I \to |\{A \in \mathcal{F} :\ B \cap A \neq \emptyset\}| = \mathfrak{c}.$$

- $[\bigcup \mathcal{F}_0]_I = [\bigcup \mathcal{F}]_I$ and
- $(\forall B \in Bor(X) \setminus I)(\forall \mathcal{A} \subseteq \mathcal{F}_0)\ B \subseteq \bigcup \mathcal{A} \to |\mathcal{A}| = \mathfrak{c}$

Enumerate
$\{B^0_\xi : \xi < \mathfrak{c}\} = \{B \in Bor(X) \setminus I : \ B \subseteq [\bigcup \mathcal{F}_0]_I \setminus]\bigcup \mathcal{F}_0[_I\}$ and
$\{B^1_\xi : \xi < \mathfrak{c}\} = \{B \in Bor(X) \setminus I : \ B \subseteq]\bigcup \mathcal{F}_0[_I\}$.
Define $\{(F^0_\xi, F^1_\xi, d_\xi) \in \mathcal{F} \times \mathcal{F} \times]\bigcup \mathcal{F}_0[_I : \xi < \mathfrak{c}\}$

- $F^0_\xi \cap B^0_\xi \neq \emptyset$,
- $F^1_\xi \cap B^1_\xi \neq \emptyset$,
- $d_\xi \in B^1_\xi$,
- $\{d_\eta : \eta < \xi\} \cap \bigcup_{\eta<\xi}(F^0_\eta \cup F^1_\eta) = \emptyset$.

Set $\mathcal{F}' = \{F^0_\xi : \xi < \mathfrak{c}\} \cup \{F^1_\xi : \xi < \mathfrak{c}\}$ then $\bigcup \mathcal{F}'$ is completely I-nonmeasurable in $[\bigcup \mathcal{F}]_I$.

Regular families

Theorem (RR and Żeberski)

Let X be a Polish space and be an I σ-ideal with Borel base with the following property

$$(\forall B \in Bor(X) \setminus I)(\exists P \in Perf(X) \setminus I)(P \subseteq B).$$

Let $\mathcal{A} \subseteq I$ be a partition of X such that

$$(\forall P \in Perf(X)) \left(\bigcup\{A \in \mathcal{A} : \ A \cap F \neq \emptyset\} \in Bor(X)\right).$$

Then there exists subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is completely I-nonmeasurable set in space X.

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Translations of small sets

Theorem

Let $P \subseteq \mathbb{R}$ be a null set such that

$$\{x \in \mathbb{R} : \bigcup\{t + P : x \in t + P\} \notin \mathscr{N}\} \in \mathscr{N}.$$

Then there exists $T \subseteq \mathbb{R}$ such that $T + P$ is completely $\mathscr{N}$-nonmeasurable.

Proof

For any real number t let us define $Z_t = \bigcup\{x + P : \ t \in x + P\}$ which is null by our assumption. Next, let us define

$$Z = \bigcup\{\{t\} \times Z_t : \ t \in \mathbb{R}\}.$$

Claim. Z is null in real plane $\mathbb{R}^2$

Proof.

Let observe that for any $t \in \mathbb{R}$ we have

$$\begin{aligned} t + Z_0 =& t + \bigcup\{x + P :\ 0 \in x + P\} = \bigcup\{(t + x) + P :\ 0 \in x + P\} \\ =& \bigcup\{(t + x) + P :\ t \in (t + x) + P\} \\ =& \bigcup\{y + P :\ t \in y + P\} = Z_t. \end{aligned}$$

$$\mathbb{R}^2(t, s) \mapsto \varphi(t, s) = (t, t + s) \in \mathbb{R}^2 \text{ is homeomorphism}$$

Wlog Z_0 is null G_δ then $Z = \varphi[\mathbb{R} \times Z_0]$ is G_δ in $\mathbb{R}^2$
Because Z is G_δ set and all sections Z_t are null in $\mathbb{R}$ for every $t \in \mathbb{R}$, then by the Fubini Theorem Z is null subset of the real plane $\mathbb{R}^2$.

□

Claim. $cov_h(\{Z_t : t \in \mathbb{R}\}) = \mathfrak{c}$

Proof.

Let $B \in Bor(\mathbb{R}) \setminus \mathcal{N}$ then $Z_B = B^2 \cap Z$ is null subset of $\mathbb{R}^2$.
By Mycielski Theorem
there is a perfect $Q \subseteq B$ s.t. $Q^2 \setminus \{(x,x) : x \in \mathbb{R}\} \subseteq B^2 \setminus Z$.
Then

$$(\forall s, s' \in Q)(s \neq s' \to Z_s \cap Z_{s'} = \emptyset)$$

and then $cov_h(\{Z_t : t \in \mathbb{R}\}) = \mathfrak{c}$

□

end of proof

Set $\mathcal{A} = \{t + P : \ t \in \mathbb{R}\}$ then

- $\bigcup \mathcal{A} = \mathbb{R}$,
- $(\forall x) \bigcup \{A \in \mathcal{A} : \ x \in A\} = Z_x \in \mathscr{N}$
- $cov_h(\{\bigcup\{A \in \mathcal{A} : x \in A\} : x \in X\}) = cov_h(\{Z_t : t \in \mathbb{R}\}) = \mathfrak{c}$.

Finally by Theorem (1) there is $\mathcal{A}' \subseteq \mathcal{A}$ s.t. $\bigcup \mathcal{A}'$ is complete $\mathscr{N}$-nonmeasurable.

Thank You