On subsets of ℓ_∞ deciding the norm convergence of sequences in ℓ_1

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Little ℓ-spaces

$$\ell_1 = \left\{x \in \mathbb{R}^\omega \colon\ \|x\|_1 = \textstyle\sum_n |x(n)| < \infty\right\}$$

$$\ell_\infty = \left\{y \in \mathbb{R}^\omega \colon\ \|y\|_\infty = \sup_n |y(n)| < \infty\right\}$$

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only if $\langle x_n, y\rangle \to 0$ for every $y \in S_{\ell_\infty}$ (*the weak convergence*)

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What about **if**? Does the weak convergence imply the norm one?

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Theorem (Schur, 1921)

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$c_0^* \cong \ell_1$

Let $e_n = (0, \ldots, 0, 1, 0, \ldots) \in c_0$

Then, $(e_n)_{n\in\omega}$ doesn't converge in norm

But for every $f \in \ell_1$, $\langle e_n, f\rangle = f(n) \to 0$

So, $(e_n)_{n\in\omega}$ converges weakly

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$\ell_1 \hookrightarrow \ell_1^{**} \cong \ell_\infty^* \cong \mathrm{ba}$ (by Riesz's representation theorem)

$$\ell_1 \ni x \mapsto \mu_x \in \mathrm{ba}$$

by the formula:

$$\mu_x(A) = \langle x, \chi_A \rangle = \sum_{n\in A} x(n)$$

for every $A \in \wp(\omega)$

Note that $\chi_A \in S_{\ell_\infty}$!

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Theorem (Phillips, 1948)

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So by Phillips: $\lim_n \sum_{j\in\omega} \left|\mu_{x_n}(\{j\})\right| = 0$

But this means exactly that: $\lim_n \left\|x_n\right\|_1 = 0$!

Phillips and Schur families

Definition

A family $\mathcal{F} \subseteq \wp(\omega)$ is **Phillips** if for every sequence $(\mu_n)_{n\in\omega} \subseteq \mathrm{ba}$ such that $\mu_n(A) \to 0$ for every $A \in \mathcal{F}$, we have

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$\wp(\omega)$ is Phillips

Quest for small Phillips families

Question

Is it consistent that there exists a Phillips family of cardinality strictly smaller than $\mathfrak{c}$?

Martin's axiom and Schur families

Theorem

Assume $\mathrm{MA}_\kappa(\sigma$*-centered*$)$ *for some cardinal number* κ*. Let* $\mathcal{F} \subseteq \wp(\omega)$ *be such that* $|\mathcal{F}| \leqslant \kappa$*.*

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In particular, $\mathcal{F}$ *is not Schur (and hence not Phillips).*

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A family $\mathcal{F} \subseteq [\omega]^\omega$ has **the strong finite intersection property** (**the SFIP**) if $\bigcap \mathcal{G}$ is infinite for every finite $\mathcal{G} \subseteq \mathcal{F}$.

A set $A \in [\omega]^\omega$ is **a pseudo-intersecton** of $\mathcal{F}$ if $A \setminus B$ is finite for every $B \in \mathcal{F}$.

$\mathfrak{p} = \min\{|\mathcal{F}| \colon \ \mathcal{F} \subseteq [\omega]^\omega \text{ has SFIP but no pseudo-intersection}\}$

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$\mathfrak{p} > \kappa$ if and only if $MA_\kappa(\sigma$-centered$)$ holds.

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2. *Under Martin's axiom, every Schur family is of cardinality $\mathfrak{c}$.*

$\mathrm{cof}(\mathcal{N})$ and Phillips families

Definition

$\mathcal{N}$ denotes the Lebesgue null ideal

$\mathrm{cof}(\mathcal{N}) = \min \{|\mathcal{F}| : \ \mathcal{F} \subseteq \mathcal{N} \ \& \ (\forall A \in \mathcal{N} \exists B \in \mathcal{F} : \ A \subseteq B)\}$

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There exists a Phillips family of cardinality $\mathrm{cof}(\mathcal{N})$.

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Bartoszyński–Judah characterization of $\mathrm{cof}(\mathcal{N})$, 1995

Let $\mathcal{C}$ denote the family of all subsets of ω^ω of the form $\prod_n T_n$ such that $T_n \in [\omega]^{n+1}$ for all $n \in \omega$. Then,

$$\mathrm{cof}(\mathcal{N}) = \min\{|\mathcal{F}|:\ \mathcal{F} \subseteq \mathcal{C}\ \&\ \bigcup \mathcal{F} = \omega^\omega\}.$$

Undecidability

Theorem

The existence of a Phillips (or Schur) family of cardinality strictly less than $\mathfrak{c}$ is independent of ZFC+$\neg$CH.

Weak* Banach–Steinhaus sets in ℓ_∞

Definition

Let $D \subseteq S_{\ell_\infty}$. A sequence $(x_n)_{n\in\omega}$ is:

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A set $D \subseteq S_{\ell_\infty}$ is **weak* Banach–Steinhaus** if every pointwise bounded on D sequence $(x_n)_{n\in\omega} \subseteq \ell_1$ is uniformly bounded.

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Weak* Banach–Steinhaus sets are uncountable and linearly weak* dense in ℓ_∞

Martin's axiom and weak* Banach–Steinhaus sets in ℓ_∞

Theorem

Assume MA_κ*(*σ*-centered) for some cardinal number* κ*. Let* $D \subseteq S_{\ell_\infty}$ *be such that* $|D| \leqslant \kappa$*. Then, there exists* $(x_n)_{n\in\omega} \subseteq \ell_1$ *such that* $\sup_n \|x_n\|_1 = \infty$ *and* $\lim_n \langle x_n, y\rangle = 0$ *for every* $y \in D$*.*

In particular, D *is not weak* Banach–Steinhaus in* ℓ_∞*.*

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In particular, D is not weak Banach–Steinhaus in* ℓ_∞*.*

Theorem

1. *Every weak* Banach–Steinhaus set in* ℓ_∞ *is of cardinality at least* $\mathfrak{p}$*.*
2. *Under Martin's axiom, every weak* Banach–Steinhaus set is of cardinality* $\mathfrak{c}$*.*

Schur families and weak* Banach–Steinhaus sets in ℓ_∞

Proposition

If $\mathcal{F} \subseteq \wp(\omega)$ is a Schur family, then $\{\chi_A \colon A \in \mathcal{F}\}$ is weak* Banach–Steinhaus.

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Thank you for the attention!