Conjectures of Rado and Chang and the Strong Tree Property

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Novi Sad Conference in Set Theory and General Topology
Iriški Venac, Fruška gora, Serbia. June 20th, 2016

Rado's Conjecture (RC)

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A family of intervals of a linearly ordered set is the union of countably many disjoint subfamilies (σ-disjoint) if and only if every subfamily of size $\aleph_1$ is σ-disjoint.

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Theorem (Feng, 1999)

Rado's Conjecture implies the presaturation of the nonstationary ideal on ω_1.

RC and MA_{ω_1}

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Theorem (Todorcevic)

RC *is equivalent to the following statement: A tree T of height ω_1 is the union of countable antichains (special) if and only if every subtree of T of size $\aleph_1$ is special.*

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Corollary

RC *and* $\mathrm{MA}_{\aleph_1}$ *are incompatible.*

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For every regular cardinal $\kappa \geq \omega_2$, there are arbitrary large λ such that for every countable $M \prec H_\lambda$ and for every $a \in [\kappa]^{\omega_1}$, there is a countable $M^* \prec H_\lambda$ and $b \in M^* \cap [\kappa]^{\omega_1}$ such that $M^* \supseteq M$ and $M^* \cap \omega_1 = M \cap \omega_1$.

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Given an ordinal λ and a set $X \subseteq [\lambda]^\omega$, we say X is *semi-stationary in* $[\lambda]^\omega$ if its $\sqsubseteq$-upward closure is stationary, i.e. if the set $\{y \in [\lambda]^\omega : \exists x \in X(x \sqsubseteq y)\}$ is stationary.

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Definition

The principle SSR asserts that the following statement $\mathrm{SSR}(\lambda)$ holds for every ordinal $\lambda \geq \omega_2$: for every semi-stationary subset $X \subseteq [\lambda]^\omega$, there is $W \in [\lambda]^{\omega_1}$ with $W \supseteq \omega_1$ such that $X \cap [W]^\omega$ is semi-stationary in $[W]^\omega$.

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Döbler and Schindler proved that both principles CC^* and SSR are equivalent.

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A regular cardinal κ has the *tree property* and we denote it by $\mathrm{TP}(\kappa)$, if every tree T of height κ, with levels of size less than κ has a cofinal branch.

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What about trees of height ω_2 and levels of size ω_1?

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A natural question is if under RC, the negation of the Continuum Hypothesis is enough to imply there are no $\aleph_2$-Aronzajn trees at all, i.e. if $\mathrm{TP}(\omega_2)$ holds.

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$\mathrm{CC}^* + \neg\mathrm{CH} \to \mathrm{TP}(\omega_2)$.

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He noticed that an inaccessible cardinal κ has the Strong Tree Property if and only if κ is strongly compact.

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We call $\mathscr{F} = \bigcup_{a \in [\kappa]^{<\lambda}} \mathscr{F}_a$ a (κ, λ)-tree, and $\mathscr{F}_a$ the level a of $\mathscr{F}$ for $a \in [\kappa]^{<\lambda}$.

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We furnish $\mathscr{F}$ with the following order: for $f, g \in \mathscr{F}$, $f \leq_{\mathscr{F}} g$ if and only if $g\restriction_{\mathrm{dom}(f)} = f$.

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Observe that in general, $\leq_{\mathscr{F}}$ is not a tree order.

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A cofinal branch trough $\mathscr{F}$ is a function $B : \kappa \to 2$ such that $B\restriction_a \in \mathscr{F}$ for every $a \in [\kappa]^{<\lambda}$.

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We say that λ has the Strong Tree Property if every (κ, λ)-tree has a cofinal branch for every $\kappa \geq \lambda$.

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CC^* *and* $\mathrm{MA}_{\omega_1}(\mathrm{Cohen})$ *together imply* $\aleph_2$ *has the Strong Tree Property.*

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RC *and* $\neg\mathrm{CH}$ *together imply* $\aleph_2$ *has the Strong Tree Property.*

We remark that CC^* is consistent with both CH and $\neg\mathrm{CH}$, and that CH implies $\neg\mathrm{TP}(\omega_2)$.

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We remark that CC^* is consistent with both CH and $\neg\mathrm{CH}$, and that CH implies $\neg\mathrm{TP}(\omega_2)$. Therefore, our result is in certain sense optimal.

Lemma (CC^*)

Let $\mathscr{F}$ be a (κ, ω_2)-tree with no cofinal branches. Then there are arbitrarily large θ such that for every countable $M \prec H_\theta$ we can find $M_0, M_1 \prec H_\theta$ countable and $a_0 \in M_0 \cap [\kappa]^{\omega_1}$, $a_1 \in M_1 \cap [\kappa]^{\omega_1}$ such that

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1. $M \cap \omega_1 = M_0 \cap \omega_1 = M_1 \cap \omega_1$,
2. $\mathscr{F}_{a_0} \cap M_0 \perp \mathscr{F}_{a_1} \cap M_1$.

We have the following:

Proposition

(CC^*) Let $\mathscr{F}$ be a (κ, ω_2)-tree with no cofinal branches. For λ sufficiently large, if the set

$$S_{\mathscr{F}} = \{M \in [H_\lambda]^\omega : \exists b \in [\kappa]^{\omega_1} \forall f \in \mathscr{F}_b \exists a \in M \cap [b]^{\omega_1} (f\restriction_a \notin M)\}$$

is nonstationary, then CH holds.

Suppose $S_{\mathscr{F}}$ is nonstationary, and let $F : [H_\lambda]^{<\omega} \to H_\lambda$ be a function such that if $M \in [H_\lambda]^\omega$ is closed under F, then $M \notin S_{\mathscr{F}}$.
As before, let $e : [\kappa]^{\omega_1} \times \omega_1 \to \mathscr{F}$ be a surjective function such that $e(a, \xi) \in \mathscr{F}_a$ for every $\xi \in \omega_1$.

Let θ be sufficiently large such that $\mathscr{F}, S_{\mathscr{F}}, F, e$ and all relevant parameters are in H_θ and where the conclusion of previous Lemma holds.

Using previous Lemma, build a binary tree $\langle M_\sigma \rangle_{\sigma \in 2^{<\omega}}$ of countable elementary submodels of H_θ with the property that for every $\sigma \in 2^{<\omega}$

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For every $r \in 2^\omega$, let $M_r = \bigcup_{n \in \omega} M_{r \restriction n}$. Let $b \in [\kappa]^{\omega_1}$ be such that $b \supseteq a$ for every $a \in M_\sigma \cap [\kappa]^{\omega_1}$ and every $\sigma \in 2^{<\omega}$.

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Claim

The application $r \mapsto f_r$ *is an injection from* 2^ω *to* $\mathscr{F}_b$ *(and therefore* CH *holds).*

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Let $r_0, r_1 \in 2^\omega$ with $r_0 \neq r_1$ and denote by f_i the node f_{r_i} for $i \in \{0,1\}$. We will find two predecessors of f_0 and f_1 that are incompatible. Let $n \in \omega$ such that $r_0\restriction_n = r_1\restriction_n = \sigma$, and $r_0\restriction_{n+1} \neq r_1\restriction_{n+1}$. Without loss of generality, suppose $r_i(n) = i$ for $i \in \{0,1\}$. By the construction of our binary tree, we can take $a_0 \in M_{r_0\restriction_{n+1}}$ and $a_1 \in M_{r_1\restriction_{n+1}}$ such that $\mathscr{F}_{a_0} \cap M_{r_0\restriction_{n+1}} \perp \mathscr{F}_{a_1} \cap M_{r_1\restriction_{n+1}}$.

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This finishes the proof of the Proposition.

We are ready to prove the main Theorem of this section.

Theorem

(CC^*) *If* CH *does not hold, then* ω_2 *has the Strong Tree Property.*

Proof of Theorem

[1] For example, let $h : X \to \omega_1$ be a bijection. So the set $\{h^{-1}[\alpha] : \alpha \in \omega_1 \setminus \omega\}$ is a club of size ω_1, and take its intersection with S.

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Assume CH does not hold, but suppose there is a (κ, ω_2)-tree $\mathscr{F}$ with no cofinal branches.

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Assume CH does not hold, but suppose there is a (κ, ω_2)-tree $\mathscr{F}$ with no cofinal branches. From Proposition, for λ sufficiently large, the set $S_{\mathscr{F}}$ is stationary in $[H_\lambda]^\omega$, and in particular it is semi-stationary. Without loss of generality, we can consider that every set in $S_{\mathscr{F}}$ is closed under e.

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Apply the Pressing Down Lemma to find $a \in [\kappa]^{\omega_1}$ and a stationary set $S'' \subseteq S'$ such that $a_x = a$ for every $x \in S''$.

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Further comments

Theorem (Sakai, Velickovic)

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Similarly, Magidor showed that an uncountable cardinal κ is supercompact if and only if it is inaccessible and has the super tree property.

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Lemma (Viale-Weiß)

If $\mathbb{P}$ is a proper standard iteration of length κ and suppose κ is inaccessible. If $\mathbb{P}$ forces κ has the super tree property, then κ is supercompact.

Theorem (Usuba)

Let κ be a strongly compact cardinal. Then there is a proper standard iteration of length κ $\mathbb{P}$ such that $\mathbb{P}$ forces Rado's Conjecture, $2^\omega = \omega_2$, and MA_{ω_1}(Cohen).

Corollary

$\mathrm{RC} + \mathrm{MA}_{\omega_1}(\textit{Cohen}) + \neg\mathrm{CH}$ *do not imply* ω_2 *has the super tree property.*

Thanks!