Pinning Down versus Density

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Novi Sad Conference in Set Theory and General Topology joint work with I. Juhász, J. van Mill and Z. Szentmiklóssy

Cardinal functions

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- X Hausdorff: $\mathrm{w}(X) \leq 2^{2^{2^{d(X)}}}$. Sharp (Juhász)

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Theorem (T. Banakh, A. Ravsky)

- *If X is T_2, $|X| < \aleph_\omega$, then* $\mathrm{pd}(X) = \mathrm{d}(X)$.
- *If $2^{2^{\mathrm{cf}(\kappa)}} > \kappa > \mathrm{cf}(\kappa)$, then there is a T_2 space X with $pd(X) < \mathrm{d}(X)$.*

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Questions

- *Regular example?*
- *ZFC example?*

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Theorem (I. Juhász, L.S., Z. Szentmiklóssy)
T.F.A.E:

(1) $2^{\kappa} < \kappa^{+\omega}$ *for each cardinal* κ,

(2) $\mathrm{pd}(X) = \mathrm{d}(X)$ *for each* T_2 *space* X,

(3) $\mathrm{pd}(X) = \mathrm{d}(X)$ *for each 0-dimensional* T_2 *space* X.

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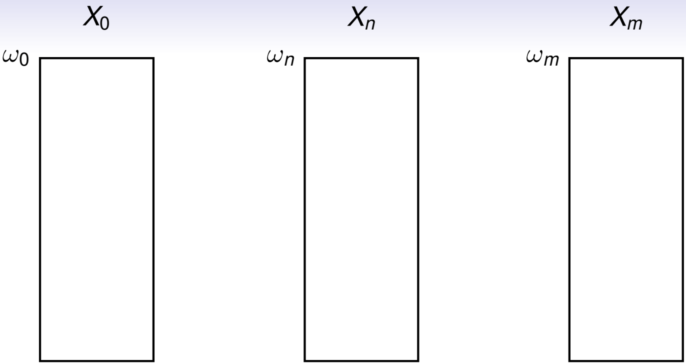
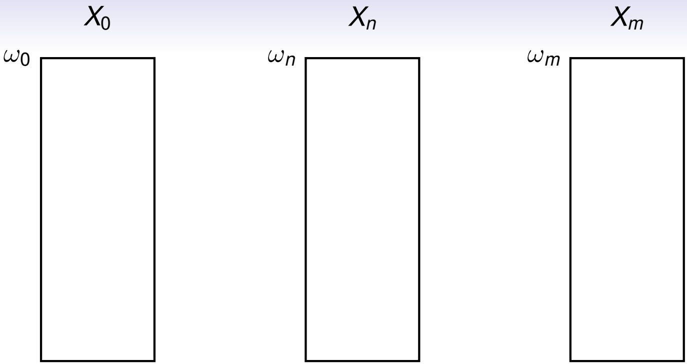
We prove:
If $2^\omega > \omega_\omega$ then there is a 0-dimensional space X with $pd(X) = \omega$ and $|X| = \Delta(X) = d(X) = \omega_\omega$.



• $X = \langle \omega_\omega \times \omega, \tau \rangle$ • $X_n = (\omega_n \setminus \omega_{n-1}) \times \omega$. • $\mathbb{P} = \prod(\omega_n \setminus \omega_{n-1})$.



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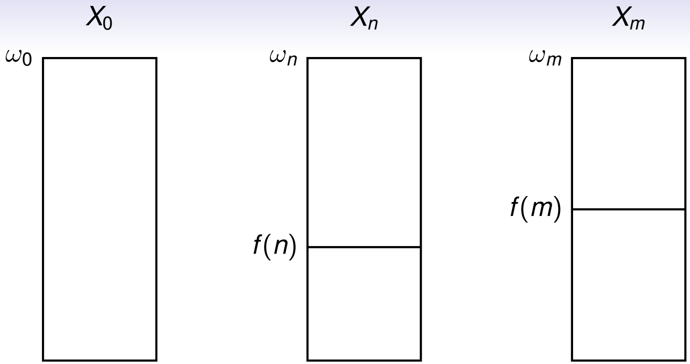
If $n \in \omega$, $f \in \mathbb{P}$, $A \subset \omega$ let $G(n, f, A) = \bigcup_{m \geq n} \Big((\omega_m \setminus f(m)) \times A \Big)$.



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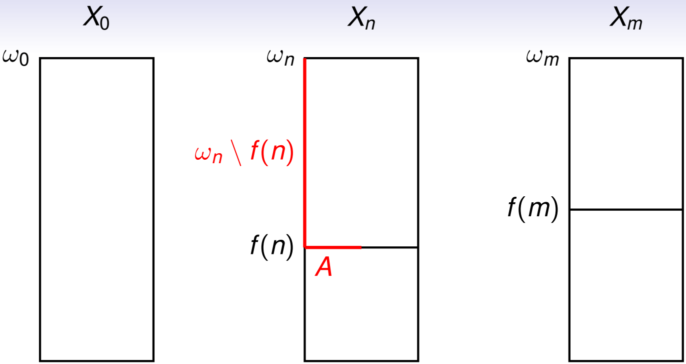

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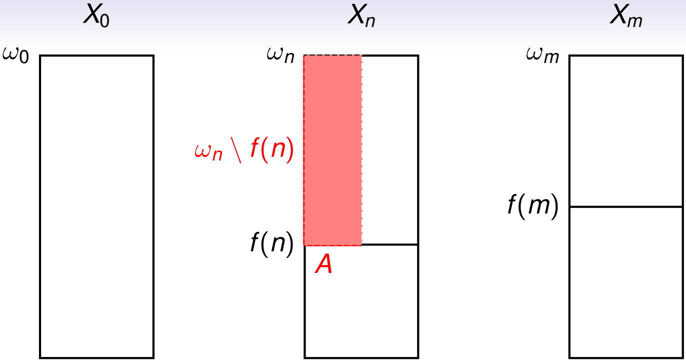

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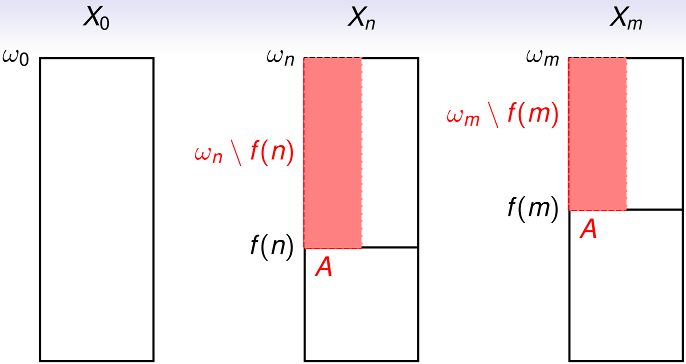

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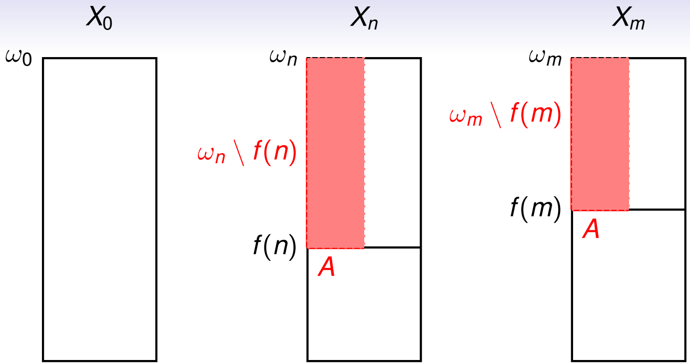

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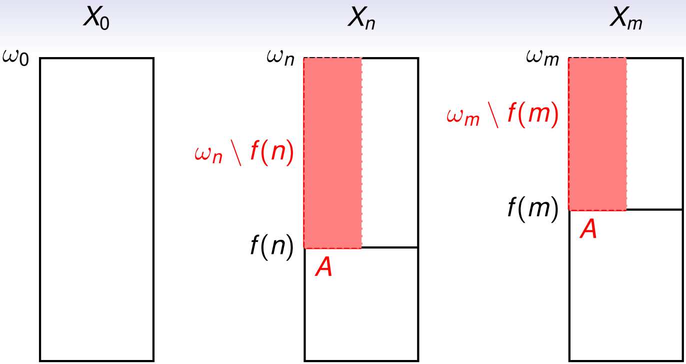

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- Then $G(n, f, A_{n,f}) \cap D = \emptyset$.

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- Then $G(n, f, A_{n,f}) \cap D = \emptyset$.
- Thus D is not dense.

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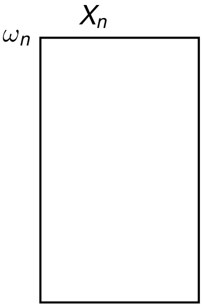

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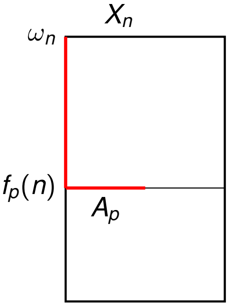

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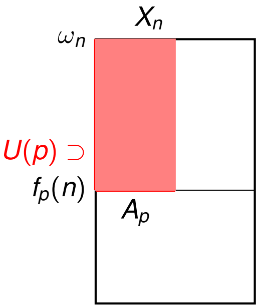

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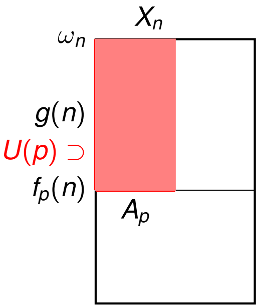

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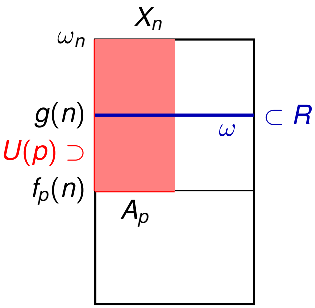

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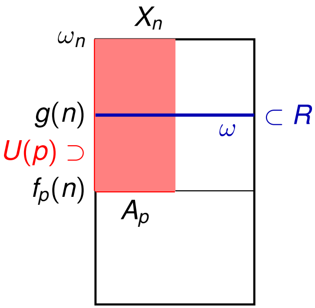

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Shelah's Strong Hypothesis:

$$\text{pp}(\mu) = \mu^+ \text{ for all singular cardinal } \mu.$$

An equiconsistency result

Theorem (I. Juhász, L.S., Z. Szentmiklóssy)

The following three statements are equiconsistent*:*

(i) *There is a singular cardinal* λ *with* $pp(\lambda) > \lambda^+$*, i.e. Shelah's Strong Hypothesis fails;*

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No equivalence:
Con(failure of SSH + the limit cardinals are strong limit)

Connected and locally connected spaces

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T:F.A.E:

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What about connected Tychonoff spaces?

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It is consistent that

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(1) *There is a singular cardinal $\mu \geq 2^\omega$ which is not a strong limit cardinal.*

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2. $pd(X) = pd(H)$,
3. *H is neat,*
4. *H is pathwise connected and locally pathwise connected.*

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Theorem (JvMSSz)

Let X be a $T_{3.5}$-space. Then

$$d(X) = d(F(X)) = d(A(X)).$$

If X is neat, then so are $A(X)$ and $F(X)$, and

$$pd(X) = pd(A(X)) = pd(F(X)).$$

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- The $O(V, \varepsilon)$ are the neighborhoods of the element $e^\bullet$ of $G^\bullet$ that generate the topology.

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$G^\bullet$ is a topological group and is pathwise connected and locally pathwise connected.
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Theorem (JvMSSz)

- $d(G) = d(G^\bullet)$.
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If X is compact then $pd(X) = \mathrm{d}(X)$.

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- *What about (regular) Lindelöf spaces?*
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Theorem (Juhász,van Mill, S, Szentmiklóssy)

It is consistent that $\mathrm{pd}(X) < \mathrm{d}(X)$ *for some hereditarily Lindelöf regular space X.*

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Sharp?

Yes.
It is consistent that $2^{\mathrm{pd}(X)}$ is as large as you wish and $\mathrm{d}(X)^+ = 2^{\mathrm{pd}(X)}$.

Inequalities

- Pospisil: $|X| \leq 2^{2^{\mathrm{d}(X)}}$ for T_2 spaces
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Problem
Does $w(x) \leq 2^{pd(x)}$ hold for regular spaces?