Nonmeasurable sets with respect to ideals defined by trees

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Trees

- $T \subseteq \omega^{<\omega}$ is a tree iff $(\forall \sigma \in T)(\forall n)(\sigma \restriction n \in T)$.
- A body of a tree T is defined by the formula

$$[T] = \{x \in \omega^\omega : \ (\forall n)(x \restriction n \in T)\}$$

Fact

For each tree T its body $[T]$ is a closed subset of ω^ω.

Definition

A tree $T \subseteq \omega^{<\omega}$ is

- a perfect tree iff $(\forall \sigma \in T)(\exists \tau \in T)(\tau \supseteq \sigma \wedge (\exists n \neq m)(\tau^\frown n, \tau^\frown m \in T)$;
- a Laver tree iff $(\exists \sigma \in T)(\forall \tau \in T)(\tau \subseteq \sigma \vee \{n \in \omega : \ \tau^\frown n \in T\}$ is infinite);
- a Miller tree iff $(\exists \sigma \in T)(\forall \tau \in T)(\tau \subseteq \sigma \vee (\exists \tau')(\tau \subseteq \tau' \wedge \{n \in \omega : \ \tau'^\frown n \in T\}$ is infinite);

Fact

A body of a perfect tree is a perfect set.

Definition of ideals defined by trees

A set $A \subseteq \omega^\omega$

- belongs to s_0 iff $(\forall T \in \mathbb{S})(\exists T' \in \mathbb{S})(T' \subseteq T \wedge [T'] \cap A = \emptyset)$;
- belongs to l_0 iff $(\forall T \in \mathbb{L})(\exists T' \in \mathbb{L})(T' \subseteq T \wedge [T'] \cap A = \emptyset)$;
- belongs to m_0 iff $(\forall T \in \mathbb{M})(\exists T' \in \mathbb{M})(T' \subseteq T \wedge [T'] \cap A = \emptyset)$;

where

- $\mathbb{S}$ denotes the family of all perfect trees,
- $\mathbb{L}$ denotes the family of all Laver trees,
- $\mathbb{M}$ denotes the family of all Miller trees.

Definition of *s*- *l*- and *m*-measurability

A set $A \subseteq \omega^\omega$

- is *s*-measurable iff
  $(\forall T \in \mathbb{S})(\exists T' \in \mathbb{S})(T' \subseteq T \wedge [T'] \cap A = \emptyset \vee [T'] \subseteq A)$;
- is *l*-measurable iff
  $(\forall T \in \mathbb{L})(\exists T' \in \mathbb{L})(T' \subseteq T \wedge [T'] \cap A = \emptyset \vee [T'] \subseteq A)$;
- is *m*-measurable iff
  $(\forall T \in \mathbb{M})(\exists T' \in \mathbb{M})(T' \subseteq T \wedge [T'] \cap A = \emptyset \vee [T'] \subseteq A)$;

where

- $\mathbb{S}$ denotes the family of all perfect trees,
- $\mathbb{L}$ denotes the family of all Laver trees,
- $\mathbb{M}$ denotes the family of all Miller trees.

Theorem (Brendle, 1995)

There are no inclusions between s_0, l_0, m_0.
In particular $s_0 \nsubseteq l_0$ and $s_0 \nsubseteq m_0$.

Brendle J., Strolling through paradise, Fundamenta Mathematicae, 148 (1), (1995), 1–25,

Fact

1. there is l-measurable set which is not s-measurable,
2. there is m-measurable set which is not s-measurable,
3. there is l-measurable set which is not m-measurable.

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Proof of 1.

- $2^\omega \subseteq \omega^\omega$.
- $2^\omega \in l_0$ and $2^\omega \notin s_0$.
- Choose $X \subseteq 2^\omega$ which is s-nonmeasurable.

$\square$

- $\mathcal{A} \subseteq \omega^\omega$ is a dominating family iff $(\forall x \in \omega^\omega)(\exists a \in \mathcal{A})(\forall^\infty n)(x(n) \leq a(n))$;
- $\mathfrak{d} = \min\{|\mathcal{A} : \ \mathcal{A} \subseteq \omega^\omega$ is a dominating family$\}$;
- $\mathcal{A} \subseteq \omega^\omega$ is an unbounded family iff $\neg(\exists x \in \omega^\omega)(\forall a \in \mathcal{A})(\forall^\infty n)(a(n) \leq x(n))$;
- $\mathfrak{b} = \min\{|\mathcal{A}| : \ \mathcal{A} \subseteq \omega^\omega$ is unbounded family$\}$.

Fact

1. If $\mathfrak{d} = \mathfrak{c}$ then there exists $A \subseteq \omega^\omega$ such that A is s-measurable and A is not l-measurable.
2. If $\mathfrak{b} = \mathfrak{c}$ then there exists $A \subseteq \omega^\omega$ such that A is s-measurable and A is not m-measurable.

Remark

To prove 1. it is enough to construct $A \in s_0 \setminus l_0$.

Proof of $\mathfrak{d} = \mathfrak{c} \implies \exists A \in s_0 \setminus l_0$

- $\mathbb{L} = \{L_\alpha : \alpha < \mathfrak{c}\}$,
- $\mathbb{S} = \{S_\alpha < \mathfrak{c}\}$.

Define a transfinite sequence:

$$((a_\xi, P_\xi) : \xi < \mathfrak{c})$$

satisfying for any $\xi < \mathfrak{c}$

1. $a_\xi \in [L_\xi]$,
2. $P_\xi \subseteq S_\xi$ and $P_\xi \in \mathbb{S}$,
3. for any $\eta < \xi$ $P_\eta \cap \{a_\beta : \beta < \xi\} = \emptyset$.

Finally, $A = \{a_\xi : \xi < \mathfrak{c}\}$.

□

Definition of $\mathcal{I}$-Luzin set

Let $\mathcal{I} \subseteq P(\omega^\omega)$ be a σ-ideal. $L \subseteq \omega^\omega$ is an $\mathcal{I}$-Luzin set iff

$$(\forall X \in \mathcal{I})(|X \cap L| < |L|)$$

Theorem (Wohofsky, WS2016)

There is no s_0-Luzin set.

Wohofsky W., There are no large sets which can be translated away from every Marczewski null set, WS2016 Hejnice, http://www.winterschool.eu/files/937...

Fact

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- There is no m_0-Luzin set.

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Proof, l_0 case

For every X such that $|X| = \mathfrak{c}$ there exists $A \subseteq X$ such that $A \in l_0$ and $|A| = \mathfrak{c}$.

$(\forall X)(|X| = \mathfrak{c} \to (\exists A \subseteq X)(A \in l_0, \wedge |A| = \mathfrak{c}))$

- $X \notin l_0$, so there is $L \in \mathbb{L}$ such that $|[L] \cap X| = \mathfrak{c}$.
- Fix a maximal antichain $\{L_\alpha : \alpha < \mathfrak{c}\}$ of Laver trees below L such that $|[L_\alpha] \cap X| = \mathfrak{c}$.
- Construct $a_\alpha \in X \setminus \bigcup_{\xi<\alpha}[L_\alpha]$.
- $A = \{a_\alpha : \ \alpha < \mathfrak{c}\}$.

□

Definition of m.e.d. familiy

- $x, y \in \omega^\omega$ are eventually different iff
$$(\forall^\infty n)(x(n) \neq y(n));$$
- A family $\mathcal{A} \subseteq \omega^\omega$ is e.d. family iff it consists of eventually different reals;
- A family $\mathcal{A} \subseteq \omega^\omega$ is m.e.d. family if it is e.d. family maximal with respect to inclusion.

Theorem (Rałowski, 2015)

It is consistent that there is a m.e.d. family $\mathcal{A} \subseteq \omega^\omega$ which is *cl*-nonmeasurable.

Rałowski R., Families of sets with nonmeasurable unions with respect to ideals defined by trees, Archive for Mathematical Logic, 54, no. 5-6, (2015), 649–658.

Theorem (Rałowski, 2015)

It is consistent that there is a m.e.d. family $\mathcal{A}$ which is *cl*-nonmeasurable and consists a dominating family of cardinality ω_1.

Rałowski R., Dominating m.a.d. families in Baire space, RIMS Kôkyûroku No.1949 (2015), pp. 73–80.

Theorem

There exists a m.e.d. family $\mathcal{A} \subseteq \omega^\omega$ such that $\mathcal{A}$ is not s, l, m-measurable and contains a dominating family of size $\mathfrak{d}$.

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Proof

- There exists an e.d. dominating family $\mathcal{D} \subseteq (4\mathbb{N})^\omega$, $|\mathcal{D}| = \mathfrak{d}$.
- Choose e.d. trees
  - $S \subseteq (4\mathbb{N}+1)^{<\omega}$, $S \in \mathbb{S}$,
  - $M \subseteq (4\mathbb{N}+2)^{<\omega}$, $M \in \mathbb{M}$,
  - $L \subseteq (4\mathbb{N}+3)^{<\omega}$, $L \in \mathbb{L}$.

Proof...

Enumerate

- $Perf(S) = \{S_\alpha : \alpha < \mathfrak{c}\}$,
- $Miller(M) = \{M_\alpha : \alpha < \mathfrak{c}\}$,
- $Laver(L) = \{L_\alpha : \alpha < \mathfrak{c}\}$.

For $\alpha < \mathfrak{c}$ we will define

$$w_\alpha = (a^s_\xi, d^s_\xi, x^s_\xi, a^m_\xi, d^m_\xi, x^m_\xi, a^l_\xi, d^l_\xi, x^l_\xi,)$$

satisfying

1. $a^s_\alpha, d^s_\alpha \in S_\alpha$,
2. $\{a^s_\xi : \xi < \alpha\} \cap \{d^s_\xi : \xi < \alpha\} = \emptyset$,
3. $\{a^s_\xi : \xi < \alpha\} \cup \{x^s_\xi : \xi < \alpha\}$ is e.d.,
4. $\forall^\infty n\, x^s_\alpha(n) = d^s_\alpha(n)$ but $x^s_\alpha \neq d^s_\alpha$.
5. ...

Proof...

$w_\alpha = (a^s_\xi, d^s_\xi, x^s_\xi, a^s_\xi, d^s_\xi, x^s_\xi, a^s_\xi, d^s_\xi, x^s_\xi,)$ satisfying

1. $a^s_\alpha, d^s_\alpha \in S_\alpha$,
2. $\{a^s_\xi : \xi < \alpha\} \cap \{d^s_\xi : \xi < \alpha\} = \emptyset$,
3. $\{a^s_\xi : \xi < \alpha\} \cup \{x^s_\xi : \xi < \alpha\}$ is e.d.,
4. $\forall^\infty n \; x^s_\alpha(n) = d^s_\alpha(n)$ but $x^s_\alpha \neq d^s_\alpha$.
5. $a^m_\alpha, d^m_\alpha \in M_\alpha$,
6. $\{a^m_\xi : \xi < \alpha\} \cap \{d^m_\xi : \xi < \alpha\} = \emptyset$,
7. $\{a^m_\xi : \xi < \alpha\} \cup \{x^m_\xi : \xi < \alpha\}$ is e.d.,
8. $\forall^\infty n \; x^m_\alpha(n) = d^m_\alpha(n)$ but $x^m_\alpha \neq d^m_\alpha$.
9. $a^l_\alpha, d^l_\alpha \in L_\alpha$,
10. $\{a^l_\xi : \xi < \alpha\} \cap \{d^l_\xi : \xi < \alpha\} = \emptyset$,
11. $\{a^l_\xi : \xi < \alpha\} \cup \{x^l_\xi : \xi < \alpha\}$ is e.d.,
12. $\forall^\infty n \; x^l_\alpha(n) = d^l_\alpha(n)$ but $x^l_\alpha \neq d^l_\alpha$.

Proof...

Now set

$$A_s = \{a_\alpha^s : \alpha < \mathfrak{c}\} \cup \{x_\alpha^s : \alpha < \mathfrak{c}\},$$

$$A_m = \{a_\alpha^m : \alpha < \mathfrak{c}\} \cup \{x_\alpha^m : \alpha < \mathfrak{c}\}$$

and

$$A_l = \{a_\alpha^l : \alpha < \mathfrak{c}\} \cup \{x_\alpha^l : \alpha < \mathfrak{c}\}$$

And finally

$$A \text{ is m.e.d. family containing } \mathcal{D} \cup A_s \cup A_m \cup A_l.$$

□

Thank You for Your Attention!

References

- Brendle J., Strolling through paradise, Fundamenta Mathematicae, 148 (1), (1995), 1–25,
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